QUESTION (HD 0504): Is it true that if I is a $*$-ideal of an integral domain D, for some star operation $*$, then the radical $\sqrt{I}$ is also a $*$-ideal?

ANSWER: (Terminology standard as in Gilmer's [Multiplicative Ideal Theory, Marcel Dekker, 1972, sections 32 and 34]).

Not really. For example if D is a rank one nondiscrete valuation domain with maximal ideal M and if x is a nonzero nonunit of $D.$ Then xD is, by definition of star operations, a $*$-ideal for any star operation $*.$ So, xD is a v-ideal. Now, since D is a rank one valuation domain, $\sqrt{xD} = M$ and M is not principal and has the following nasty property: If for $a,b \in D\backslash\{0\},\ M \subseteq \frac{a}{b}D$ then $a \mid b.$ (For a quick proof, note that D is a valuation domain, so $a \mid b$ or $b \mid a.$ Suppose that $a \nmid b.$ Then $b \mid a$ properly i.e. $\frac{a}{b} = c$ is a nonunit of $D.$ This gives $M \subseteq cD$ and since M is the maximal ideal of D we have $M = cD$ contradicting the fact that M is not principal.) Now $D \supseteq M_v = \bigcap\limits_{M \subseteq \frac{a}{b}D} \frac{a}{b}D \supseteq D,$ because in each case $a \mid b$ and so $\frac{a}{b}D \supseteq D.$ So $M,$ the radical of the v-ideal $xD,$ is not a v-ideal.

However, there is a result that is so closely related to this question that it would be unfair not to mention it. Recall that if $*$ is a star operation then the operation $*_f$ defined for each nonzero fractional ideal A by $A^{*_f} = \cup\{F^*$: where F ranges over nonzero finitely generated subideals of $A\}$ is again a star operation and that this operation is of finite character and that each $*$-ideal is a $*_f$-ideal.

Theorem 1. If I is a $*$ ideal for some star operation $*,$ then $\sqrt{I}$ is a $*_f$-ideal.

Proof. According to Hedstrom and Houston's [Proposition 1.1(5), J. Pure Appl. Algebra 18(1980) 37-44] if X is a $*_f$ ideal then every minimal prime ideal of X is a $*_f$-ideal and as we have mentioned each $*$-ideal is a $*_f$-ideal. Now set $X = I$ and recall that $\sqrt{I} = \cap P$ where P ranges over minimal prime ideals of I and so each P is a $*_f$-ideal. Now recall from Gilmer's [Proposition 32.2, Multiplicative Ideal Theory, Marcel Dekker, 1972] that if $\{A_i\}$ is a family of $*$-ideals, for some star operation $*,$ such that $\cap A_i \neq (0)$ then $\cap A_i$ is a $*$-ideal.

Corollary 2. For any star operation $*$ the radical of a $*$-invertible $*$-ideal I is a t-ideal and so the radical of an invertible ideal is a t-ideal.

The proof is easy once you realize that $I \in F(D)$ is $*$-invertible if there is $J \in F(D)$ such that $(IJ)^* = D$ and that J^* can be shown to be $I^{-1}.$ Thus $(II^{-1})^* = D$ and so I^{-1} is $*$-invertible. This leads to $I = I^* = (I^{-1})^{-1} = I_v.$ Now note that for $I \in F(D),\ I_t = \cup\{F_v$: where F ranges over nonzero finitely generated subideals of $A\} = A^{v_f}.$