QUESTION (HD1001) You refer to Conrad's F-condition a lot, in lattice ordered groups G, their generalizations, and in the so called multiplicative ideal theory; and it confuses me. I keep worrying about a situation in an l.o. group G where the condition F holds yet for some $0 < a \in G$ we have that for every $n \in N$ there is a set E_n consisting of pairwise disjoint elements below a. I would like to see a direct proof or an explicit reference where it is shown that the above situation cannot occur.

ANSWER. Let us agree to use G for a lattice ordered group. Conrad's F-condition for G a lattice ordered group reads: Each $0 < x \in G$ exceeds at most a finite number of disjoint positive elements.

Here's the reference: [C]. P. Conrad, Some structure theorems for lattice ordered groups, Trans. Amer. Math. Soc. 99(1961), 212-240. You can find it mentioned on page 212 of the paper. Theorems 5.2 and 6.2 of that Conrad paper and their proofs say the following.

$(*)$. If G satisfies F then every (strictly) positive element is greater than or equal to at least one and at most a finite number of mutually disjoint basic elements.

Let us note that a basic element in G is a strictly positive element b such that the interval $[0, b]$ is a chain, i.e. for every pair $x, y \in [0, b]$, $x \leq y$ or $y \leq x$, two elements x, y are disjoint if $\inf(x, y) = 0$, i.e., $x \wedge y = 0$. It is easy to see that

$(*_1)$: if $x \wedge y = 0$, $0 < r \leq x$ and $0 < s \leq y$ then $r \wedge s = 0$.

$(*_2)$: If x is basic and $b_1, b_2, ..., b_n$ are disjoint then x can be non-disjoint with at most one of the b_i.

$(*_3)$: An element $0 < x \in G$ is non-basic if and only if there are at least two disjoint positive elements preceding x.

In the condition F the key is the phrase "at most a finite number" which, can be interpreted to mean that there is a finite number m such that the number of mutually disjoint elements below x would have to be less than or equal to m. But as you do not seem to like that interpretation I provide below some proofs of $(*)$.

(1). Suppose that $0 < x \in G$ where G satisfies F. By Theorem 5.1 of [C] x exceeds at least one basic element. This means that if $S = \{x_1, x_2, ..., x_m\}$ is a set of mutually disjoint elements then there is correspondingly a set $T = \{b_1, b_2, ..., b_n\}$ of mutually disjoint basic elements, where each of b_i is less than or equal to (at least) one x_i (cf. $(*_1)$), where $| T | \geq | S |$. (Call this process association of basic elements.)

Let $x > 0$ and let S be the set of mutually disjoint basic elements that are less than or equal to x. If S is not maximal then there is a strictly positive element u_1 that is disjoint with every member of S and is less than or equal to x. Then there is a basic element $b_1 \leq u_1$ by [C, Theorem 5.1]. By $(*_1)$ b_1 is disjoint with every member of S. Take $S_1 = S \cup \{b_1\}$. We can repeat this procedure to improve S_1 to S_2 etc. but because of condition F we cannot continue this procedure of improvements indefinitely. So there is an n such that there is no improvement on S_n. Obviously S_n is a maximal set of basic elements that are less than or equal to x. This n is the largest size any set of mutually disjoint elements can have. The reason is that if U is another set of mutually disjoint elements that are less than or equal to x, then by association of basic elements there is a corresponding set V of basic elements. Every member of V must be non-disjoint with at least (and hence exactly) one member of S_n for if not and some $v \in V$ is disjoint with every member of S_n then our conclusion that S_n cannot further be improved, is violated. So, $|V| \leq |S_n|$. Since $\mid U \mid \leq \mid V \mid$ we have the conclusion.

Comment (A). From Proof (1) we can also conclude that there is a set $\{x_1, x_2, ..., x_n\}$ of largest size n for some n with $x_i \leq x$ and x_i mutually disjoint and that each of x_i is necessarily basic. This was essentially the conclusion in Griffin's proof of (1) $\Rightarrow$ (2) of Theorem 6 of [G]: Griffin, Some results on v-multiplication rings, Canad. J. Math. 19 (1967) 710-722.

Proof (2). Suppose that G satisfies F and $0 < x \in G$ and suppose that there is for every $n \in N$ a set $E_n = \{x_1^{(n)}, x_2^{(n)}, ..., x_n^{(n)}\}$ of n mutually disjoint elements with $0 < x_i^{(n)} < x$. Associate to each E_n a set $B_n = \{b_1^{(n)}, b_2^{(n)}, ..., b_n^{(n)}\}$ of mutually disjoint basic elements preceding x, associating exactly one of $b_i^{(n)}$ to each of $x_i^{(n)}$. Now with B_i we proceed as follows: B_{n+1} has one more basic element than B_n so there is at least one basic element say b_{n+1} in B_{n+1} that is disjoint with all the elements of B_n. (The previous statement can be established using ($*_2$). Alternatively assume that each of the $n+1$ mutually disjoint basic elements of B_{n+1} is non-disjoint with at least one member of B_n. This results in two disjoint basic elements, x, y, of B_{n+1} being non-disjoint with one basic element z of B_n which is impossible in view of the definition of a basic element.)

Now form $S_{n+1} = B_n \cup \{b_{n+1}\}$. Now B_{n+2} is larger in size than S_{n+1}, so there is at least one basic element $b_{n+2} \in B_{n+2}$ such that b_{n+2} is disjoint with every element of S_{n+1}, set $S_{n+2} = S_{n+1} \cup \{b_{n+2}\}$. This process can be continued indefinitely to obtain an ascending chain: $B_n \subseteq S_{n+1} \subseteq S_{n+2} \subseteq ... \subseteq S_{n+r}$.... But then $S = \cup S_{n+i}$ is an infinite set of basic elements below x a contradiction.

Comment (B). Proof (2) contains the answer to your question.

Recall that a basis B of an l.o. group G is a maximal set (in size) of mutually disjoint elements which are necessarily basic. According to [C, Theorem 5.1] G has a basis if and only if every $x > 0$ exceeds at least one basic element. The proof of (2) above can be used to prove the following result.

Theorem C. Let G be a lattice ordered group with a basis. If there is $0 < x \in G$ such that for each n there is a set E_n consisting of n mutually disjoint elements preceding x then x exceeds infinitely many disjoint elements.

I have a feeling that the assumption about G having a basis, in the above theorem, is not necessary. But cannot come up with a proof at present.

Let's note that by the condition F the basic elements are ubiquitous in that every strictly positive element exceeds at least one. This observation is at the base of the following proof.

Proof (3). Let $\mathcal{S}$ be the set of sets of mutually disjoint basic elements preceding x. Then $\mathcal{S}$ is not empty, because of F and $*_3$. Being a set of sets $\mathcal{S}$ can be partially ordered by inclusion. Let C be a chain of members of $\mathcal{S}$, and let

$T = \bigcup_{A \in C} A$. Then every element of T, being in some member of C, precedes x and is a basic element. Also for any two distinct $x, y \in T$, x, y belong to some member of C and hence are disjoint. Thus the union of each chain of $\mathcal{S}$ is in $\mathcal{S}$. Hence by Zorn's Lemma there is a maximal set U in $\mathcal{S}$. By condition F, U must be finite; say $U = \{x_1, x_2, ..., x_n\}$. This establishes the result.

Remark D. Proof (3) is interesting but it does not work if we replace "mutually disjoint basic elements" by mutually disjoint elements". Before the example let me point out that the clue is that a singleton can pass as a set of mutually disjoint elements.

Example E. Note that the group of divisibility $G(D) = \{\frac{a}{b}D : a, b \in D\backslash\{0\}\}$ of any GCD domain D is a lattice ordered group, with the order defined by $\frac{a}{b}D \geq \frac{c}{d}D$ if and only if $\frac{a}{b}D \subseteq \frac{c}{d}D$ see e.g. page 5 of [A]: D.D. Anderson, GCD domains, Gauss' lemma, and contents of polynomials. Non-Noetherian commutative ring theory, 1–31, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000. Note that the strictly positive elements of $G(D)$ are the nonzero integral principal ideals of D, different from D. Also "precedes" turns into contains and positive powers of primes are basic elements. Let us take Z the ring of integers as our GCD domain and consider $x = 2^2 5^2$ in Z. Then $\mathcal{S} = \{\{(2^2 5^2)\}, \{(25^2)\}, \{(2^2 5)\}\{(2^2)\}, \{(5^2)\}, \{(2^2), (5^2)\}, \{(2)\}, \{(5)\}, \{(2), (5^2)\}, \{(2^2), (5)\}, \{(2), (5)\}\}$. In this case, while $\mathcal{S}$ includes legitimate maximal elements: $\{(2^2), (5^2)\}$, $\{(2), (5^2)\}$, $\{(2^2), (5)\}, \{(2), (5)\}$ it also includes $\{(2^2 5^2)\}, \{(25^2)\}, \{(2^2 5)\}$ which fit the definition of maximal elements.

Notes:

(1) Warren McGovern who read the above question and answer has come up with the following result.

Proposition E. Suppose G is an l-group and $0 < x \in G$. If x has the property that for each natural n, there is a set E_n consisting of n pairwise disjoint elements, then x exceeds an infinite set of pairwise disjoint elements.

Proof. Suppose $0 < x \in G$ has the property that for each natural n, there is a set E_n consisting of n pairwise disjoint elements. However, suppose by way of contradiction that x does not exceed an infinite set of pairwise disjoint elements. Let H be the convex l-subgroup generated by x. Recall that $H = \{g \in G : \exists n \in N$ such that $| g | \leq nx\}$.

I claim that H has a basis. If not, then there is a $0 < y \in H$ which does not exceed a basic element by Theorem 5.1 of Conrad. Now, Conrad's argument in the proof of Theorem 5.2 yields that y exceeds an infinite set of elements, say $\{y_i\}_{i \in N}$, which are pairwise disjoint. Choose $n \in N$ such that $y \leq nx$. Now for each i, $y_i \leq y \leq nx$. By the Riesz Decomposition, $y_i = y_{i1} + ... + y_{in}$ such that $0 \leq y_{ij} \leq x$. The collection $\{y_{i1}\}_{i \in N}$ is an infinite set of elements which are pairwise disjoint and beneath x; a contradiction. Consequently, H has a basis.

Now, the fact that x satisfies the property in the hypothesis with respect to G implies that it satisfies the property in the hypothesis with respect to H. But H has a basis and so by Theorem C, we get that x exceeds an infinite pairwise disjoint set of elements in H, and hence in G.

(2) For the information of general readers, Conrad's F-condition yields sim-

ilar results for Riesz groups as can be seen in a paper by Mott, Rashid and Zafrullah [MRZ, J. Group Theory, 11(1)(2008), 23-41], with applications in ring theory as can be seen in recent papers [Z, J. Pure Appl. Algebra 214(2010), 654-657], [DZ, J. Pure Appl. Algebra, 214 (2010), 2087-2091] and in a paper to appear in Comm. Algebra [DZ, t-Schreier domains]. (Preprints of the above mentioned papers can be found at: http://www.lohar.com/mit.html

(3) Remark D and Example E must be heeded. If you see such an erroneous application of Zorn's Lemma, see if something like Proof (3) can rectify the error.

(4) I am thankful to Warren McGovern and Yichuan Yang for taking interest, pointing out typos and errors and pointing to better proofs. I am particularly grateful to G.M. Bergman for catching a stupid error on my part. (He thinks he has forgotten all about l.o. groups!)

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