QUESTION (HD 1506): Please give an example to show that an Almost-Schreier domain is not a pre-Schreier domain generally.

ANSWER: Almost Schreier (AS) domains were studied by Dumitrescu and Khalid in [Comm. Algebra 38(2010) 2981–2991]. They call D an AS-domain if, whenever $a \mid b_1b_2$ in $D\backslash\{0\}$, there exist an integer $k \geq 1$ and $a_1, a_2 \in D$ such that $a^k = a_1a_2$, $a_i \mid b_i^k$, $i = 12$.

Here D is a pre-Schreier domain if, whenever a $|b_1b_2$ in $D\backslash\{0\}$ we have $a = a_1a_2$ where $a_i \mid b_i$, $i = 1, 2$.

In their own words: "While a pre-Schreier domain is clearly an AS-domain, there exist simple examples of AS-domains which are not pre-Schreier such as $Z[\sqrt{-3}]$." Also any Almost GCD (AGCD) domain is an AS-domain. This suggests a simple way of finding AS-domains that are not pre-Schreier.

Observation A. Any atomic AS-domain that is not a UFD is an AS-domain that is not pre-Schreier.

This follows from the fact that an atomic pre-schreier domain is a UFD, see Corollary 1.8 of [Comm. Algebra

15 (1987), 1895–1920].

Observation A can be used to line up a number of easy examples of Almost GCD domains that are not pre-Schreier.

(1) A Dedekind domain with torsion class group that is not a PID, is an AS-domain that cannot be Schreier (Recall that a Schreier domain is integrally closed pre-Schreier.)

(2) A Krull domain with torsion divisor class group (almost factorial ring) that is not a UFD is an AS-domain that is not Schreier.

This much about integrally closed examples. Let's take the non-integrally closed ones.

(3) $Z[\sqrt{-3}]$, as mentioned above, is a Noetherian AS-domain that is not integrally closed and hence cannot be a UFD and so cannot be pre-Schreier. Then there's Corollary 6.3 in the above mentioned paper of Dumitrescu and Khalid. The Corollary says: Let D be a Noetherian domain such that $D = \bigcap_{P \in X^1(D)} D_P$. Then D is an AS-domain if and only if D has torsion t-class group. Indeed a Noetherian one-dimensional local domain is AS. In particular, a one dimensional Noetherian domain is an AS-domain if and only if it has torsion Picard group.

(4) Example 2.13 of [Manuscripta Math. 15(1985) 29-62]. This is an example of a non integrally closed atomic domain that is also an AGCD domain. This is one of the examples that you can play with, to get more ideas.

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