QUESTION: (HD0803) In your article, "Putting t-invertibility to use", you mention on page 443 that you have an example of a t-linked extension that is not t-compatible and doesn't satisfy any of (a)-(d) on pages 442 and 443, and that this example would be included in another article. (This question was asked by Jesse Elliott of CSU Channel Island.)

ANSWER: I am learning, painfully, not to make promises about future work that has not been put on paper or computer. In any case I looked into my old e-files that have survived and I found something that was a sort of spin-off from "Putting t-invertibility to use". In it there was an example which I put down below. But first a little bit of introduction.

The article referred to above is: Putting t-invertibility to use, by Zafrullah [Z1, non-Noetherian commutative ring theory, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000]. The conditions (a)-(d) are to do with extensions of domains $D \subseteq R$. Those conditions, from pages 442-443, are listed below.

(a): $A^{-1}R = (AR)^{-1}$, for all nonzero finitely generated fractional ideals of D

(b): $(A^{-1}R)_v = (AR)^{-1}$, for all nonzero finitely generated fractional ideals of D

(c): $A^{-1}R = (AR)^{-1}$, for all nonzero fractional ideals of D

(d): $(A^{-1}R)_v = (AR)^{-1}$, for all nonzero fractional ideals of D.

Also in an extension of domains $D \subseteq R$, the domain R is said to be t-linked over D if for each finitely generated ideal A of D with $A^{-1} = D$ we have $(AR)^{-1} = R$. (The above sentence is often abbreviated as: R is t-linked over D and usually, now, R is not required to be an overring of D.)

On page 443 of the above mentioned article I stated the following, "We note that in each of the extensions $D \subseteq R$, discussed above, R is t-linked over D, i.e., for every $A \in f(D), A^{-1} = D$ implies $(AR)^{-1} = R$ (Dobbs, Houston, Lucas and Zafrullah [DHLZ, Comm. Algebra 17(1989), 2835-2852] (*)). So in each case there is a homomorphism $\theta : Cl_t(D) \longrightarrow Cl_t(R)$ defined by $\theta([A]) = [(AR)_t]$ (Anderson-Houston-Zafrullah [AHZ, JPAA 86(1993), 109-124]). However, if R is t-linked over D, the extension $D \subseteq R$ may not satisfy any of (a)- (d) and may not satisfy any of the equivalent conditions. (These facts will be included in a detailed account in the promised article.)"

The equivalent conditions, referred to in the above cited passage, are given in my version of Proposition 1.1 of Barucci-Gabelli-Roitman [BGR, Comm. Algebra 22(1994), 173-211] and I list them below.

Proposition 0. ([Z 1, Proposition 2.6]) Let $D \subseteq R$ be an extension of integral domains then the following statements are equivalent:

Proposition 1 *(1). $I_{v_D}R \subseteq (IR)_{v_R}$ for each nonzero finitely generated $I \in F(D)$.*

(2). $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ for each finitely generated $I \in F(D)$.

(3). $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ for each $I \in F(D)$.

(4). $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ for each $I \in F(D)$.

(5). If A is an integral t-ideal of R such that $A \cap D \neq (0)$, then $A \cap D$ is a t-ideal of D.

(6). $I_{t_D} R \subseteq (IR)_{t_R}$ for each $I \in F(D)$.

Moreover, under these equivalent conditions, the map $[A] \longrightarrow [(AR)_{t_R}]$ is a homomorphism from $Cl_t(D)$ to $Cl_t(R)$.

If these equivalent conditions hold for an extension $D \subseteq R$, we say that $D \subseteq R$ is *t-compatible.*

Here's a limited example.

Example 1 (from "spin-off"). Let D be a one-dimensional (Noetherian) local domain that is not a DVR and let R be the integral closure of D. Then the extension $D \subset R$ is t-linked according to [DHLZ], because the maximal ideal of D is a t-ideal, and it is well known that R is a PID.

Now, since D is not a DVR, there is at least one non-invertible nonzero ideal A in D, which can well be M. So $AA^{-1} \subseteq M$ where M is the maximal ideal of D. Thus $AA^{-1}R = dR$ where d is a nonunit of the PID R. Next as $AA^{-1}R = (AR)(A^{-1}R)$, we have $(AR)(A^{-1}R) = dR$. Multiplying both sides by the (principal) fractional ideal $(AR)^{-1}$ we get $A^{-1}R = dR(AR)^{-1} \neq (AR)^{-1}$, because d is a nonunit.

This takes care of (a) and (c) because for a Noetherian D, $f(D) = F(D)$. Next because R is a PID, every nonzero (fractional) ideal of R is principal and hence divisorial, so $A^{-1}R = (A^{-1}R)_v$. This of course takes care of (b) and (d).

The example looks and sounds like one of those tongue in the cheek examples (that do not seem right) and it does not do the job promised in the above mentioned paper as we shall see below. We have now another example that, hopefully, will do the job completely.

To get that example we need to quote an example from Mimouni [M, Comm. Algebra, 33(2005), 1345-1355), Example 2.10]. (The purpose of this example is to show that there are some w-ideals that are not t-ideals. For details read [M].)

Example 2. Set $T = Q(\sqrt{2})[[X, Y, Z]]$ where X, Y, Z are indeterminates over Q. Then $T = Q(\sqrt{2}) + M$. Now set $R = Q + M$. By Gabelli and Houston [GH, Michigan Math. J. 44(1997), 99-112], $R = Q + M$ is a local noetherian domain with integral closure T. Since the maximal ideal M is common to both R and T, $M = MT$ and so are the following prime ideals contained in M. $P_1 = XT$, $P_2 = (X, Y)T$. Then we have $P_1 \subsetneq P_2 \subsetneq M$. We claim that P_2 is not a t-ideal of R while M is a t-ideal of R. This follows from the the following observations. Since $ht_T(P_2) = 2$ we have $T = T : P_2 = (P_2 : P_2) = R : P_2$. Similarly $T = T : M = M : M = R : M$.

Now as, with respect to R, $M^{-1} \supsetneq R$ we must have $M_v \subsetneq R$. But since $R = Q + M$ is local $M_v = M$. Next as, with respect to R, $P_2^{-1} = M^{-1}$, we have $(P_2)_v = M_v = M$. But as $P_2 \subsetneq M$ we conclude that P_2 is not a t-ideal. From this example we also conclude that every overring of R is t-linked [DHLZ, Theorem 2.6], because the only maximal ideal M of R is a v-ideal and hence a t-ideal.

Now reason as follows.

By Proposition 1.1 of Barucci-Gabelli-Roitman [BGR] (**) an extension of domains $R \subseteq Z$ is such that for every nonzero finitely generated ideal A of R we have $(AZ)_{v_Z} = ((A_{v_R})Z)_{v_Z}$, if and only if every t-ideal of Z contracts to a t-ideal of D or to 0. So if there is a Z with $R \subseteq Z$ an extension of domains and a nonzero t-ideal I of Z with $I \cap R$ a nonzero non-t-ideal then for some finitely generated ideal A of R we have $(AZ)_{v_Z} \neq ((A_{v_R})Z)_{v_Z}$. If Z is also t-linked over R then we have our example, that was promised in [Z1].

Example 3. Let us go back to Example 2. In $R = Q + M$ we have a chain of prime ideals $P_1 \subsetneq P_2 \subsetneq M$ of which P_2 is not a t-ideal. Now by Corollary 19.7 of Gilmer [G, Multiplicative Ideal Theory, Dekker 1972], there is a valuation overring Z of R with a chain of proper prime ideals $K_1 \subsetneq K_2 \subsetneq L$, where L is a maximal ideal of Z, such that $K_i \cap R = P_i$ and $L \cap R = M$. Now, it is well known that in a valuation domain every nonzero ideal is a t-ideal. So, K_2 is a t-ideal in Z that contracts to a non-t-ideal P_2. Now why is this observation important? As we noted in the explanation of the Mimouni example, Z is t-linked over $R = Q + M$. So, as we reasoned above, there is a (finitely generated) non zero ideal A of $R = Q + M$, which we do not know anything about, such that $(AZ)_{v_Z} \neq ((A_{v_R})Z)_{v_Z}$.

Finally, will Example 3 take care of (a) - (d)? Let us check. First off, the example we have is a Noetherian domain, so we need only to take care of (a) and (b). Next, every fractional ideal I of a domain D is expressible as $I = \frac{A}{x}$ where A is an integral ideal of D, so (a)-(d) can be stated for integral ideals; because the fractions cancel out in each case. Note also the usual (old) convention that the inverses and hence the v-operations, in (a)-(d), are with respect to the relevant rings (rings to whose (fractional) ideals the operation is applied).

Now we need a result from the "spin-off".

Lemma 4. Let $D \subseteq R$ be an extension of domains. If for a nonzero (integral) ideal A of D, $(AR)^{-1} = A^{-1}R$ or $(AR)^{-1} = (A^{-1}R)_v$ then $(AR)_v = (A_vR)_v$.

Proof. Note that in both cases $(AR)_{v_R} = (A^{-1}R)^{-1} (= R : (D : A))$. Now let $x \in A_v$ then $xA^{-1} \subseteq D$, which gives $xA^{-1}R \subseteq R$ and this forces $x \in (A^{-1}R)^{-1} = (AR)_v$. So $A_v \subseteq (AR)_v$, which gives $A_vR \subseteq (AR)_v$ and hence $(A_vR) \subseteq (AR)_v$. Finally since $A \subseteq A_v$ which gives $AR \subseteq A_vR$ and hence $(AR)_v \subseteq (A_vR)_v$ and so the equality follows.

From Lemma 4 we note that if for some integral ideal A of D we have $(AR)_v \neq (A_vR)_v$, then $(AR)^{-1} \neq A^{-1}R$ and $(AR)^{-1} \neq (A^{-1}R)_v$.

Example 5. Going back to our Example 3, the ideal A for which $(AZ)_{v_Z} \neq ((A_{v_R})Z)_{v_Z}$ is precisely the ideal for which $(AZ)^{-1} \neq A^{-1}Z$ and for which $(AZ)^{-1} \neq (A^{-1}Z)_v$. So Example 3 also serves as the example of a domain extension for which (a)-(d) do not hold.

Now as shown in Example 5, thanks to taking contrapositives, $(AZ)_{v_Z} \neq ((A_{v_R})Z)_{v_Z} \Rightarrow ((AZ)^{-1} \neq A^{-1}Z) \wedge (AZ)^{-1} \neq (A^{-1}Z)_v$, what if $(AZ)^{-1} \neq A^{-1}Z$ and/or $(AZ)^{-1} \neq (A^{-1}Z)_v$ hold(s)?

We show that in either case $(AZ)_{v_Z} \neq ((A_{v_R})Z)_{v_Z}$ may not hold (i.e. $(AZ)_{v_Z} = ((A_{v_R})Z)_{v_Z}$ may hold).

Before we give an example let us recall the following lemma From Fossum [F, The divisor class group of a Krull domain, Ergebnisse der Mathematik und ihrer grenzgebiete B. 74, Springer-Verlag, Berlin, Heidelberg, New York, 1973]

Lemma 6 [F, Lemma 4.5 (Beck)]. Let A be the integral closure of of the Noetherian integral domain B. Let $\{a_1, a_2, ..., a_n\}$ be a set of elements in A. Then $B:(B:\sum Ba_i) \subseteq A:(A:\sum Aa_i)$.

Now look at it this way, the elements a_i can well be in B. So let $I = (a_1, a_2, ..., a_n) \subseteq B$. Then Lemma 6 says that $(I)_{v_B} \subseteq (IA)_{v_A}$. This gives $(I)_{v_B}A \subseteq (IA)_{v_A}$ and hence $((I)_{v_B}A)_{v_A} \subseteq (IA)_{v_A}$. The reverse containment is easy once we note that $I \subseteq I_{v_B}$, so $IA \subseteq I_{v_B}A$, which leads to $(IA)_{v_A} \subseteq (I_{v_B}A)_{v_A}$. Thus we can say that for every nonzero ideal I of B with integral closure A, A and B as in Lemma 6, we have $(IA)_{v_A} = (I_{v_B}A)_{v_A}$. Now note that for an ideal I of a domain R and for every nonzero element of $qf(R)$ $(xI)_{v_R} = xI_{v_R}$. So if J were a nonzero fractional ideal with $J = \frac{I}{x}$ for I integral and $x \in D\backslash\{0\}$ we can have $(IA)_{v_A} = (I_{v_B}A)_{v_A}$ and dividing the equation by x and using the property of the v-operation we shall have $(\frac{I}{x}A)_{v_A} = (\frac{I_{v_B}}{x}A)_{v_A}$. This gives us the following statement.

Lemma 7. Let Z be the integral closure of the Noetherian integral domain R. Let A be a nonzero fractional ideal of R. Then $(AZ)_{v_Z} = (A_{v_R}Z)_{v_Z}$. (***)

Going back to Example 1 we can see that there is a one dimensional local domain R with integral closure Z such that for any non invertible ideal A of R we have $(AZ)^{-1} \neq (A^{-1}Z)$ where $A^{-1}Z$ is a v-ideal. Thus we have the following Proposition.

Proposition 8. There exist integral domains R and Z such that for every ideal A we have $(AZ)_{v_Z} = ((A_{v_R})Z)_{v_Z}$ but for some ideals B we may have $(BZ)^{-1} \neq B^{-1}Z$ and/or $(BZ)^{-1} \neq (B^{-1}Z)_v$, with the usual convention about inverses and v-operations.

Finally you may ask the question that I made the promise in [Z1] in the year 2000, but I am using an Example that was published in 2005. For the life of me, I cannot answer that question, for I do not remember what I had in mind when I was writing the paper. (One of the perks of getting old, perhaps.) The only thing that I can recall is the feeling that Proposition 1.1 of [BGR] was very restrictive and so there would have to be a t-linked overring/extension that does satisfy the conditions of that proposition.

(*) A better reference for this kind of t-linked extensions is [AHZ].

(**) The importance of [BGR, Proposition 1.1] is in the fact that it subsumes the following results to produce something remarkable. (1) If A is a finitely generated nonzero ideal and S a multiplicative set of a domain D then $(AD_S)_v = (A_vD_S)_v$ (Zafrullah [Z2, Manuscripta Math. 24(1978), 191-203]) and (2) Lemma 3.4 of Kang [K, J. Algebra 123(1989), 151-170] which says: If S is a multiplicative set of D then $(AD_S)_t = (A_tD_S)_t$for every nonzero ideal A of D.

(***) Evan Houston has provided a direct proof of Lemma 7 as follows. (About this proof he said that it might be known, perhaps but I got it from him.)

Houston's alternate for Lemma 7. Let D be Noetherian with integral closure R, and let A be a nonzero ideal of D. Then $(A_vR)_v = (AR)_v$ (where the outer v's are taken with respect to R.

Proof: Let $x \in (AR)^{-1}$, so that $xA \subseteq R$. Since R is integral over D and A is finitely generated, there is a (necessarily) finitely generated ideal J of D with $xAJ \subseteq J$. Hence xA_vJ_v subset J_v. Since J_v is finitely generated, this yields $xA_v \subseteq R$, i.e., $xA_vR \subseteq R$. Hence $(AR)^{-1} \subseteq (A_vR)^{-1}$. The reverse inclusion is obvious, and, taking inverses, we have the conclusion.

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