QUESTION HD0903: How do you construct a PVMD that is not Krull nor Prufer nor GCD? What's the simplest such example known?

ANSWER: Here's a very easy construction. Note that a Krull domain D is a UFD if and only if D is a GCD domain. Let D be a Krull domain that is not a UFD nor Dedekind, let K be the quotient field of D and let X be an indeterminate over K. Then $D + XK[X] = \{f \in K[X] : f(0) \in D\}$ is a PVMD according to Theorem 4.43 of [CMZ, J. Algebra 53(1978), 423-439]. This $D+XK[X]$ cannot be Krull because for any nonzero nonunit a in D we have in $R = D+XK[X]$ the strictly ascending chain of principal ideals: $XR \subset (X/a)R \subset (X/a^2)R \subset (X/a^3)R \subset ... \subset (X/a^n)R \subset (X/a^{n+1})R$ .... In other words $D + XK[X]$ does not satisfy Ascending Chain Condition on Principal ideals, but a Krull domain must satisfy ACCP, because say according to Bourbaki's [Commutative Algebra Chapter 7, section 1, No. 3, Theorem 2] D is a Krull domain if and only if D is completely integrally closed and satisfies ACC on integral divisorial ideals. (Nonzero principal ideals are divisorial.) To see that $D+XK[X]$ is neither GCD nor Prufer note that by Corollary 4.15 and Theorem 1.1 of [CMZ], $D+XK[X]$ is Prufer (GCD, Bezout) if and only if D is. So if D is not a GCD (or Prufer) domain to start with then $D+XK[X]$ is not a GCD domain nor Prufer.

Next, note that the integral closure of a Noetherian domain is a Krull domain (see Theorem 4.3 of Fossum [F, The Divisor Class Group of a Krull domain, Springer, New York, 1973] and so an integrally closed Noetherian domain is Krull, which is a PVMD. Let D be an integrally closed Noetherian domain that is not a UFD, nor a Dedekind domain, S a multiplicative set in D, X an indeterminate over D_S and consider the domain $D^{(S)} = D + XD_S[X] = \{f \in D_S[X] : f(0) \in D\}$. We claim that $D^{(S)}$ is a PVMD and if there is a nonunit $s \in S$ then $D^{(S)}$ is not Krull. To verify this claim, note that (a) $D^{(S)}$ is integrally closed [CMZ, remark before Theorem 1.1] (b) $D^{(S)}$ is coherent by [CMZ, 4.32] and (c) a coherent integrally closed domain is a PVMD [Z, Manuscripta Math. 24(1978), 191-204]. Now by Theorem 1.1 of [CMZ] for $D^{(S)}$ to be GCD it is necessary that D be a GCD domain but, as we have assumed in our case that D is not a UFD, we conclude that $D^{(S)}$ is not GCD. Now $D^{(S)}$ can be Prufer if and only if D is Prufer and $D_S = K$ by Theorem 3.6 of [AAZ, Commutative algebra. Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001), no. 1, 3–16] So if D is not Dedekind there is no chance for $D^{(S)}$ to be Prufer.

Finally if S contains a nonunit s then every power of s divides X in $D^{(S)}$ and so we can construct a strictly ascending infinite chain of integral principal ideals: $XD^{(S)} \subset (X/s)D^{(S)} \subset (X/s^2)D^{(S)} \subset (X/s^3)D^{(S)} \subset ... \subset (X/s^n)D^{(S)} \subset (X/s^{n+1})D^{(S)}$ ..., making it impossible for $D^{(S)}$ to be Krull.

By Corollary 2.7 of [AAZ] you can, in the above example, choose D to be a non-UFD, non-Dedekind, Krull domain with multiplicative set S that contains at least one nonunit. The resulting $D^{(S)}$ is a PVMD that is not Krull nor GCD nor Prufer.

Note: I am thankful to Said El Baghdadi for suggestions towards the improvement of this response.