QUESTION (HD 0601): Call an integral domain D an irreducible divisor finite (idf) domain if every nonzero element of D is divisible by at most a finite number of non-associated irreducible elements. Let K be a field, let Q^+ be the set of non-negative rationals and let $R = K[X; Q^+]$ be the monoid ring construction. Is there a field K such that $K[X; Q^+]$ is not an idf domain?

ANSWER: I have some partial answers. If K is algebraically closed then $K[X; Q^+]$ has no irreducible elements and so $K[X; Q^+]$ is vacuously an idf domain. The proof is hidden in the construction of $R = K[X; Q^+] = \{\sum_{i=1}^{n} \alpha_i X^{q_i} :$ $a_i \in K$ and $q_i \in Q^+\}$ and its interpretation. Here we note a few important points.

1. R can be regarded as an ascending union of the polynomial rings $R_{n!} = K[X^{\frac{1}{n!}}]$ where $n!$ denotes the factorial of the natural number n. That is $R = \bigcup R_{n!}$, where obviously, $R_1 \subset R_{2!} \subset R_{3!} \subset .. \subset R_{n!} \subset R_{(n+1)!} \subset ...$ (Note: $K[Q^+]$ can also be viewed as a directed union of $Q[X^{\frac{1}{n}}]$ where n varies over natural numbers.)

2. As shown in HD 0308, R is a Bezout domain (every finitely generated ideal is principal) and it is easy to show that in a Bezout domain every irreducible element is prime.

3. If K is algebraically closed then $R = K[X; Q^+]$ has no irreducible elements. (Take $f(X) = \sum_{i=1}^{n} \alpha_i X^{q_i}$ where $a_i \in K$ and $q_i \in Q^+$ and suppose that f is irreducible and let n be the least natural number such that f is a polynomial in $R_{n!}$. Then $f(0) \neq 0$, because any power of X is hopelessly factorizable and because K is algebraically closed f must be a linear polynomial in $R_{n!}$. But as $R_{n!} \subset R_{(n+1)!}$ and in $R_{(n+1)!}$ f is a higher degree polynomial and hence factorizable as a product of liner factors in $R_{(n+1)!}$.)

There are other situations in which $K[X; Q^+]$ has no atoms, for instance if K is real closed or perfect with characteristic $p > 0$ and there are situations in which $K[X; Q^+]$ has some atoms, for instance when $K = Q$ the field of rational numbers. For details on these examples you may look up the paper "Monoid domain constructions of antimatter domains", by Anserson, Coykendall, Hill, and myself. The paper is to appear, yet you can find a copy on this web page at: http://www.lohar.com/mit.html.

Next, call an integral domain D a GD(1) domain if every nonzero element of D belongs to at most a finite number of principal primes. The GD(1) domains were introduced by Okoh and Malcolmson in [Rocky Mountain J. Math. 35 (2005), no. 5, 1689–1706]. Call an element $x \in D$ a GD(1) element if $x \neq 0$ and x belongs to at most a finite number of principal prime ideals. Now here is a little Kaplansky type theorem which is included here just as FYI.

Theorem. An integral domain D is a GD(1) domain if and only if every nonzero prime ideal of D contains a GD(1) element.

Proof. Indeed if D is GD(1) then every nonzero element in a prime ideal is a GD(1) element. Conversely, let S be the set of all GD(1) elements of D. Obviously S is multiplicative and saturated. Claim: $S = D\backslash\{0\}$. For if not and let $x \in D\backslash S$, then there is a nonzero prime ideal P containing xD maximal with respect to being disjoint with S. But then by the condition P contains a GD(1) element a contradiction.

An obvious consequence of this theorem is the following statement.

Corollary. An integral domain D is not a GD(1) domain if and only if there is a prime ideal P of D such that no nonzero element of P is a GD(1) element, if and only if there is a nonzero element that is contained in an infinite number of principal primes.

Because $K[X; Q^+]$ is a Bezout domain and because in a Bezout domain an irreducible element is a prime we conclude that for $K[X; Q^+]$, being idf is equivalent to being GD(1).

So, the question of whether $K[X; Q^+]$ may not be idf for some field K actually is whether we can construct a field K such that $K[X; Q^+]$ contains a nonzero element that is divisible by infinitely many primes.

Comments: (1) This question was proposed by Frank Okoh, who is a Professor of Mathematics at Wayne State University, Detroit. Having seen my response to his question, he wrote: "Perhaps you should mention that it is more suggestive to call GD(1) PPF for principal primes finite.

I chose GD(1) because I thought that it was generalizing Dedekind domains. Several people pointed out that there are oodles of generalizations of Dedekind domains."

(2) As an afterthought, the basic problem is to answer the following question: Is $Q[X; Q^+]$ idf? An answer to this question might either settle the problem or give us clues to the solution of the main problem. (If you have an easy answer do let me know.)

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