Practice take home exam solutions, typed by Anna Hoskins.

1. Rate of change = rate in –rate out. Therefore the differential equation is:

$$\frac{dx}{dt} = 12 - .4x(t)$$

2. Find the general solution

(i)

$$y' + \frac{t}{1+t^2} y = (1+t^2)^{\frac{1}{2}}$$

$$I.F. = e^{\frac{1}{2}\int \frac{2t}{1+t^2}} = e^{\frac{1}{2}\ln(1+t^2)} = e^{\ln\sqrt{1+t^2}} = \sqrt{1+t^2}$$

$$\sqrt{1+t^2} + \frac{t}{\sqrt{1+t^2}} y = 1+t^2$$

$$(y\sqrt{1+t^2})' = 1+t^2$$

$$y\sqrt{1+t^2} = t + \frac{t^3}{3} + c$$

$$y = \frac{t}{\sqrt{1+t^2}} + \frac{t^3}{3\sqrt{1+t^2}} + \frac{c}{\sqrt{1+t^2}}$$

(ii)

$$y' = (1-t)(1+y^2)$$

$$\frac{dy}{dt} = (1-t)(1+y^2)$$

$$\frac{dy}{1+y^2} = (1-t)dt$$

$$\tan^{-1} y = t - \frac{t^2}{2} + c$$

3.

$$y'+\frac{t^2}{4-t^2}\,y'=\frac{3t}{4-t^2}\quad ,y(0)=\frac{5}{2}$$

$$4-t^2=(2+t)(2-t)$$

$p(t)$ and $q(t)$ are discontinuous at $t=\pm 2$. This splits the real line as follows :

———————|———————|———————

-2 0 2

Of these the longest interval containing t_0 =0 is (-2, 2).

$$y'+\frac{2t}{4-t^2}\,y=\frac{3t}{4-t^2}$$

$$I.F.=e^{\int\frac{2t}{4-t^2}dt}=e^{-\ln(4-t^2)}=e^{\ln\frac{1}{(4-t^2)}}=\frac{1}{4-t^2}$$. Multiplying by IF we have

$$\frac{1}{4-t^2}\,y'+\frac{2t}{(4-t^2)^2}\,y=\frac{3t}{(4-t^2)^2}\ \text{ or }\ \left(\frac{1}{4-t^2}\,y\right)'=-\frac{3}{2}-\frac{2t}{(4-t^2)^2}$$

(Integrating both sides) $\frac{1}{4-t^2}\,y=-\frac{3}{2}\left(-\frac{1}{4-t^2}\right)+c$

$\frac{1}{4-t^2}\,y=(\frac{3}{2})\frac{1}{4-t^2}+c$ or $y=\frac{3}{2}+c(4-t^2)$. Using $y(0)=\frac{5}{2}$ we get $4c=1$ or $c=\frac{1}{4}$

$\therefore$ Which gives $y=\frac{3}{2}+\frac{1}{4}(4-t^2)$ or $y=\frac{5}{2}-\frac{1}{4}t^2$

4.

$\frac{dx}{dt}+12x=24x^{\frac{3}{4}}$. Dividing through by $x^{\frac{3}{4}}$ we get $x^{-\frac{3}{4}}\frac{dx}{dt}+12x^{\frac{1}{4}}=24$

Putting $v=x^{\frac{1}{4}}$ and differentiating we get $\frac{dv}{dt}=\frac{1}{4}x^{-\frac{3}{4}}\frac{dx}{dt}$ or $4\frac{dv}{dt}=x^{-\frac{3}{4}}\frac{dx}{dt}$

Substituting, the equation becomes,

$4\frac{dv}{dt}+12v=24$ or $\frac{dv}{dt}+3v=6$, which reduces to

$\frac{dv}{dt}=-3(v-2)$ or $\frac{dv}{v-2}=-3dt$ Integrating $\ln|v-2|=-3t+c_1$ or $v-2=ce^{-3t}$,

or $v=2+ce^{-3t}$. Now $x^{\frac{1}{4}}=2+ce^{-3t}$ or $x=(2+ce^{-3t})^4$

5.

(i)

$$(2yx^2+2y)dx+(2xy^2+2x)dy=0$$

M N

$$\frac{\partial M}{\partial y}=2x^2+2, \quad \frac{\partial N}{\partial x}=2y^2+2$$

$$\frac{\partial M}{\partial t}\neq\frac{\partial N}{\partial t}$$

is not exact

(ii)

$$(2x^2+2t+1)dt+(4x^3+4tx)dx=0$$

M N

$$\frac{\partial M}{\partial x}=4x, \ \frac{\partial N}{\partial t}=4x$$

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t} \ \therefore \ exact.$$

$$Now \ \ M=\frac{\partial f}{\partial t}, \quad N=\frac{\partial f}{\partial x}$$

$$f=\int(2x^2+2t+1)dt$$

$$=2x^2t+t^2+t+\Psi(x)$$

$$N=\frac{\partial f}{\partial x}=4xt+\Psi'(x)=4x^3+4tx$$

$$\Rightarrow \Psi'(x)=4x^3 \text{ intgrating } \Psi(x)=x^4$$

and the solution is

$$f=2x^2t+t^2+t+x^4=c$$

6.

(i)

$$y''-2y'+y=0$$

Characteristic equation : $r^2-2r+1=0$, which gives $(r-1)^2=0$

the roots are : $r=1$ repeated twice. So, $y_1=e^t$, $y_2=te^t$

are fundamental solutions and the general solution is:

$$y=c_1e^t+c_2te^t$$

6. (ii)

$$2y'' - 2y' + y = 0$$

Characteristic Equation

$$2r^2 - 2r + 1 = 0$$

$$r = \frac{2 \pm \sqrt{4-8}}{4}$$

$$r = \frac{2 \pm 2i}{4} = \frac{1 \pm i}{2}$$

$$\therefore y_1 = e^{(\frac{t}{2} - \frac{it}{2})} = e^{\frac{t}{2}}(\cos\frac{t}{2} + i\sin\frac{t}{2})$$

$$y_2 = e^{\left(\frac{t}{2} - \frac{it}{2}\right)} = e^{\frac{t}{2}}\left(\cos\frac{t}{2} + i\sin\frac{t}{2}\right)$$

Fundamental solutions: $e^{\frac{t}{2}}\cos(\frac{t}{2}),\quad e^{\frac{t}{2}}\sin(\frac{t}{2})$

General Solution: $y = e^{\frac{t}{2}}(c_1\cos\frac{t}{2} + c_2\sin\frac{t}{2})$

7.

$$ty'' + 2y' = 0, \quad t > 0. \text{ Dividing by } t \quad y'' + \frac{2}{t}y' = 0$$

$$W(y_1, y_2) = ce^{-\int \frac{2}{t}dt} = ce^{-2\ln t} = ce^{\ln\frac{1}{t^2}} = \frac{c}{t^2}$$

$$W(y_1, d) = \begin{vmatrix} y_1, d \\ y_1', 0 \end{vmatrix} = \frac{c}{t^2} \quad or \quad dy_1' = \frac{c}{t^2} \text{ or } y_1' = -\frac{c_1}{t^2}$$

$$\text{Integrating we get } y_1 = \frac{c_1}{t}$$

8.

(a)

$$\frac{dS}{dt} = rS + k = r(S + \frac{k}{r})$$

$$\frac{dS}{S + \frac{k}{r}} = rdt. \text{ Integrating we get } \ln(S + \frac{k}{r}) = rt + c \text{ which gives}$$

$$S + \frac{k}{r} = ce^{rt} \text{ or } S = -\frac{k}{r} + ce^{rt}$$

Initially S(0)=0 c=k/r

$$\therefore S = -\frac{k}{r} + \frac{k}{r}e^{rt}$$

8. (b)

$$rS = k(e^{rt} - 1) \quad or \quad k = \frac{rS(t)}{e^{rt} - 1}$$

$$k = \frac{(.075)10^6}{e^{(.075)40} - 1} = \frac{75(10^3)}{e^3 - 1} \approx 3929.68$$

8. (c)

$$r10^6 = 2000(e^{40r} - 1)$$

$500r = e^{40r} - 1$ Solving, using a graphic calculator, we get

$r = .0927,\ i.e,\ 9.77\%$

9.

$$15u'' - 2u' - u = 0, u(0) = u'(0) = 1$$

Characteristic Equation

$$15r^2 - 2r - 1 = 0 \quad or \quad r = \frac{2 \pm \sqrt{4 + 60}}{30}$$

or $r = \frac{1}{3}, -\frac{1}{5}$ which gives $u = c_1 e^{\frac{t}{3}} + c_2 e^{-\frac{t}{5}}$

$$u(0) = 1 = c_1 + c_2$$

$$u'(0) = \frac{1}{3}c_1 - \frac{1}{5}c_2 = 1$$

$$c_1 = \frac{9}{4} \quad and \quad c_2 = -\frac{5}{4} \ \therefore u = \frac{9}{4}e^{\frac{t}{3}} - \frac{5}{4}e^{-\frac{t}{5}}.$$