QUESTION:(HD0314) What is a pullback? Give some examples.

ANSWER:

The notion of a pullback comes from category theory where it shows up as a special case of the inverse limit problem. A good description of it can be found on pages 51 and 52 of J.J. Rotman's book [An introduction to homological algebra, Academic Press, 1979]. Briefly if f and g are maps from objects B and C to object A as shown:

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ B & \overrightarrow{f} & A \end{array}$$

The pullback is an object L and maps α, β such that the diagram

$$\begin{array}{ccc} L & \overrightarrow{\alpha} & C \\ \downarrow \beta & & \downarrow g \\ B & \overrightarrow{f} & A \end{array}$$

commutes and has the property that if there were another object M with maps $u : M \to C$ and $v : M \to B$ such that

$$\begin{array}{ccc} M & \overrightarrow{u} & C \\ \downarrow v & & \downarrow g \\ B & \overrightarrow{f} & A \end{array}$$

commutes then there is a unique map $\theta : M \to L$ such that

$$\begin{array}{ccccc} M & - & u & \longrightarrow & C \\ | & \theta \searrow & & & \parallel \\ v & & L & \overrightarrow{\alpha} & C \\ \downarrow & & \downarrow \beta & & \downarrow g \\ B & = & B & \overrightarrow{f} & A \end{array}$$

commutes.

As indicated on page 53 of Rotman's book, if we complete

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ B & \overrightarrow{f} & A \end{array} \quad \text{with} \quad \begin{array}{ccc} D & \overrightarrow{\alpha} & C \\ \downarrow \beta & & \downarrow g \\ B & \overrightarrow{f} & A \end{array} \quad \text{such that}$$

$D = \{(b,c) \in B \times C : f(b) = g(c),\ \alpha : (b,c) \mapsto c$ and $\beta : (b,c) \mapsto b\}$

then D is a pullback. Let us call the above commutative diagram the Cartesian product construction of a pullback.

Here is another scenario: Let in the following diagram

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ B & \overrightarrow{f} & A \end{array}$$

g be injective and f be surjective. To make life easier we can assume that g is an inclusion. Then $L = f^{-1}(C)$ satisfies the above definition of a pullback. For we can take the map $\alpha = f|_L$ and the map β is clearly inclusion. So, with this description the diagram:

$$\begin{array}{ccc} L & \overrightarrow{\alpha} & C \\ \downarrow \beta & & \downarrow g \\ B & \overrightarrow{f} & A \end{array}$$

obviously commutes. Next, let M be an object with maps $u : M \to C$ and $v : M \to B$ such that

$$\begin{array}{ccc} M & \overrightarrow{u} & C \\ \downarrow v & & \downarrow g \\ B & \overrightarrow{f} & A \end{array} \quad \text{commutes}$$

define $\theta : M \to L$ by $\theta(m) = v(m)$. Indeed as g is an inclusion we have for each $m \in M$, $u(m) = f(v(m))$, which forces $v(m) \in f^{-1}(C) = L$. Now θ is unique because β is an inclusion. Thus we have the following rule from the ring theory point of view.

Proposition. If A, B, C are objects in a category of R-modules such that such that $C \subseteq A$ and $f : B \to A$ is a surjective homomorphism, then

$L = f^{-1}(C)$ is the pullback of :

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ B & \overrightarrow{f} & A \end{array} \quad \text{where } g \text{ is an inclusion.}$$

Just to indicate that the above proposition does not represent the only scenario, here is another scenario. Let B and C be subsets of an object A then $B \cap C$ is a pullback of the injective maps B to A and C to A. This remark has been used by Grothendieck to define

Here are a few examples of pullbacks commonly used in the context of integral domains:

(1). Let M be a maximal ideal of an integral domain R. Then R/M is a field. Let A be a subring of R/M. Now we have $f : R \longrightarrow R/M$ the canonical surjection and $g : A \longrightarrow R/M$ the injection (inclusion). In pictures these functions can be represented by

$$\begin{array}{ccc} & & A \\ & & \downarrow g \\ R & \overrightarrow{f} & R/M \end{array}$$

Now let $D = f^{-1}(A)$ which can be shown to be a ring. Then by the above Proposition

$D = f^{-1}(A)$ is a pullback

$$\begin{array}{ccc} D & \vec{\alpha} & A \\ \downarrow \beta & & \downarrow g \\ R & \vec{f} & R/M \end{array}$$

commutes, where α is is surjective and β is injective. So D can be identified with a subring of the original ring R.

Based on the same theme here are some examples: Consider for R, the polynomial ring $Q[X]$ where Q is the the field of rational numbers and X is an indeterminate over Q. Now $R = Q[X]$ is a PID, so every nonzero prime ideal of R is a maximal ideal of R. Take $M = XQ[X]$. Then $R/M \cong Q$. Choose Z the ring of integers to serve for A. As a subring of R/M, $Z \cong \{z + M : z \in Z\}$. Next take the canonical surjection $f : Q[X] \to Q[X]/XQ[X] \cong Q$ and consider $f^{-1}(z + M : z \in Z\} =$ polynomials whose images under f are in the set of cosets $\{z + M : z \in Z\}$. Now these are precisely the polynomials in $Q[X]$ whose constant terms are in the set of integers Z. Thus

$f^{-1}(\{z + M : z \in Z\}) = \{h(X) \in Q[X] : h(0) \in Z\} = \{a_0 + \sum a_i X^i : a_0 \in Z, a_i \in Q\}$. Now it is easy to see that each $\sum_{i=1}^{i=n} a_i X^i \in XQ[X]$. So we can write $f^{-1}(\{z + M : z \in Z\}) = Z + XQ[X]$ and it is easy to see that $Z + XQ[X]$ is a subring of $Q[X]$. So we have the following picture:

$$\begin{array}{ccc} Z + XQ[X] & \vec{\alpha} & \{z + M : z \in Z\} \\ \downarrow \beta & & \downarrow g \\ R = Q[X] & \vec{f} & \{q + M : q \in Q\} \end{array} \qquad \text{............(I)}$$

Now this is what happens: Take $h(X) = a_0 + \sum_{i=1}^{i=n} a_i X^i \in Z + XQ[X]$, the map α assigns to $h(X)$ the coset $\alpha_0 + M$ and g being the inclusion maps this coset into $\{q + M : q \in Q\} = R/M$ as the coset $\alpha_0 + M$. So, we have $g\alpha(a_0 + \sum_{i=1}^{i=n} a_i X^i) = \alpha_0 + M$. Next as $Z + XQ[X]$ is a subring of $Q[X]$ we can take β to be the inclusion map. Then for $h(X) = a_0 + \sum_{i=1}^{i=n} a_i X^i \in Z + XQ[X]$ we have $\beta(a_0 + \sum_{i=1}^{i=n} a_i X^i) = a_0 + \sum_{i=1}^{i=n} a_i X^i \in Q[X]$ and f being the canonical surjection takes $a_0 + \sum_{i=1}^{i=n} a_i X^i$ into $\alpha_0 + M$. But then for all $h(X)$ in $Z + XQ[X]$ we have $g\alpha(h(X)) = f\beta(h(X))$, which means that the diagram

$$\begin{array}{ccc} Z + XQ[X] & \vec{\alpha} & \{z + M : z \in Z\} \\ \downarrow \beta & & \downarrow g \\ R = Q[X] & \vec{f} & \{q + M : q \in Q\} \end{array} \qquad \text{...........(II)}$$

commutes. Thus we see that $Z + XQ[X]$ is a pullback of

$$\begin{array}{ccc} & & \{z + M : z \in Z\} \\ & & \downarrow g \\ R = Q[X] & \vec{f} & \{q + M : q \in Q\} \end{array} \qquad \text{..........(III)}$$

where $M = XQ[X]$, $\{q + M : q \in Q\}$ is the set of cosets of $Q[X]/XQ[X]$, $\{z + M : z \in Z\}$ is the set of cosets (which is a subring of $Q[X]/XQ[X]$ that represent polynomials with constant term in Z), g is the inclusion map from $\{z+M : z \in Z\}$ into the quotient $Q[X]/XQ[X]$ and f is the canonical surjection.

Note: Now that the description part is over, we can do away with some of the details. Noting that $\{z+M : z \in Z\} \cong Z$ and that $Q[X]/XQ[X] \cong Q$ we can replace (II) by

$$\begin{array}{ccc} Z+XQ[X] & \overrightarrow{\alpha} & Z \\ \downarrow \beta & & \downarrow g \\ R = Q[X] & \overrightarrow{f} & Q \cong Q[X]/XQ[X] \end{array}$$

and say that $Z+XQ[X]$ is a pullback of

$$\begin{array}{ccc} & & Z \\ & & \downarrow g \\ R = Q[X] & \overrightarrow{f} & Q \cong Q[X]/XQ[X] \end{array}$$

Now looking at this example you can make your own pullbacks of this kind. For example, if D is an integral domain which is contained (as a subring) in a field L then $D+XL[X]$ is a pullback of

$$\begin{array}{ccc} & & D \\ & & \downarrow g \\ R = L[X] & \overrightarrow{f} & L \cong L[X]/XL[X] \end{array}$$

where g is the canonical injection and f is the canonical surjection.

My description has been lengthy because, I do not know how much the reader already knows. Usually, as already indicated, a pullback (domain) is described as: Let M be a maximal ideal of a domain R, $f : R \to R/M$ the canonical surjection. If A is a subring of R/M then $f^{-1}(A)$ is a pullback of A in R over R/M. This approach is responsible for a number of interesting pullback constructions:

1. The $D+M$ construction: Let $V = k+M$ be a valuation domain, where k is a field and M is the maximal ideal of V. For each subring D of k then $D+M$ is a pullback of D in V over k. This kind of pullback constructions, being a good source of examples of a certain type, have been extensively used by Gilmer and his students and quite a few other Mathematicians. Gilmer's book, [Multiplicative Ideal Theory, Marcel Dekker, 1972] is a good source for these constructions. Another relevant reference is R. Gilmer and E. Bastida [Michigan Math. J. 20 (1973), 79–95].

2. The generalized $D+M$ construction: This notion was introduced by J. Brewer and E. Rutter in [Michigan Math. J. 23 (1976), no. 1, 33–42]. This construction requires a domain R expressible as $R = k+M$ where k is a field and M is a maximal ideal of R. Pick a suitable subring D of k to make the ring $D+M = \{d+m : d \in D$ and $m \in M\}$. The generalized $D+M$ construction includes the $D+XL[X]$ construction and is a pullback for similar reasons. However, it is a bit more versatile in that it also allows constructions like $D+XL[[X]]$, the ring of power series over a field L with constant terms in a subring D.

There is another, more general, approach to consructing pullbacks. It goes as follows: Let R be a domain, let M be a nonzero prime ideal of R and let A be a subring of the domain

R/M. Consider:

$$\begin{array}{ccc} & & A \\ & & \downarrow g \\ R & \overrightarrow{f} & R/M \end{array}$$

where g is injective. Then the subring D of R is a pullback if the diagram

$$\begin{array}{ccc} D & \overrightarrow{\alpha} & A \\ \downarrow \beta & & \downarrow g \\ R & \overrightarrow{f} & R/M \end{array}$$

commutes. Indeed this D turns out to be $f^{-1}(A)$ by the Proposition above.

One of the easier constructions using this general approach is the, so called, $A+XB[X]$ construction where $A \subseteq B$ is an extension of domains and $A+XB[X]$ denotes the ring of polynomials over B with constants in A.

Indeed there appears to be a common thread in the above examples. Each of these examples consists of a pair of rings $R \subseteq S$ such that R and S contain a common ideal. It turns out that in such a pair $R \subseteq S$, R is a pullback.

I have used the simplest possible exampes to explain the theory. For more theoretical treatment consult Marco Fontana's Topologically defined classes of commutative rings, Ann. Mat. Pura Appl., IV. Ser. 123(**1980**), 331-355.

There is extensive literature on pullbacks. Yet from the Multplicative Ideal Theory point of view the following references will be useful.

i.Cahen, Paul-Jean. Couples d'anneaux partageant un ideal. Arch. Math. **1988**, *51*, 505–514.

ii. Fontana, Marco; Gabelli, Stefania. On the class group and the local class group of a pullback. J. Algebra **1996**, *181 (3)*, 803–835.

iii. Gabelli, Stefania; Houston, Evan. Coherentlike conditions in pullbacks. Michigan Math. J. **1997**, *44 (1)*, 99–123.

On the $A+XB[X]$ constructions the reader may want to consult the following articles and references there:

iv. Lucas, Thomas G. Examples built with $D+M$, $A+XB[X]$ and othe pullback constructions. Non-Noetherian Commutative Ring Theory; Chapman, S., Glaz, S., Eds.; Kluwer Academic Publishers, **2000**; 341–368.

v. Picavet, Gabriel. About composite rings. Commutative ring theory (Fès, **1995**); Lecture Notes in Pure and Appl. Math., 185; Dekker: New York, **1997**; 417–442.

vi. Zafrullah, Muhammad, Various facets of rings between $D[X]$ and $K[X]$, Comm. Algebra 31(5) **2003**, 2497-2540.

(The following have helped in preparing this answer: David Anderson, Gabriel Picavet and Martine Picavet.)