Quasi-Schreier domains II

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Abstract

We study a class of integral domains characterized by the property that every nonzero finite intersection of principal ideals is a directed union of invertible ideals.

Key Words: Schreier domain, invertible ideal

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Let D be a domain with quotient field K. Call $a \in D\setminus\{0\}$ *primal* if for all $b_1, b_2 \in D \setminus \{0\}$ $a \mid b_1b_2$ implies that $a = a_1a_2$ for some $a_1, a_2 \in D$ with $a_i \mid b_i$ for $i = 1, 2$. Cohn (1968) called D a *Schreier domain* if D is integrally closed and if every nonzero element of D is primal. He showed that a GCD domain is Schreier, cf. Cohn (1968, Theorem 2.4) and that if D is Schreier then so is each polynomial ring over D. Schreier domains were also studied in McAdam and Rush (1978). In Zafrullah (1987) D was called *pre-Schreier* if every nonzero element of D is primal. So a Schreier domain is an integrally closed pre-Schreier domain. In Zafrullah (1987) it was indicated that if D is pre-Schreier then so is any fraction ring of D, cf. Zafrullah (1987, Corollary 1.3). By Bouvier and Zafrullah (1988, Proposition 1.4), a pre-Schreier domain D has zero t-class group hence it has zero Picard group (see definitions recalled below).

In Dumitrescu and Moldovan (2003a), the following generalization of the class of pre-Schreier domains was studied. A domain D is a *quasi-Schreier domain* if whenever A, B_1, B_2 are invertible ideals of D and $A \supseteq B_1B_2$, then $A = A_1A_2$ for some (invertible) ideals A_1, A_2 of D with $A_i \supseteq B_i$ for $i = 1, 2$.

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As proved in Dumitrescu and Moldovan (2003), every generalized GCD domain is quasi-Schreier. Recall that, according to Anderson and Anderson (1979), a *generalized GCD domain (GGCD) domain* is a domain in which the intersection of any two invertible ideals is again an invertible ideal. Since a Prüfer domain that is Schreier is GCD (Cohn, 1968), any Dedekind domain that is not a PID is an example of a quasi-Schreier domain that is not pre-Schreier. We show that D is quasi-Schreier if and only if every finite intersection of nonzero principal ideals of D is a directed union of invertible ideals. This enables us to show that if D is quasi-Schreier then so is every ring of fractions of D. We also show that $D[X]$ is quasi-Schreier if and only if D is integrally closed quasi-Schreier, which is quite reminiscent of the following result: the polynomial ring $D[X]$ is pre-Schreier if and only if D is Schreier (Cohn, 1968, Theorem 2.7 and McAdam and Rush, 1978, Theorem 3). As a by-product of our study we show that D is a GGCD domain if and only if every upper to zero in the polynomial ring $D[X]$ is an invertible ideal; a result reminiscent of the well known fact that D is a GCD domain if and only if every upper to zero in $D[X]$ is principal. (Recall that an *upper to zero* in $D[X]$ is a nonzero prime ideal P of $D[X]$ such that $P \cap D = (0)$). We also prove that a domain D is integrally closed quasi-Schreier if and only if its Nagata ring $D(X)$ is Schreier.

For a fractional ideal I of D, consider the fractional ideals $I^{-1} = \{x \in K;\ xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \cup J_v$ for J running over the set of all finitely generated fractional subideals of I (see Gilmer, 1972, Sections 32 and 34). Call I a *t-ideal* if $I = I_t$. Say that I is *invertible* (resp., *t-invertible*) if $IJ = D$ (resp., $(IJ)_t = D$) for some fractional ideal J. A domain D is a *Prüfer v-multiplication domain (PVMD)* if I_t is t-invertible for each finitely generated fractional ideal I of D (see Mott and Zafrullah, 1981). The *Picard group* of D, $Pic(D)$, is the multiplicative group of invertible fractional ideals of D modulo the subgroup of principal ideals. The *t-class group* of D, $Cl_t(D)$, is the group of all t-invertible fractional t-ideals of D under t-multiplication (i.e., the operation sending a pair of t-ideals I, J of D to $(IJ)_t$) modulo the subgroup of principal ideals (see Bouvier and Zafrullah, 1988). $Pic(D)$ is a subgroup of $Cl_t(D)$ and $G(D) = Cl_t(D)/Pic(D)$ is the so-called *local class group* of D (see Bouvier, 1982). If D is a Krull domain, $Cl_t(D)$ is the usual divisor class group, and $Cl_t(D) = 0$ (resp., $G(D) = 0$) if and only if D is factorial (resp., locally factorial) (see Fossum, 1973).

Throughout this paper, all rings are commutative and unitary. Our standard reference for any undefined notation or terminology is Gilmer (1972).

RESULTS

We begin by recalling, in the next theorem, some results from Dumitrescu and Moldovan (2003a).

Theorem 1 *Let D be a domain.*

(a) A domain D is pre-Schreier if and only if it is quasi-Schreier with zero Picard group.

(*b*) *If* D *is quasi-Schreier, then* D *is locally pre-Schreier with zero local class group (that is, every* t*-invertible* t*-ideal of* D *is invertible). Moreover, a locally pre-Schreier* h*-local domain is quasi-Schreier.*

(*c*) *A domain is a GGCD domain if and only if it is a quasi-Schreier PVMD.*

(*d*) *If the polynomial ring* $D[X]$ *is quasi-Schreier, then so is* D.

(*e*) *If* D *is seminormal and* $D[X]$ *is quasi-Schreier, then* D *is integrally closed.*

(*f*) *If* D *is an integrally closed quasi-Schreier domain, then* $D[X]$ *is locally pre-Schreier.*

(*g*) D *is a* π*-domain if and only if* D *is quasi-Schreier and has ACC on invertible ideals.*

Our first result improves part (*b*) of the preceding theorem. Let D be a domain and $\mathcal{F}$ a family of nonzero prime ideals of D. According to (Anderson and Zafrullah, 1999), $\mathcal{F}$ is called an *IFC family* if $D = \cap_{P\in\mathcal{F}} D_P$, every nonzero $x \in D$ belongs to only finitely many members of $\mathcal{F}$, and if no two members of $\mathcal{F}$ contain a nonzero prime ideal of D. Say that D is an $\mathcal{F}$*-IFC domain* if D has an IFC family $\mathcal{F}$. The class of $\mathcal{F}$-IFC domains includes the h-local domains of Matlis (1972), Noetherian domains whose grade-one primes are of height one, Krull domains, the generalized Krull domains of Ribemboim (1956), and independent rings of Krull type of Griffin (1968). Let $\mathcal{F}$ be an IFC family. The mapping $I \mapsto I^* = \cap_{P\in\mathcal{F}} ID_P : F(D) \to F(D)$ is a finite character star-operation and, if $I, J \in F(D)$, then $I^* = J^*$ if and only if $ID_P = JD_P$ for each $P \in \mathcal{F}$, cf. Anderson (1988).

Theorem 2 *An* $\mathcal{F}$*-IFC domain* D *is quasi-Schreier if and only if* D *is locally pre-Schreier.*

Proof. The only if part is clear, cf. part (*b*) of the preceding theorem. For the if part, let $I \supseteq AB$ with I, A, B invertible ideals of D and let $M_1, M_2, ..., M_n$ be the members of $\mathcal{F}$ containing I. For each i, set $D_i = D_{M_i}$, $I_i = ID_i$, $A_i = AD_i$, $B_i = BD_i$. Clearly, I_i, A_i, B_i are principal ideals of D_i and $I_i \supseteq A_iB_i$. Since D_i is pre-Schreier, $I_i = a_ib_iD_i$ for some $a_i, b_i \in D$ with $a_iD_i \supseteq A_i$ and $b_iD_i \supseteq B_i$. Since D is an $\mathcal{F}$-IFC-domain, the only member of $\mathcal{F}$ which can contain $A'_i = a_iD_i \cap D$ (resp. $B'_i = b_iD_i \cap D$) is M_i, cf. Anderson and Zafrullah (1999, Lemma 2.3). Let $*$ be the star-operation induced by $\mathcal{F}$. Set $A' = A'_1A'_2 \cdots A'_n$ and $B' = B'_1B'_2 \cdots B'_n$. As I, A, B are invertible ideals, they are $*$-ideals. Then the relations $I = (A'B')^*$, $A'^* \supseteq A$ and $B'^* \supseteq B$ hold locally (in every D_P with $P \in \mathcal{F}$), so they hold globally. As $*$ is a star-operation of finite character, we get $I = (A'B')_t$. So A'_t and B'_t are t-invertible t-ideals, hence they are invertible, because D being locally pre-Schreier it has zero local class group, cf. Bouvier and Zafrullah (1988, Corollary 2.10). Thus the relations $I = A'_tB'_t$, $A'_t \supseteq A$ and $B'_t \supseteq B$ hold, as desired. •

We give new characterizations for quasi-Schreier domains.

Theorem 3 *For a domain D, the following assertions are equivalent:*

(a) D is a quasi-Schreier domain,

(b) every finite intersection of invertible ideals is a directed union of invertible (sub-) ideals,

(c) every finite intersection of nonzero principal ideals is a directed union of invertible ideals,

(d) for every nonzero finitely generated ideal I of D, I^{-1} is a directed union of invertible fractional ideals, and

(e) if $a_1,...,a_n$, $b_1,...,b_m$ are nonzero elements of D such that every a_i divides every b_j, there exists an invertible ideal I such that $(b_1,...,b_m) \subseteq I \subseteq \cap_i a_i D$.

Proof. The equivalence of (a) and (b) was noted in Dumitrescu and Moldovan (2003). The equivalence of (b) and (c) is a consequence of the well-known fact that every invertible ideal is isomorphic to a finite intersection of principal ideals. Finally, (d) and (e) are just restatements of (c). •

Remark 4 Recall that a domain D is called *v-coherent* if every intersection of two nonzero principal ideals is an ideal of v-finite type. Using the theorem above, it can be shown that a domain D is a GGCD domain if and only if D is v-coherent and quasi-Schreier. This observation improves part (c) of Theorem 1. It may also remind one of the already known result: an integral domain D is a GCD domain if and only if D is pre-Schreier and v-coherent, cf. (Anderson and Zafrullah, to appear, Proposition 2.3).

The following result has been proven in Dumitrescu and Moldovan (2003a) under the additional hypothesis that every invertible ideal of D_S is an extension of an invertible ideal of D. Thanks to the previous characterization of quasi-Schreier domains, this hypothesis can be dropped.

Theorem 5 *If D is a quasi-Schreier domain, then every fraction ring of D is also quasi-Schreier.*

Proof. Let S be a multiplicative set in D. Let nonzero $a_1,...,a_n$, $b_1,...,b_m$ belong to D_S such that every a_i divides every b_j in D_S. Multiplying by suitable elements from S, we may assume that $a_1,...,b_m$ are in D and every a_i divides every b_j in D. Since D is quasi-Schreier, there exists an invertible ideal I of D which contains all elements b_j and is contained in the intersection of all principal ideals $a_i D$. Then ID_S is an invertible ideal of D_S and $(b_1,...,b_m)D_S \subseteq ID_S \subseteq a_1 D_S \cap \cdots \cap a_n D_S$. •

In Cohn (1968, Theorem 2.7) and McAdam and Rush (1978, Theorem 3), it was shown that, for a domain D, $D[X]$ is pre-Schreier if and only if D is pre-Schreier and integrally closed (i.e., D is Schreier). We use this opportunity to give a direct proof for the implication that when $D[X]$ is a pre-Schreier domain, D must be integrally closed.

Proposition 6 *Let D be a domain and $0 \neq a, b \in D$ with b primal in $D[X]$. If a/b is integral over D, then $a/b \in D$.*

Proof. As a/b is integral over D, we can write an equality $f = (X - a/b)g$ with $f \in D[X]$ monic and $g \in D[a/b][X]$. There exists an integer $n \geq 1$ such that $b^{n-1}g = G \in D[X]$. So $b^n f = (bX - a)G$, hence $b^n \mid_{D[X]} (bX - a)G$. As b^n is primal in $D[X]$, cf. Cohn (1968, Lemma 2.5), it follows that $b^n = uv$ for some $u, v \in D$ such that $u \mid_{D[X]} bX - a$ and $v \mid_{D[X]} G$. So $f = ((b/u)X - a/u)G/v$, hence b/u is a unit of D, because f is monic. Thus $b \mid_D u \mid_D a$, that is, $a/b \in D$. •

Corollary 7 *For a domain D, $D[X]$ is Schreier if and only if every element of D is primal in $D[X]$.*

Proof. The direct implication is clear. Conversely, assume that every element of D is primal in $D[X]$. By Proposition 6, D is integrally closed. So $D[X]$ is a Schreier domain by the proof of Cohn (1968, Theorem 2.7). •

Our next theorem extends the fact mentioned above that $D[X]$ is a pre-Schreier domain if and only if D is Schreier, to the case of quasi-Schreier domains, thus improving part (e) of Theorem 1. We need the following lemma which is probably already known.

Lemma 8 *Let D be an integral domain. If the local class group of $D[X]$ is zero, then D is integrally closed.*

Proof. Let P be an upper to zero in $D[X]$ that contains a polynomial f with $c(f) = D$, where $c(f)$ is the content ideal of f. It suffices to show that P is invertible, cf. Anderson and Zafrullah (1993, Theorem 3.2). By Houston and Zafrullah (1989, Theorem 1.4), P is t-invertible. As $D[X]$ has zero local class group, P is invertible. •

Theorem 9 *Let D be an integral domain. Then $D[X]$ is a quasi-Schreier domain if and only if D is an integrally closed quasi-Schreier domain.*

Proof. The "only if" part follows from Lemma 8 and parts (b) and (d) of Theorem 1. Conversely, assume that D is an integrally closed quasi-Schreier domain. By part (d) of Theorem 3, it suffices to show that for every nonzero finitely generated ideal J of $D[X]$, J^{-1} is a directed union of invertible fractional ideals. Let J be such an ideal. As D is integrally closed, there exist a nonzero fractional ideal I of D and $f \in D[X]$ such that $J_v = fI_vD[X]$, cf. Querre (1980, Lemme 2). Moreover, we may assume that I is finitely generated, cf. Theorems 2.1 and 3.1 of Anderson et al. (1995). Hence $J^{-1} = f^{-1}I^{-1}D[X]$, cf. Hedstrom and Houston (1980, Lemma 4.1). By part (d) of Theorem 3, I^{-1} is a directed union of invertible fractional ideals of D. So J^{-1} is a directed union of invertible fractional ideals of $D[X]$. •

Let D be an integrally closed domain and I a finite intersection of nonzero principal ideals of $D[X]$. The key point of the preceding proof is that I is isomorphic, as a $D[X]$ module, to $JD[X]$ for some finite intersection of nonzero principal ideals J of D, cf. Theorems 2.1 and 3.1 of Anderson et al. (1995).

Now McAdam and Rush (1978, Theorem 3) can be derived from our results.

Corollary 10 *When D is an integral domain, $D[X]$ is pre-Schreier if and only D is Schreier.*

Proof. We combine the preceding theorem, part (*a*) of Theorem 1 and the fact that $\text{Pic}(D) \simeq \text{Pic}(D[X])$ when D is integrally closed, cf. Gilmer and Heitmann (1980). ∎

Example 11 Let A be a Prüfer domain which has two nonzero nonunits a, b such that $a \in \cap_n b^n A$. Consider the domain $B = A + XA[1/b][X]$. By Theorem 9, $A[X]$ is a quasi-Schreier domain. Hence B is quasi-Schreier, because B is the directed union of its subrings $A[X/b^n]$, $n \geq 1$. Moreover, B is not a GGCD domain, because the ideal $aA[1/b] \cap A = aA[1/b]$ is not finitely generated and hence not invertible, so (Anderson et al., 2001, Theorem 3.3) applies. Note that if D is an integrally closed quasi-Schreier domain and S is a multiplicative set in D, then $D + XD_S[X]$ is quasi-Schreier (use the direct union argument given above).

We prove that a domain D is integrally closed quasi-Schreier if and only if its Nagata ring $D(X)$ is Schreier. Recall that $D(X)$ is the quotient ring of $D[X]$ with respect to the multiplicative system of polynomials whose coefficients generate the unit ideal. The proof needs a lemma that is a variation of the well known result that D is integrally closed if and only if $(c(fg))_v = (c(f)c(g))_v$ for all nonzero $f, g \in D[X]$, cf. Gilmer (1972, Theorem 34.8), where $c(h)$ is the content ideal for h. Compare with Mott et al. (1990, Theorem 1.5).

Lemma 12 *For a domain D, the following are equivalent.*

(1) *D is integrally closed.*
(2) *Whenever $f = f_1 f_2$ where $c(f)$ is invertible, then $c(f_1)$ is invertible.*
(3) *Whenever $f = f_1 f_2$ where $c(f)$ is principal, then $c(f_1)$ is invertible.*
(4) *Whenever $f = f_1 f_2$ where $c(f)$ is invertible, then $c(f) = c(f_1)c(f_2)$.*
(5) *Whenever $f = f_1 f_2$ where $c(f)$ is principal, then $c(f) = c(f_1)c(f_2)$.*

Proof. (1) $\Rightarrow$ (2). $c(f) = (c(f))_v = (c(f_1)c(f_2))_v \supseteq c(f_1)c(f_2) \supseteq c(f)$. So $c(f_1)$ being a factor of an invertible ideal is invertible. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear. (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Since $c(f_1)$ is invertible, $c(f) = c(f_1 f_2) = c(f_1)c(f_2)$. (5) $\Rightarrow$ (1). Let α in K be integral over D with monic polynomial f. So in $K[X]$, $f = (X - \alpha)f_2$. Choose nonzero b in D with $b(X - \alpha), bf_2$ in $D[X]$. So $b^2 D = c(b^2 f) = c(b(X - \alpha)bf_2) = c(b(X - \alpha))c(bf_2) = b^2 c(X - \alpha)c(f_2)$. So $\alpha \in (1, \alpha)c(f_2) = c(X - \alpha)c(f_2) = c(f) = D$ since f_2 has leading coefficient 1. ∎

Theorem 13 *For a domain D, the following are equivalent.*
(1) *D is integrally closed quasi-Schreier.*
(2) *$D(X)$ is Schreier.*

Proof. (1) $\Rightarrow$ (2). D integrally closed quasi-Schreier implies $D[X]$ is integrally closed quasi-Schreier implies $D(X)$ is integrally closed quasi-Schreier, cf. Theorems 5 and 9. But $Pic(D(X)) = 0$, cf. Anderson (1977, Theorem 2), so $D(X)$ is integrally closed pre-Schreier, that is, Schreier. (2) $\Rightarrow$ (1). Suppose that $D(X)$ is Schreier. Then $D(X)$ is integrally closed gives that $D = D(X) \cap K$ is integrally closed. Let A, B_1, B_2 be invertible ideals of D with $A \supseteq B_1B_2$. Choose $f \in D[X]$ with $c(f) = A$. Then $AD(X) = fD(X)$ since $c(f)$ is invertible, cf. Anderson (1977). Likewise choose $b_i \in D[X]$ with $c(b_i) = B_i$ and so $B_iD(X) = b_iD(X)$. Then $fD(X) = AD(X) \supseteq B_1D(X)B_2D(X) = b_1D(X)b_2D(X)$. So $f|b_1b_2$ in $D(X)$. Since $D(X)$ is Schreier, we can write $f = a_1a_2$ where $a_i|b_i$ with $a_i \in D(X)$. Write $a_i = a_i'/a_i''$ where $a_i', a_i'' \in D[X]$ with $c(a_i'') = D$. Then $fa_1''a_2'' = a_1'a_2'$. Now $c(a_1'a_2') = c(fa_1''a_2'') = c(f)c(a_1'')c(a_2'') = c(f) = A$ is invertible. Put $A_i = c(a_i')$, so $A = A_1A_2$ by the preceding lemma. Now $a_i|b_i$, so $B_iD(X) = b_iD(X) \subseteq a_iD(X) = a_i'D(X) = A_iD(X)$ and hence $B_i = B_iD(X) \cap D \subseteq A_iD(X) \cap D = A_i$. •

We may look at the preceding proof from the following point of view. A domain D is a quasi-Schreier domain if and only if its group of invertible fractional ideals $Inv(D)$ has the Riesz interpolation property (see Anderson (2000)). Now, it follows easily that if G is an ordered abelian group satisfying the Riesz interpolation property, then this property is inherited by every convex subgroup of G. Assume that D is an integrally closed domain and let $\delta : Inv(D) \to Inv(D(X))$ be the canonical morphism. Restricting, we get an isomorphism of ordered abelian groups $\delta : Inv(D) \to Im(\delta) = H$ and, clearly, the positive cone of H consists of the principal ideals of $D(X)$ generated by polynomials $f \in D[X]$ with invertible content. In the proof of Theorem 13, it is shown implicitly that H is a convex subgroup of $Inv(D(X))$.

Example 14 We give an example of a domain D which is not integrally closed, such that $D(X)$ is pre-Schreier. Suppose that (V, M) is a valuation domain with $M = M^2 \neq 0$ such that V has nonperfect residue field k. Let $\pi : V \to k$ be the canonical map and consider the pull-back domain $W = \pi^{-1}(k^p)$ where p is the characteristic of k. By Anderson and Dobbs (1980, Proposition 2.6), W is a PVD which is not integrally closed and, clearly, $V^p \subseteq W$. By Zafrullah (1987, Theorem 4.4), W is pre-Schreier. Since V and W are quasilocal with maximal ideal M, $V(X) = V[X]_{M[X]}$ and $W(X) = W[X]_{M[X]}$. Let $\pi' : V(X) \to k(X)$ be the canonical map. Then $W(X) = \pi'^{-1}(k^p(X))$, because W is a pullback and the pullbacks are preserved by flat extensions, as it is the Nagata ring extension. As $V(X)$ is a valuation domain with idempotent maximal ideal, we can repeat the argument above to show that $W(X)$ is a pre-Schreier domain.

Recall that a domain D is called a *UMT-domain* (Houston and Zafrullah, 1989) if the uppers to zero in $D[X]$ are t-invertible. By Fontana et al. (1998, Theorem 2.4), D is a UMT-domain if and only if $D[X]$ is. Thus, if D is a UMT-domain such that $D[X]$ is quasi-Schreier, then the uppers of zero in $D[X]$ are invertible. Hence D is a GGCD domain, as our next result shows.

Theorem 15 *For a domain D, the following assertions are equivalent.*
(1) *D is a GGCD domain.*
(2) *Every upper to zero in $D[X]$ is invertible.*
(3) *For every linear polynomial f the prime ideal $f(X)K[X] \cap D[X]$ is invertible.*

Proof. (1)⇒(2). Since D is a GGCD domain, for every $f \in K[X]\backslash\{0\}$ we have $(c(f))^{-1}$ invertible (Anderson and Anderson, 1979, Theorem 1). Next since for every upper P to zero there is $f(X) \in K[X]$ such that $P = f(X)K[X] \cap D[X]$ and since a GGCD domain is integrally closed we have $P = f(X)K[X] \cap D[X] = f(X)c(f)^{-1}[X]$, cf. Querre (1980). (2)⇒(3). Obvious. (3)⇒(1). We first show that D is integrally closed. For this let $\alpha \in K$ be integral over D and let $g(X)$ be the monic polynomial of minimal degree in $D[X]$ such that $g(\alpha) = 0$. Now $P = (X - \alpha)K[X] \cap D[X]$ is an upper to zero and so is invertible because of (3). But since $g(X) \in P$ we conclude by Gilmer et al. (1987) that P is principal. This forces $\alpha \in D$. Next, D being integrally closed, for each pair $a, b \in D\backslash\{0\}$ we have $P = (aX + b)K[X] \cap D[X] = (aX + b)c(aX + b)^{-1}[X]$. Since P is invertible, so must be $c(aX + b)^{-1}[X]$ and hence so is $c(aX + b)^{-1} = (a, b)^{-1}$. Having shown that for each pair $a, b \in D\backslash\{0\}, aD \cap bD$ is invertible we quote Anderson and Anderson (1979, Theorem 1). •

Remark 16 Theorem 15 provides a GGCD analogue of the well known fact that D is a GCD domain if and only if every prime upper to zero in $D[X]$ is principal if and only if for every linear polynomial $f(X)$ the prime upper containing $f(X)$ is principal. As the following corollary shows, under certain conditions not equivalent to the pre-Schreier condition in general, the GCD property just takes over.

Corollary 17 *If D is a semi-quasilocal domain such that the prime uppers to zero in $D[X]$ are invertible, then D is a GCD domain.*

Proof. If the prime uppers to zero in $D[X]$ are invertible, then by Theorem 15 D is a GGCD domain and a semilocal GGCD domain is GCD, see e.g. Zafrullah (1977). •

Recall that according to Zafrullah (1987) an integral domain is said to have the $(*)$ property if for every set $a_1, ..., a_m; b_1, ..., b_n \in D\backslash\{0\}$ we have $(\cap a_i D)(\cap b_j D) = \cap a_i b_j D$. It was shown in Zafrullah (1987) that D has the $(*)$ property if and only if D locally has the $(*)$ property and that a locally pre-Schreier domain has the $(*)$ property. It was also shown in Zafrullah (1987) that a PVMD D is a GGCD domain if and only if D has the $(*)$ property. Now that we have shown that a v-coherent domain which is also quasi-Schreier is a GGCD domain the following questions arise. Is the quasi-Schreier property equivalent to the above mentioned (*)? And, is the quasi-Schreier condition equivalent to being locally pre-Schreier? All we have at present is the following observation. A locally pre-Schreier domain D is quasi-Schreier if and only if for all $a_1, ..., a_n$, $b_1, ..., b_m$ in D with $(b_1, ..., b_m)D \subseteq \cap_i a_i D$, there is a t-invertible ideal I such that $(b_1, ..., b_m)D \subseteq I \subseteq \cap_i a_i D$. The proof is straightforward and so is left to the reader.

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