QUESTION (HD0304). If (V,M) is a valuation domain and X an indeterminate over V then how is $V[X]_{M[X]}$ a valuation domain?

This is a somewhat elementary question. So we shall approach it the most elementary way.

Recall that an integral domain V is a valuation domain if for all (nonzero) $x,y \in V$ we have $x \mid y$ or $y \mid x$. (So, a field is a valuation domain!). Using this defining property we can show that if a valuation domain V is not a field then the set of all nonunits of V forms an ideal M which is the maximal ideal. Using the same defining property we can show that if $\{v_1, v_2, \ldots v_n\}$ is a finite set of elements of a valuation domain V, then there is a $v_k \in \{v_1, v_2, \ldots v_n\}$ such that $v_k \mid v_i$ for $i = 1, 2, \ldots, n$. (From this we can also conclude that in a valuation domain V the ideal generated by a finite set $\{v_1, v_2, \ldots v_n\}$ of elements is principal i.e. $\{v_1, v_2, \ldots v_n\}V = v_k V$ and so a valuation domain is a Bezout domain.)

Now let V be a valuation domain and let X be an indeterminate over V. A nonzero element $f(X)$ of the polynomial ring $V[X]$ can be expressed as $f(X) = \sum_{i=0}^{i=n} f_i X^i$ where $f_i \in V$. Using the above observation we conclude that there is f_k such that $f_k \mid f_i$ for $i = 0, 1, 2, \ldots, n$. But then $f(X) = f_k \sum_{i=0}^{i=n} \frac{f_i}{f_k} X^i = f_k g(X)$ where at least one coefficient of $g(X)$ is a unit (in this case the coefficient of X^k).

From these observations we make the following observation:

Observation A. If V is a valuation domain then every nonzero element of $V[X]$ can be written as $v(g(X))$ where $v \in V$ and at least one coefficient of $g(X)$ is a unit.

Now recall that $M[X] = \{f(X) \in V[X]$: every coefficient of $f(X)$ comes from $M\} = \{f(X) \in V[X]$: every coefficient of $f(X)$ is a nonunit of $V\}$. Thus $S = V[X] \backslash M[X] = \{g(X) \in V[X]$: at least one coefficient of $g(X)$ is a unit$\}$. Now recall that $V[X]_{M[X]} = V[X]_S = \{\frac{f(X)}{g(X)} : f(X), g(X) \in V[X]$ where $g(X) \in S\}$. So in $V[X]_{M[X]}$ the units are all the polynomials with at least one coefficient a unit . Now let a and b be two elements of $V[X]_{M[X]}$. Then $a = \frac{f(X)}{g(X)}$, $b = \frac{h(X)}{k(X)}$, where $f(X), h(X) \in V[X]$, and $g(X)$, $k(X)$ are units in $V[X]_{M[X]}$. By Observation A, $f(X) = (u)(r(X))$ and $h(X) = (v)(s(X))$ where $u, v \in V$ and each of $r(X), s(X)$ has at least one coefficient a unit and so each of $r(X), s(X)$ is a unit in $V[X]_{M[X]}$. This gives us $a = \frac{u(r(X))}{g(X)}$, $b = \frac{v(s(X))}{k(X)}$. Thus $a = u\lambda$ and $b = v\mu$ where λ and μ are units of $V[X]_{M[X]}$. Since in V, $u \mid v$ or $v \mid u$ we conclude that in $V[X]_{M[X]}$ either $a \mid b$ or $b \mid a$. But as a and b are any two nonzero elements of $V[X]_{M[X]}$ we conclude that $V[X]_{M[X]}$ is a valuation domain.

Note: The valuation domain $V[X]_{M[X]}$ is also denoted by $V(X)$ and it is called a trivial extension of V (to $K(X)$).