QUESTION: (HD 1301) Anh, Marki and Vamos in [Trans. Amer. Math. Soc. 364 (8) (2012), 3967-3992] seem to suggest that what they call Bezout monoids provide the best set up for studying GCD domains and UFDs. Any comments?

ANSWER: Well, they expressly state that their monoids are suited for the study of **divisibility** in GCD domains. But as there is a danger of someone mistaking the monoids as the panacea for all ills of integral domains, I make the following remarks. For the general case, I believe that if we were to solely depend upon monoids we would miss some results on GCD domains and UFDs. The reason is, monoids have one operation and rings have two. Of course the multiplicative structure of a domain is a monoid but staying within that monoid structure we may not be able to capture some fine points. It appears that some examples are in order. Yet before that it seems pertinent to provide necessary introductions for general readers. The paper you mention defines a Bezout monoid as a commutative monoid S with 0 such that under the natural partial order (for $a,b \in S$, $a \leq b \in S \iff bS \subseteq aS$), S is a distributive lattice, multiplication is distributive over both meets and joins, and for any $x,y \in S$, if $d = x \wedge y$ and $dx_1 = x$, then there is a $y_1 \in S$ with $dy_1 = y$ and $x_1 \wedge y_1 = 1$. Let's recall another definition from another expert Franz Halter-Koch [HK]. If we look at the definition of a monoid at page 5 of [HK] we find that Halter-Koch allows a zero in a multiplicative monoid M such that $0 \cdot m = 0$ for all $m \in M$. He also calls an element x in a monoid M cancellative if $x \cdot a = x \cdot b$ implies $a = b$ for all $a,b \in M$ and he calls a monoid M cancellative if all elements in $M \backslash \{0\}$ are cancellative.

Restricting to cancellative monoids Halter-Koch defines, in Chapter 10 of [HK], a GCD of a (non-empty) finite set $B \subseteq M$ as d such that (1) $d \mid b$ for all $b \in B$ and (2) if $c \in M$ such that $c \mid b$ for all $b \in B$ then $c \mid d$. This is pretty much the same as the definition of a GCD of a finite set in an integral domain. By Halter-Koch's definition $GCD(B)$ is a set of associates of a GCD and $GCD(B) = dM^{\times}$ where d is a GCD and $M^{\times}$ is the set of units of M. He defines a GCD monoid as a cancellative monoid such that for every finite (nonempty) $B \subseteq M$, $GCD(B) \neq \Phi$. Halter-Koch denotes $GCD(a,b)$ by $a \wedge b$ and calls it $\inf$. Next he shows that $x(GCD(a,b)) = GCD(xa,xb)$ or $x(a \wedge b) = xa \wedge xb$, that is multiplication distributes over $\wedge$. He also shows that M is a GCD monoid if and only if $GCD(a,b)$ is nonempty for every pair $a,b \in M$. He also states that M is a GCD monoid if and only if $M_{red} = M/M^{\times}$ is and that an integral domain D is a GCD domain if D is a GCD monoid (under the multiplication of D). Now let M be a reduced GCD monoid and let $a,b \in M$ then $GCD(a,b) = dM^{\times}$ for some $d \in M$. Since M is reduced $M^{\times}$ is a singleton and so can be represented by 1. So we can write $a \wedge b = d$. Now as $d \mid a,b$ we can write $a = a_1d$ and $b = b_1d$. Now as $d = a_1d \wedge b_1d = d(a_1 \wedge b_1)$ and as M is cancellative we get $1 = a_1 \wedge b_1$. If for some x_1 we have $a = dx_1$ then by cancellation $x_1 = a_1$ and we already have a b_1 such that $b = b_1d$ and $a_1 \wedge b_1 = 1$. So a reduced GCD monoid is indeed a Bezout monoid of [AMV]. Let's also note that Given a GCD monoid M we can take the quotient $M_1 = M/M^{\times}$ to reduce it and we can by a direct product with a group K construct a GCD monoid $M_2 = M_1 \times K$. This indicates that the multiplicative structure of a GCD monoid does not have any control over the size of its group of units.

With this introduction let us take the following simple result, that apparently cannot be proved using the multiplicative monoids and by a monoid we shall mean a cancellative monoid.

Proposition A. Every valuation domain that is not a field has infinitely many units.

Proof. Let V be a valuation domain that is not a field and let x be a nonzero nonunit in V and consider the set $\{x^n + 1,\ n \in N\}$. Each of $x^n + 1$ is a unit and all are distinct. For if we put $x^n + 1 = x^m + 1$ for $n > m$ then $x^{n-m} = 1$ forcing x to be a unit. So V has at least a countably infinite set of units.

There is nothing new in Proposition A. Using the same proof one can prove that if D is a domain with nonzero Jacobson radical then D has infinitely many units. This result too is not as important as its contrapositive which says that if D has only a finite number of units then the Jacobson radical of D is zero. We shall make a use of this in a generalization of GCD domains and in GCD domains. But for this we need to prepare a little. Let us use D to denote an integral domain with quotient field K. Let $F(D)$ be the set of nonzero fractional ideals of D. For $A \in F(D)$ the inverse of A, that is $A^{-1} = \{x \in K : xA \subseteq D\} \in F(D)$. The function on $F(D)$ defined by $A \mapsto A_v = (A^{-1})^{-1}$ is a star operation called the v-operation. Another, related, star operation is the so called t-operation defined by $A \mapsto A_t = \cup F_v$ where F ranges over nonzero finitely generated subideals of A. A fractional ideal $A \in F(D)$ is called a v-ideal (resp., t-ideal) if $A = A_v$ (resp., $A = A_t$), a nonzero principal fractional ideal is both a v-ideal and a t-ideal. For a review of the star operations look up sections 32 and 34 of [G]. We recall the notions and results necessary for our study here. Note that $A \in F(D)$ is a t-ideal if and only if $x_1, x_2, \ldots, x_n \in A$ implies that $(x_1, x_2, \ldots, x_n)_v \subseteq A$, if $(x_1, x_2, \ldots, x_n) \neq (0)$. Also an integral ideal that is maximal among integral t-ideals is a prime ideal called a maximal t-ideal. Every nonzero prime t-ideal is contained in a maximal t-ideal.

Let's start with a sort of tongue-in-the-cheek statement.

Proposition B. If an integral domain $D \neq K$ has only a finite number of maximal t-ideals then D has infinitely many units.

The proof depends upon Proposition 7 of [Z2] which says that if D has finitely many maximal t-ideals then D is semi-quasilocal. Now of course the Jacobson radical of a semiquasi-local domain is nonzero. The point of interest in this case too is that if D has only a finite number of units then D has an infinite number of maximal t-ideals. (Though D having an infinite number of maximal t-ideals does not mean that we have a finite number of units, nor does it mean that the Jacobson radical of D is zero. In $Z + XQ[[X]]$ every maximal ideal is principal and hence is a t-ideal yet this ring has infinitely many units.)

One may remark that in the proof of Proposition B, as in the proof of Proposition A, which serves as the proof of the contrapositive also, we had to use the addition, in addition to multiplication. Now noting that most of the ideal theoretic notions known for integral domains have been translated to the language of monoids. We leave the following as a problem.

Problem C. Prove or disprove: It can be shown using monoid theoretic techniques that if an integral domain D has finitely many units then D has infinitely many maximal t-ideals.

Let's now turn to GCD monoids to show similar results. We go a bit general to include some more notions. For more general results we need some introductions. Call D an almost GCD domain if for all $x, y \in D$ there is a positive integer $n = n(x,y)$ such that $x^nD \cap y^nD$ is principal ($(x^n, y^n)_v$ is principal), [Z]. These domains generalize GCD domains in that an almost GCD domain is a GCD domain if for each pair $x, y \in D,$ we can reduce $n = n(x, y)$ to $1.$ Some good examples of almost GCD domains are Krull domains with torsion divisor class groups. These domains were called almost factorial domains in Fossum [F] and were studied by Storch in [S] as fastfactorielle ringe. In [AZ] an integral domain D was called an almost Bezout domain if for all $x, y \in D$ there is a positive integer $n = n(x, y)$ such that (x^n, y^n) is principal. An almost Bezout domain is an almost GCD domain and obviously it generalizes a Bezout domain. It was shown in [AZ] (Corollary 4.5) that an AGCD domain D is almost Bezout if and only if every maximal ideal of D is a t-ideal. Generally two elements x, y of a domain D are called v-coprime if $xD \cap yD = xyD$ (equivalently if $(a, b)_v = D$).

Theorem D. Let D be an almost GCD domain in which there can at most be finite sets of mutually v-coprime nonunits. Then D has at least a countable infinity of units. Equivalently, an almost GCD domain with finitely many units must have an infinite number of mutually v-coprime nonunits.

Proof. "There can at most be finite sets of mutually v-coprime nonunits" can be translated into "there is no infinite sequence of mutually v-coprime nonunits". So, by Proposition 2.1 of [DLMZ] D is an almost Bezout semilocal domain and this forces D to have finitely many maximal t-ideals and consequently a nonzero Jacobson radical, which forces D to have an infinity of units.

Corollary E. Let D be a GCD domain in which there can at most be finite sets of mutually coprime nonunits. Then D has at least a countable infinity of units.

Now note that both Theorem D and Corollary E need both the addition and multiplication for their proofs and so do their contrapositives namely: if D is an AGCD (or a GCD) domain with only a finite number of units then D contains infinitely many mutually v-coprime (coprime) nonunits. This seems to be a sort of analogue of Euclid's Theorem about infinitude of primes in the ring of integers. A similar result can be proven in much more generality. But I restrict it here to GCD domains as the above mentioned paper restricts its claim to GCD domains.

At this stage we can say that multiplicative monoids are a tool that can be used to settle questions about divisibility, or questions related to divisibility, by disregarding units and often the zero. Most of multiplicative ideal theory does just that. But as the monoids are more general, proving results for monoids does not mean proving results about rings, except in some limited cases. The "limited cases" are essentially linked with the Krull-Jaffard-Ohm Theorem that says that given a lattice ordered group G one can construct a Bezout domain with a group of divisibility isomorphic with $G.$ But there are well known examples of directed partially ordered groups that are not the groups of divisibility (see e.g. Example 4.7 on page 110 of [FS]). I have brought directed partially ordered groups here because the group of divisibility of an integral domain is a directed partially ordered group and the positive cone of a partially ordered group is a reduced monoid.

Then there are some special constructions, such as the polynomial and power series ring constructions that integral domains permit but monoids do not. The situation is worsened by the so-called $A + XB[X]$ constructions of [AAZ] and various other constructions mimicking them. Let me include a simple description of the $A + XB[X]$ construction. Let $A \subseteq B$ be an extension of rings and let X be an indeterminate over $B.$ Then

$A+XB[X]=\{f\in B[X]:f(0)\in A\}$ is an integral domain which is obviously a subring of $B[X]$. In ring theory we can study the multiplicative properties of $A+XB[X]$ with reference to those of A and $B,$ using monoids if we have to but there is no monoid construction that takes care of $A+XB[X],$ as the notion of a polynomial requires addition and so cannot be mimicked without some extra macinery.

One could come up with the suggestion: let us study some conditions on monoids and then see if those properties of monoids can define some domains. This would be very interesting. Yet this reminds me of the first time that I ran into a problem trying to transfer a defining property of an Abelian Riesz group to that of pre-Schreier domains. Let us recall that a directed partially ordered group G is a Riesz group if for all $x,y,z\in G^{+}$, $x\leq yz$ implies that $x=rs$ where $r\leq y$ and $s\leq z.$ Cohn [C] called an integrally closed integral domain D a Schreier domain if for all $x,y,z\in D\backslash\{0\}$ $x\mid yz$ implies that $x=rs$ where $r,s\in D$ are such that $r\mid y$ and $s\mid z.$ If we define the group of divisibility of D as $G(D)=\{xD:x\in qf(D)\backslash\{0\}\}$, ordering $G(D)$ by $xD\leq yD$ if and only if $xD\supseteq yD$ (i.e $y=xd$ for some $d\in D$) and defining the group operation as $(xD)(yD)=xyD$ we see that D is the identity of $G(D)$ and $G(D)^{+}=\{xD:x\in D\backslash\{0\}\}$ and because order being containment related is conformable with the product. From this one can conclude that an integrally closed domain is a Schreier domain if its group of divisibility is a Riesz group. Now if you ditch the integrally closed condition you get what was termed as a pre-Schreier domain in [Z1]. So a domain D is pre-Schreier if $G(D)$ is a Riesz group. Now, for $x_1,x_2,\ldots,x_n\in G$ where G is a partially ordered group, let $U(x_1,x_2,\ldots,x_n)=\{t\in G:t\geq x_i\}.$ So if $x_1D,x_2D,\ldots,x_nD\in G(D)$ then $U(x_1D,x_2D,\ldots,x_nD)=\{tD:tD\geq x_iD\}=\{tD:x_i\mid t\}.$ Thus $U(x_1,x_2,\ldots,x_n)\cup\{0\}=\cap x_iD.$ Since multiplication is involved we can associate $\cap x_iD$ with $U(x_1,x_2,\ldots,x_n).$ Now by (3) of Theorem 2.2 of [F], $G(D)$ is a Riesz group if and only if for all $a_1D,a_2D,\ldots,a_mD,$ $b_1D,b_2D,\ldots,b_nD\in G(D)$ we have $U(a_1D,a_2D,\ldots,a_mD)U(b_1D,b_2D,\ldots,b_nD)=U(a_1b_1D,\ldots a_ib_jD,\ldots,a_mb_nD).$ This translates, in ring theoretic terms, to $(\bigcap a_iD)(\bigcap b_jD)=\bigcap a_ib_jD.$ In [Z1], where the above observation was made (pages 1905-6), a domain D was said to have the $*$-property if for for all $a_1,a_2,\ldots,a_m;\ b_1,b_2,\ldots,b_n\in qf(D)\backslash\{0\}$ we have $(\bigcap a_iD)(\bigcap b_jD)=\bigcap a_ib_jD.$ It is easy to see that in the definition of the $*$-property we can restrict a_i,b_j in $D\backslash\{0\}.$ It was shown in [Z1] (Corollary 1.7) that a pre-Schreier domain has the $*$-property yet a domain with the $*$-property may not be a pre-Schreier domain. This discrepancy was blamed on involvement of addition in the definition of products of ideals in an integral domain. While we are at it let me mention that if D is a pre-Schreier domain that does not allow an infinite sequence of mutually coprime nonunits (in a pre-Schreier domain v-coprime is the same as coprime) then D is semi-quasi-local and so must have an infinity of units. (I could not prove that if a general domain D has zero Jacobson radical then D must have a an infinite sequence of mutually comaximal elements.)

Next, here's an anecdote that might indicate that results proved for monoids may be very hard to establish for domains. Around February 1995, just days before my kidneys failed, I wrote to Scott Chapman asking about the possibility of an n-UFD. By an n-UFD I meant an atomic domain in which every product of n irreducible elements is unique up to associates etc. The answers resulted in [ACH-KZ] and it turned out that Franz Halter-Koch was able to find an example of a 2 -UF monoid that was not a UFD and to my knowledge we have, to date, no example of a 2-UFD that is not a UFD.

To sum up, it appears to me that monoids are an efficient tool but they come in handy for integral domains only when the dust has settled after an interaction between addition and multiplication and that the monoids deal mostly with divisibility related problems.

Remark 1. There are some other reasons given on pages 148 and 149 of [FZ].

Remark 2. I exposed my co-authors of [ACH-KZ] to a version of the above material. Of them Franz Halter-Koch responded. He seems to agree that monoids cannot fill completely in for rings and is quite enthusiastic about writing with me a more detailed article on indicating where monoids are more efficient and where rings are. I refrain from including here his thoughts because I have not got his permission yet and because they would need a lot of introduction. So you will have to wait until something emrges in the form of an article.

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