$6.4,\ page\ 321,\ problems\ 2, 3, 4, 10$

$2.\ y'' + 2y' + 2y = h(t) \qquad y(0) = 0,\ y'(0) = 1$

$$h(t)\begin{cases} 0 & 0 \le t < p \\ 1 & p \le t < 2p \\ 0 & t \ge 2p \end{cases}$$

$$L(h(t)) = \int_{p}^{2p} e^{-st}dt = \frac{e^{-ps} - e^{-2ps}}{s}$$

$$\therefore\ h(t) = u_p(t) - u_{2p}(t)$$

$$L(y'') + 2L(y') + 2L(y) = \frac{e^{-ps} - e^{-2ps}}{s}$$

or

$$s^2 L(y) - sy(0) - y'(0) + 2(sL(y) - y(0)) + 2L(y) = \frac{e^{-pp} - e^{-2ps}}{s}$$

or, as $y(0) = 0$ and $y'(0) = 1$,

$$\left(s^2 + 2s + 2\right)L(y) - 1 = \frac{e^{-ps} - e^{-2ps}}{s}$$

$$L(y) = \frac{1}{s^2 + 2s + 1 + 1} + \frac{(e^{-ps} - e^{-2ps})}{s}\left[\frac{1}{s(s^2 + 2s + 2)}\right] = \frac{1}{(s+1)^2 + 1} + (e^{-ps} - e^{-2ps})[F(s)]$$

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

$$1 = A(s^2 + 2s + 2) + (Bs + C)s$$

$$1 = (A + B)s^2 + (2A + C)s + 2A$$

$$A + B = 0;\ \ 2A + C, A = \frac{1}{2}$$

$$B = -\frac{1}{2};\ C = -2\left(\frac{1}{2}\right) = -1.\text{ Consequent ly}$$

$$F(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{\frac{1}{2}s + 1}{(s+1)^2 + 1} = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{\frac{1}{2}s + \frac{1}{2} + \frac{1}{2}}{(s+1)^2 + 1} = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s+1}{(s+1)^2 + 1}\right) - \frac{1}{2}\left(\frac{1}{(s+1)^2 + 1}\right)$$

$$\therefore K(t) = L^{-1}(F(s)) = \frac{1}{2} - \frac{e^{-t}}{2}\cos t - \frac{1}{2}e^{-t}\sin t$$

Now

$$L(y) = \frac{1}{(s+1)^2 + 1} + e^{-ps}F(s) - e^{-2ps}F(s)$$

$$L(y) = L(e^{-t}\sin t) + L(K(t)) - e^{-2ps}L(K(t))$$

So,

$$y = e^{-t}\sin t + \mathsf{L}^{-1}\left(e^{-ps}K(t) - \mathsf{L}^{-1}e^{-2ps}\,\mathsf{L}(K(t))\right) = e^{-t}\sin t + u_p\,K(t-p) - u_{2p}\,K(t-2p)$$

Now

$$K(t-p) = \frac{1}{2} - \frac{1}{2}e^{-t+p}\cos t + \frac{1}{2}e^{-t+p}\sin t$$

$$K(t-2p) = \frac{1}{2} - \frac{1}{2}e^{-t+2p}\cos t + \frac{1}{2}e^{-t+2p}\sin t$$

So the solution is

$$y = e^{-t}\sin t + u_p(t)\left(\frac{1}{2} + \frac{1}{2}e^{-t+p}\cos t + \frac{1}{2}e^{-t+p}\sin t\right) - u_2(t)\left(\frac{1}{2} - \frac{1}{2}e^{-t+2p}\cos t + \frac{1}{2}e^{-t+2p}\sin t\right)$$

3. $y'' + 4y = \sin t - u_{2p}(t)\sin(t-2p) \quad y(0)=0,\ y'(0)=0$

$$(s^2+4)L(y) = \frac{1}{s^2+1} - e^{-2ps}\frac{1}{s^2+1} = (1-e^{-2ps})\frac{1}{s^2+1}$$

$$\mathrm{L(y)} = (1-e^{-2ps})\frac{1}{(s^2+1)(s^2+4)} \quad \text{or}$$

$$L(y) = (1-e^{-2ps})[F(s)] \quad \text{where}$$

$$F(s) = \frac{1}{(s^2+1)(s^2+4)}$$

$$= \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4} = L(h(t))$$

(as in #4) $\quad \therefore \quad h(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$

$$L(y) = (1-e^{-2ps})[L(h(t))]$$

$$= L(h(t)) - e^{-2ps}L(h(t))$$

$$y = h(t) - L^{-1}(e^{-2ps}L(h(t)))$$

$$= h(t) - u_{2p}(t)h(t-2p)$$

$$= h(t) - u_{2p}(t)h(t)$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - u_{2p}(t)\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)$$

4. $y'' + 4y = \sin t + u_p(t)\sin(t - p); \quad y(0) = 0, \ y'(0) = 0$

$$L(y'') + 4L(y) = L(\sin t) + L(u_p(t)\sin(t - p))$$

$$(s^2 + 4)L(y) = \frac{1}{s^2 + 1} + e^{-ps}\frac{1}{s^2 + 1}$$

$$\therefore \ L(y) = (e^{-ps} = 1)\left[\frac{1}{(s^2 + 1)(s^2 + 4)}\right]$$

$$= (e^{-ps} + 1)F(s)$$

$$F(s) = \frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{(s^2 + 1)} + \frac{Cs + D}{(s^2 + 4)}$$

$$1 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1)$$

$$1 = (A + C)s^3 + (B + D)s^2 + (4A + C)s + 4B + D$$

$$A + C = 0 \ \ (1) \ \ B + D = 0 \ (2) \ \ 4A + C = 0 \ (3) \ \ 4B + D = 1(4)$$

$$From(1)\&(3) \ A = C = 0; \ \ From \ (2)\& \ (4) \ B = \frac{1}{4}, \ D = -\frac{1}{3}$$

$$\therefore \frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3}\left(\frac{1}{(s^2 + 1)}\right) - \frac{1}{6}\left(\frac{2}{(s^2 + 4)}\right)$$

$$L^{-1}(F(s)) = h(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

$$L(y) = (e^{-ps} + 1)[L(h)] = e^{-ps}L(h) + L(h)$$

$$\therefore \ y = L^{-1}(e^{-ps}L(h(t))) + h(t)$$

$$y = u_p(t)h(t - p) + h(t)$$

$$y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u_p(t)\left(\frac{1}{3}\sin(t - p) - \frac{1}{6}\sin 2t\right)$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u_p(t)\left(\frac{1}{3}(-\sin t) - \frac{1}{6}\sin 2t\right)$$

10. $y'' + y' + \frac{5}{4}y = g(t) = \begin{cases} \sin t & 0 \le t < p \\ 0 & t \ge p \end{cases} \qquad y(0) = 0, \ y'(0) = 0$

$$(s^2 + s + \frac{5}{4})L(y) = L(g(t)) = \int_0^p \sin(t)e^{-st}dt = \left(\frac{e^{-pt} + 1}{s^2 + 1}\right)$$

So

$$L(y) = (e^{-pt} + 1)\left[\frac{1}{(s^2+1)(s^2+s+\frac{5}{4})}\right]$$

$$\frac{1}{(s^2+1)(s^2+s+\frac{5}{4})} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+s+\frac{5}{4}}$$

$$1 = (As+B)\left(s^2+s+\frac{5}{4}\right) + (Cs+D)(s^2+1)$$

$$1 = (A+C)s^3 + (A+B+D)s^2 + \left(\frac{5}{4}A+B+C\right)s + \left(\frac{5}{4}B+D\right)$$

$A + C = 0$ (1); $\qquad A + B + D = 0$ (2);

$\frac{5}{4}A + B + C = 0$ (3); $\qquad \frac{5}{4}B + D = 1$ (4)

$(3) - (1)$ *gives* $\qquad (4) - (2)$ *gives*

$\frac{1}{4}A + B = 0$ (5); $\qquad -A + \frac{1}{4}B = 1$

$(5) + (6)$ *gives* $\frac{17}{4}B = 1$ *or* $B = \frac{4}{17}$

From$(5)\ A = -\frac{16}{17}$ *is from* $C = \frac{16}{17}$

Now $D = -A - B = \frac{16}{17} - \frac{4}{17} = \frac{12}{17}$

$$L(y) = (e^{-ps} + 1)\left[\frac{-\frac{16}{17}s + \frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s + \frac{12}{17}}{\left(s^2+\frac{1}{2}\right)^2 + 1}\right]$$

$$L(y) = (e^{-ps} + 1)\{L(h(t))\}$$

$$Where\ L(h) = -\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{\frac{16}{17}\left(s+\frac{1}{2}\right) + \frac{4}{17}}{\left(s+\frac{1}{2}\right)^2 + 1}$$

$$\therefore L(h(t)) = -\frac{16}{17}L(\cos t) + \frac{4}{17}L(\sin t)$$

$$L(h) == -\frac{16}{17}L(\cos t) + \frac{4}{17}L(\sin t) + \frac{16}{17}L(e^{-\frac{1}{2}t}\cos t) + \frac{4}{17}L(e^{-\frac{1}{2}t}\sin t)$$

or

$$L(h(t)) = L\left[-\frac{16}{17}\cos t + \frac{4}{17}\sin t + \frac{16}{17}e^{-\frac{1}{2}t}\cos t + \frac{4}{17}e^{-\frac{1}{2}t}\sin t\right]$$

$$h(t) = -\frac{16}{17}\cos t + \frac{4}{17}\sin t + \frac{16}{17}e^{-\frac{1}{2}t}\cos t + \frac{4}{17}e^{-\frac{1}{2}t}\sin t$$

Now

$$L(y) = (e^{-ps} + 1)L(h(t))$$

$$or\ \ L(y) = e^{-ps}L(h(t)) + L(h(t))$$

$$or\ \ y = u_p(t)h(t-p) + h(t)$$