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A NOTE ON TWO GENERATOR FINITE GROUPS WITH TWO RELATIONS

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In this note we give a presentation of a class of finite groups with two generators and two defining relations, that is, we prove the following :—

Theorem : The group

$$G(n, r) = gp\{a, b;\ a^n = 1,\ (a^{-1}b)^r = (ba^{-1})^{r+1}\} \quad (1)$$

is finite :

First we observe that the relation $(a^{-1}b)^r = (ba^{-1})^{r+1}$ implies that a and b have the same order :

$$(a^{-1}b)^r = b\,(a^{-1}b)^r a^{-1}$$

or $\qquad (a^{-1}b)^r\, a\,(a^{-1}b)^{-r} = b$

Also from the same relation we have

$$a^{-1}(ba^{-1})^r\, a = (ba^{-1})^{r+1}$$

If $\qquad ba^{-1} = x$, then

$$a^{-1}x^r a = x^{r+1} \quad .. \quad (2)$$

Taking rth power of both sides of (2)

$$a^{-1}x^{r^2}a = x^{r(r+1)}$$

That is

$$a^{-2}x^{r^2}a^2 = a^{-1}x^{r(r+1)}a = (a^{-1}x^r a)^{r+1}$$

$$= (x^{r+1})^{r+1} = x^{(r+1)^2}$$

Suppose that

$$a^{-k}x^{r^k}a^k = x^{(r+1)^k} \quad .. \quad (3)$$

is true.

Taking rth power of both sides of (3)

$$a^{-k}\ x^{r^{k+1}}a^k = x^{r(r+1)^k}$$

That is

$$a^{-k-1}\, x^{r^{k+1}}\, a^{k+1} = (a^{-1}\, x^{r}\, a)^{(r+1)^k}$$

$$= (x^{r+1})^{(r+1)^k} = x^{(r+1)^{k+1}}$$

Hence by Mathematical induction

$$a^{-n}\, x^{r^n}\, a^{n} = x^{(r+1)^n} \quad \text{for all positive integers } n.$$

Thus if $a^n=1$, then the order m of x is a divisor of $(r+1)^n - r^n$

Now both r, $(r+1)$ are coprime to $(r+1)^n - r^n$ and so x^r and $x^{(r+1)}$ genrate the same cyclic group as x.

Consider now the group

$$G = gp\{a, x \,;\, a^n = x^m = 1,\, a^{-1}\, x^r\, a = x^{r+1}\}$$

As r, $(r+1)^n - r^n$ are coprime, there exists an integer k such that $x^{kr}=x$. Then $a^{-1}\, x\, a = a^{-1}\, x^{kr}\, a = a^{-1}\, x^r\, a \ldots a^{-1}\, x^r\, a$ (k times).

$$= x^{(r+1)k} = x^{rk+k} = x^{k+1}$$

However since $(x^{k+1})^r = x^{kr+r} = x^{r+1}$, $gp\{x^{k+1}\} = gp\{x\}$ and therefore G is an extension of $gp\{x\}$ of order $m \mid (r+1)^n - r^n$ by $gp\{a\}$ of order n.

But as $x = ba^{-1}$, G contains together with a also the element b and so $G = G(n, r)$. Hence the theorem.

We note that when $n=2$, then for different values for r we get all the dihedral groups of order $2m$, m odd.