QUESTION: (HD 1210) What should we expect from an ideal A of grade 1 with $A_t = D$?

ANSWER: I wrote a short note a while ago. It is sort of incomplete, but hopefully it will answer your question.

Let D be an integral domain with quotient field K. Let $F(D)$ denote the set of nonzero fractional ideals of D and denote, for $A \in F(D)$, the set $D :_K A = \{x \in K : xA \subseteq D\}$ by A^{-1}. It is well known that for $A \in F(D)$, A^{-1} is a fractional ideal and of course so is $A_v = (A^{-1})^{-1}$. The function v on $F(D)$ defined by $A \mapsto A_v$ is called the v-operation (see section 34 of Gilmer [G]). By an ideal we mean a fractional ideal contained in D. Call two elements $a, b \in D \backslash \{0\}$ v-coprime if $(a, b)_v = D$. It is easy to show that a, b are v-coprime if and only if $(a, b)^{-1} = D$ if and only if $aD \cap bD = abD$. This notion started with the paper of Gilmer and Parker [GP] and is stronger than the notion of coprimality, as indicated by Zafrullah [Z]. H. Uda , in section 7 of [U] gave an example of an ideal A such that $G(A) = 1$ but $A_t = D$. The purpose of this short note is to explore the properties a finitely generated nonzero proper ideal A must have if $A_v = D$ and $G(A) = 1$. Here $G(A)$ denotes the classical grade of A and it is known, as shown below, that if A is a nonzero proper ideal in a Noetherian domain D then $A_v = D$ if and only if $G(A) \geq 2$. We actually show that for a nonzero ideal A in D $G(A) \geq 2$ if and only if there are two v-coprime elements $a, b \in A$ and that in a so called TV domain (to be defined later) which generalizes Noetherian domains, $G(A) \geq 2$ if and only if A contains a pair of v-coprime element.

Proposition A. Let A be a nonzero proper ideal in a domain D and let $G(A)$ denote the classical grade of A. Then $G(A) \geq 2$ if and only if A contains a pair of v-coprime elements.

Let's now include the tools needed to prove the above proposition, and prove it. A sequence of elements $x_1, x_2, ..., x_n$ in a ring R is called a regular sequence or an R-sequence if (a) $(x_1, x_2, ..., x_n) R \neq R$, (b) x_1 is a nonzero divisor on R and x_i is a nonzero divisor on $R/(x_1,x_2, ..., x_{i-1})$ for $i > 1$. The classical grade of an ideal I is the (maximum) length of an R-sequence in I. I have picked these notions from Kaplansky's book [K]. According to Theorem 116 of Kaplansky [K] $x_1, x_2, ..., x_n$ is an R-sequence on a ring R if and only if for each $i < n$, $x_1, ..., x_i$ is an R-sequence and $x_{i+1}, ..., x_n$ is an R-sequence on $R/(x_1, ..., x_i)$. So, if $x_1, x_2, ..., x_n$ is an R sequence on R and $n \geq 2$, then x_1, x_2 is an R-sequence. Now let $a, b,$ ... be an R-sequence on an integral domain D then a is a nonzero divisor on $D/(b)$ which means that $b \mid ax$ implies $b \mid x$. This in turn means that if $y \in aD \cap bD$ then $y \in abD$ which means that $aD \cap bD = abD$ i.e. $(a, b)_v = D$. Thus we have the following Lemma

Lemma B. In an integral domain D, the first two elements of every R-sequence of length > 1 are v-coprime.

Now if I is a proper ideal of D and if I contains a pair of v-coprime elements a, b. Then as $aD \cap bD = abD$ we have from $b \mid ax$ that $ax \in aD \cap bD = abD$ which means that $ax = aby$ cancelling a from both sides we have $b \mid x$. So if $a.b$ are v-coprime, $b \mid ax$ implies $b \mid x$ and so b is a nonzero divisor on $D/(a)$

and a, b is an R-sequence in D. Of course by Theorem 116 of [K] again, we can construct an R-sequence contained in I by selecting $x_3 \in I$ such that x_3 is a nonzero divisor in $D/(a, b)$ and so on ... . This gives us the following Lemma.

Lemma C. Let I be an ideal of an integral domain D such that $I \subsetneq D$. If I contains a pair of v-coprime elements then $G(I) \geq 2$.

The two lemmas above constitute the proof of Proposition A. Next, using some properties of the v-operation, we indicate what it means to say that for a proper ideal I of a domain D, $G(I) \geq 2$.

Corollary D. Let I be a proper nonzero ideal of an integral domain D. Then the following are equivalent.

(1) There exist two elements $a, b \in I$ such that $(a, b)_v = D$.

(2) There exist two elements $a, b \in I$ such that $I_v = (a, b)_v = D$.

(3) There exist two elements $a, b \in I$ such that $I^{-1} = (a, b)^{-1} = D$

(4) There exist two elements $a, b \in I$ such that $(a, b)^{-1} = D$.

(5) There exist two elements $a, b \in I$ such that $aD \cap bD = abD$.

(6) $G(I) \geq 2$.

Proof. (1) $\Rightarrow$ (2). All we need to note is that if $I \subsetneq D$ is an ideal then $I_v \subseteq D_v = D$ and that $(a, b) \subseteq I$ implies $(a, b)_v \subseteq I_v$. So we have $D = (a, b)_v \subseteq I_v \subseteq D$. Of course (2) $\Rightarrow$(1) is obvious and (3) $\Leftrightarrow$ (2) because for any nonzero fractional ideal A we have $(A_v)^{-1} = A^{-1}$ and $A_v = (A^{-1})^{-1}$ and for the same reason (1) $\Leftrightarrow$ (4). Again (4) and (5) are equivalent because they are the restatements of each other. That (1) $\Leftrightarrow$ (6) is Proposition A.

The "properties" of the v-operation used above can be verified from the definition or the reader may look up sections 32 and 34 of [G]. However we provide some introduction here so that the reader does not need to refer to [G] for minor details.

A *star operation* $*$ *on* D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$, and

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal A is of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-invertible for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The best known examples of star operations are the d-*operation* defined by $A \mapsto A_d = A$, the v-*operation* defined by $A \mapsto A_v = (A^{-1})^{-1}$, and the t-*operation* defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$.

In some integral domains D, such as in Krull domains, given an ideal I and an element $a \in I$ there is an element $b \in I$ such that $I_v = (a, b)_v$ (see Mott and

Zafrullah [MZ]). Taking a hint from Theorem 2.2 of Anderson and Zafrullah [AZ], we state and prove the following result.

Proposition E. Let A be a proper finitely generated ideal of an integral domain D. Suppose that A contains a nonzero element a which belongs to at most a finite number of maximal t-ideals. Then $G(A) \geq 2$ if and only if $A^{-1} = D$.

Proof. Note that if $G(D) \geq 2$ then $A^{-1} = D$. So all we need to prove is the converse. For that let $Q_1, Q_2, ..., Q_n$ be the set of all the maximal t-ideals of D that contain a. Since A is finitely generated and $A_v = D$ we conclude that A is not contained in any maximal t-ideal M of D. In particular $A \subsetneq Q_1 \cup Q_2 \cup ... \cup Q_n$. Choose $b \in A \backslash Q_1 \cup Q_2 \cup ... \cup Q_n$ and proceed as follows. We have $(a, b) \subseteq A$. Now for any maximal t-ideal M of D we have $(a, b)D_M = D_M = AD_M$ if M is one of Q_i (because of b) and we have $(a, b)D_M = D_M = AD_M$ if $M \notin \{Q_1, Q_2, ..., Q_n\}$ because of a. Thus we have $D = \bigcap_{M \in t-\max(D)} D_M = \bigcap_{M \in t-\max(D)} (a, b)D_M = \bigcap_{M \in t-\max(D)} AD_M$ which implies that $(a, b)_v = A_v$ (see Griffin [Gr]) where $a, b \in A$ and this by Proposition A means that $G(A) \geq 2$.

Corollary F. (1). Let M be a non-maximal maximal t-ideal of D and let x be a nonzero nonunit element of $D \backslash M$. If x belongs to at most a finite number of maximal t-ideals then there exists an element $y \in M$ such that $D = (M, x)_t = (x, y)_v$, i.e., $G(M, x) \geq 2$, if (M, x) is proper. Consequently if M is a maximal t-ideal in a domain of finite t-character and x is a nonzero nonunit of D with $x \in D \backslash M$, with $(M, x) \neq D$ then $G(M, x) \geq 2$.

(2). Let M be a non-maximal maximal t-ideal of D and let $(M, x) \neq D$. If there is a nonzero $y \in (M, x)$ such that y is contained in at most a finite number of maximal t-ideals then $G(A) \geq 2$.

Proof. Since $D = (M, x)_t$ there exist $x_1, x_2, ..., x_n \in (M, x)$ such that $(x_1, x_2, ..., x_n)_v = D$. Since $x \in (x_1, x_2, ..., x_n)_v$ we can write $(x_1, x_2, ..., x_n)_v = (x_1, x_2, ..., x_n, x)_v = D$. The ideal $(x_1, x_2, ..., x_n, x)$ meets the requirements of Proposition E and so there is $y \in (x_1, x_2, ..., x_n, x)$ such that $D = (x_1, x_2, ..., x_n)_v = (x_1, x_2, ..., x_n, x)_v = (x, y)_v$. Now as $y \in (x_1, x_2, ..., x_n, x)$ and as $x_i = m_i + \alpha_i x$, where $m_i \in M$ and $\alpha_i \in D$, we have $y = m + \alpha x$ where $m \in M$ and $\alpha \in D$. So we have $D = (x_1, x_2, ..., x_n, x)_v = (m + \alpha x, x)_v = (m, x)_v$ But that means $G(M, x) \geq 2$.

(2) is a direct corollary of Proposition E.

Recall that an integral domain D is a Mori domain if D satisfies ACC on integral divisorial ideals. So Noetherian domains are Mori domains and it is well known that a Mori domain is of finite t-character see, e.g., Proposition 1.4 Houston and Zafrullah [HZ] . In [HZ] the authors introduced TV-domains as those domains whose t-ideals are v-ideals. It was shown in [HZ] that a TV-domain is of finite t-character; a Mori domain is a TV-domain.

Corollary G. If D is a TV-domain and A is a finitely generated proper nonzero ideal then $A^{-1} = D$ if and only if $G(A) \geq 2$. Consequently in a Mori domain (or in a Noetherian domain), for a nonzero proper ideal A, $A^{-1} = D$ if and only if $G(A) \geq 2$.

To my knowledge, the above corollary is only known for Noetherian domains.

Corollary H. If A is a nonzero proper ideal of a domain D such that $A_t = D$ and $G(A) = 1$ then there is no nonzero element a in A such that a is contained in at most a finite number of maximal t-ideals.

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