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A GENERAL THEORY OF ALMOST FACTORIALITY

Muhammad Zafrullah

Let R be a commutative integral domain with 1. The non-zero elements a, b of R may be called v-coprime if $aR \cap bR = abR$. A Krull domain is called almost factorial if for all f, g in R there is $n \in N$ such that $f^nR \cap g^nR$ is principal. From this it is easy to establish that if R is almost factorial then for all x in R there is $n \in N$ such that $x^n = p_1p_2 \ldots p_r$ where p_i are mutually v-coprime primary elements and that this expression is unique. In this article we drop the requirement that R be Krull and replace the primary elements by elements called prime blocks and develope a theory of almost factoriality, a special case of which is the theory of almost factorial Krull domains.

We start with the definition that a non-zero non-unit $p \in R$ is a prime block if for all $x, y \in R$ with x, y both v-coprime to p there exist $n(x,y) \in N$ and $d \in R$ such that $(x^n, y^n) \subseteq dR$ and at least one of x^n/d, y^n/d is v-coprime to p. Using this definition we show that if a non-unit x is expressible as a product of finitely many prime blocks then it is expressible uniquely as a product of mutually v-coprime prime blocks. Thus a general almost factorial ring is an integral domain R such that for all $x \in R - \{0\}$ there is $n(x) \in N$ such that x^n is expressible as a product of prime blocks. Here we take a unit as an empty product. We show that a general almost UFD, R is an almost GCD

(AGCD) domain that is for $x,y \in R$ there is $n \in N$ such that $x^nR \cap y^nR$ is principal. Unlike the UFD's and GCD-domains a general almost UFD and hence an almost GCD-domain may not be integrally closed. An example of a non-integrally closed general almost UFD is constructed to establish this point. We also study the relationship between an almost GCD-domain and its integral closure. It is interesting to note that the integral closure of an almost GCD-domain is an almost GCD-domain.

To explain our results on integrally closed general AUFD's and AGCD-domains we shall be using terminology with which a general reader may not be familiar. So before mentioning our results we explain, briefly, some of the terms involved. The function defined on $F(R)$, the set of fractional ideals of R, by $A \mapsto (A^{-1})^{-1} = A_v$ is called the v-operation on R. For $A,B \in F(R)$, $(AB)_v=(A_vB)_v=(A_vB_v)_v$; these equations represent what may be called v-multiplication. $A \in F(R)$ is called a v-ideal if $A = B_v$ for some B in $F(R)$ and A is said to be a v-ideal of finite type if $A = B_v$ where B is finitely generated. An integral domain R is said to be a Prüfer v-multiplication domain (PVMD) if the set $H(R)$ of v-ideals of finite type of R is a group under v-multiplication. We show that an integrally closed AGCD-domain is a PVMD with the property that for every finitely generated ideal A, $(A^n)_v$ is principal for some $n \in N$. In [7] it is noted that the theory of PVMD's runs along lines remarkably similar to that of Krull domains. This may be demonstrated once more by the introduction of the t-class group which is defined as follows. The t-operation on $F(R)$ is defined as $A \mapsto A_t = \bigcup B_v$, where B ranges over finitely generated fractional ideals contained in A. An ideal $X \in F(R)$ is said to be <u>t-invertible</u> if there is Y in $F(R)$ with $(XY)_t = R$. In this case $Y_t = Y_v = X^{-1}$ and X_v and X^{-1} are both v-ideals of finite type (cf.[3]). Denote by $T(R)$ the set of all v-ideals of R which are t-invertible. Then it is easy to establish that $T(R)$ is a group under v-multiplication for any integral domain R. Clearly the set $P(R)$ of principal fractional ideals of R is a subgroup of $T(R)$. We call the quotient group $C_T(R) = T(R)/P(R)$ the <u>t-class group</u> of R. Noting that a Krull domain R is a PVMD in which every v-ideal is of finite type we conclude that for R Krull $T(R) = H(R) = \mathrm{div}(R)$ and consequently for R Krull $C_T(R) = Cl(R)$ the divisor class group. This result follows from the fact that R is a PVMD if and only if every finitely generated ideal of R is t-invertible (cf.[3]). So for a PVMD also, $T(R) = H(R)$. Noting also that a PVMD is a GCD-domain if and only if every t-invertible v-ideal of R is principal ([7] Proposition 6.1) we conclude that a PVMD is a GCD-domain if and only if its t-class group is trivial. Thus for R a PVMD $C_T(R)$ measures how far is R from being a GCD-domain. Now it is easy to see that a PVMD with a torsion t-class group is an almost GCD-domain. We show that an integrally closed general almost UFD is a ring of finite character called an <u>independent ring of Krull type</u> (cf.[2]) with torsion t-class group. We also study the conditions under which an <u>atomic</u> integrally closed general AUFD becomes an almost factorial ring or <u>fastfaktoriell</u> ring of Storch [11]. Here R is atomic if every non-zero non-unit of

R is expressible as a product of finitely many irreducible elements. In what follows we shall use the term 'almost UFD' to mean 'general almost UFD' and when we have to mention the existing notion due to Storch we shall call it fastfaktoriell ring. We do this to economise on space.

This article is split into five sections, apart from this rather lengthy introduction. In the first section we study the notion of a prime block and define almost UFD's (AUFD's). In section 2 we study the properties of AUFD's which enable us to describe their structure. For example we show that an AUFD is an AGCD-domain and that an AGCD-domain R is an AUFD if and only if every non-zero non-unit of R belongs to finitely many maximal t-ideals of R and no pair of maximal t-ideals of R contains a non-zero prime ideal. Also in this section we construct an example to show that an AUFD may not necessarily be integrally closed. In section 3 we study integrally closed AGCD-domains and show that an integrally closed AGCD-domain is a PVMD. In section 4 we study integrally closed AUFD's and establish their relationship with fastfaktoriell rings. Finally in section 5 the usual ring formations from integrally closed AGCD-domains. Indeed we show that if R is an integrally closed AGCD-domain then so is $R[X]$. Using the notion of t-class groups we can state this result as: for R integrally closed R is a PVMD with torsion t-class group if and only if the polynomial ring $R[X]$ is.

Finally, for introduction, if I am to gain any positive credit for writing this article, part of it should go to Professor Ulrich Krause who wrote [4] and [6] and then asked me if there was a general theory of factorization of this kind. In fact, although my German is extremely poor, [6] helped form the theory of almost factoriality described above - in one way or another.

1. PRIME BLOCKS

Let R be a commutative integral domain with 1, and let K be its field of fractions. Two elements $a,b \in R - \{0\}$ are called v-coprime if $aR \cap bR = abR$. We denote v-coprimeness of a,b by $(a,b)_v = 1$. When $aR \cap bR \neq abR$ we write $(a,b)_v \neq 1$. The idea of a v-coprime pair comes from the notion of v-operation which is defined in the following lines.

Let F(R) denote the set of fractional ideals of R. Then the function from F(R) to F(R) defined by $A \mapsto (A^{-1})^{-1} = A_v$ is called the v-operation on R. The v-operation has the following properties. For $A,B \in F(R)$

(1) if A is principal $A_v = A$, $(AB)_v = AB_v$,

(2) $A \subseteq A_v$ and if $A \subseteq B$ then $A_v \subseteq B_v$,

(3) $(A_v)_v = A_v$.

Moreover if $A,B \in F(R)$ then $(AB)_v = (A_vB)_v = (A_vB_v)_v$. For details on the v-operation see sections 32 and 34 of [1] . For our purposes we refer the reader to the introduction and note in addition that A^{-1} is a v-ideal, $(A_v)^{-1} = A^{-1}$ and $A^{-1} = R$ if and only if $A_v = R$.

In this section we study those properties of prime blocks which lead to the definition of an AUFD. Using the above definitions we indicate some properties of v-coprimeness via the following lemma.

LEMMA 1.1. *Let* $a,b \in R$. *Then the following hold*

(1) *a, b are v-coprime if and only if* $(a,b)_v = R$,

(2) *if* $(p,q)_v = 1$ *and* $p \mid xq$ *then* $p \mid x$ ($u \mid v \equiv u$ *divides* v),

(3) *if* $(a,b)_v = 1$ *and* $x \mid a$, $y \mid b$ *then* $(x,y)_v = 1$,

(4) $(a,b)_v = 1$ *if and only if* $(a,b^m)_v = 1$ *for all* $m \in N$,

(5) $(a,b)_v = 1$ *if and only if* $(a^n,b)_v = 1$ *for all* $n \in N$.

PROOF. (1). If $(a,b)_v = R$ then $(a,b)^{-1} = \dfrac{(a) \cap (b)}{ab} = R$ which gives $(a) \cap (b) = (ab)$. Conversely if $(a) \cap (b) = (ab)$ then $R = \dfrac{(a) \cap (b)}{ab} = (a,b)^{-1}$ and so $(a,b)_v = R^{-1} = R$.

(2). Here $p \mid xq$ and obviously $q \mid xq$. But $pR \cap qR = pqR$ requires that $pq \mid xq$ which gives the result.

(3). Let $z \in xR \cap yR$. We show that $xy \mid z$. Since $x \mid a$ and $y \mid b$ we have $a = rx$ and $b = sy$. Now consider rsz. Since $x \mid z$ we have $a \mid rsz$ and since $y \mid z$ we have $b \mid rsz$. But as $(a,b)_v = 1$, $ab \mid rsz$ or, $rxsy \mid rsz$ which gives $xy \mid z$.

(4). Suppose $aR \cap bR = abR$. Consider $z \in aR \cap b^mR$. Then as $a \mid z$ we have $z = az_1$. Now $b^m \mid z$ and it is sufficient to show that $b^m \mid z_1$. But this can be done by using (2) for single powers of b. The converse follows from (3).

(5). This can be proved in the same way as (4).

DEFINITION 1.2. *A non-zero non-unit* $p \in R$ *is a prime block if for all* $x, y \in R$ *with* $(x,p)_v \neq 1$ *and* $(y,p)_v \neq 1$ *there exist* $n \in N$ *and* $d \in R$ *such that* $(x^n, y^n) \subseteq dR$ *with* $(x^n/d, p)_v = 1$ *or* $(y^n/d, p)_v = 1$.

PROPOSITION 1.3. *Let* $p \in R$ *be a prime block. Then for any* $m \in N$ *and for all* $x, y \mid p^m$ *there exists* $n(x,y) \in N$ *such that* $x^n \mid y^n$ *or* $y^n \mid x^n$.

PROOF. Let p be a prime block and let $x, y \mid p^m$ for some $m \in N$. If any one of x, y is a unit we have nothing to prove. If not then $pR \cap xR \neq pxR$ and $pR \cap yR \neq pyR$. By the definition $(x^n, y^n) \subseteq dR$ such that $(x^n/d, p)_{\underline{v}} = 1$ or $(y^n/d, p)_{\underline{v}} = 1$. If $(x^n/d, p) = 1$ then x^n/d must be a unit because x^n divides p^{nm}. Similarly we deal with the case when $(y^n/d, p)_{\underline{v}} = 1$ and from both we draw the conclusion that $x^n \mid y^n$ or $y^n \mid x^n$.

DEFINITION 1.4. *Two prime blocks* b_1, b_2 *will be called similar if* $b_1 R \cap b_2 R \neq b_1 b_2 R$.

PROPOSITION 1.5. *Two prime blocks* b_1, b_2 *are similar if and only if for some* $n \in N$, $b_1^n \mid b_2^n$ *or* $b_2^n \mid b_1^n$.

PROOF. Clearly $b_1 R \cap b_1 R \neq b_1^2 R$ and $b_1 R \cap b_2 R \neq b_1 b_2 R$. Since b_1 is a prime block there exist $n \in N$ and $d \in R$ such that b_1^n/d, $b_2^n/d \in R$ and one of b_i^n/d is v-coprime to b_1. If $(b_1^n/d, b_1)_{\underline{v}} = 1$ then b_1^n/d is a unit and so $b_1^n \mid b_2^n$. So we suppose that $(b_1^n/d, b_1)_{\underline{v}} \neq 1$. Then $(b_2^n/d, b_1)_{\underline{v}} = 1$ and we claim that $b_2^n \mid b_1^n$. For $h = b_2^n/d$ and d both divide b_2^n and as b_2 is a prime block there exists m such that $h^m \mid d^m$ or $d^m \mid h^m$(cf. Proposition 1.5). Now if $h^m \mid d^m$ then as $d^m \mid b_1^{nm}$ and $(h, b_1)_{\underline{v}} = 1$ we conclude that h is a unit. But then $b_2^n \mid b_1^n$. Finally $d^m \mid h^m$ is impossible because then $(h^m, b_1^n)_{\underline{v}} \neq 1$ which contradicts $(h, b_1)_{\underline{v}} = 1$ by (4) and (5) of Lemma 1.1. The converse of course is obvious.

COROLLARY 1.6. *The relation 'is similar to' in the set of prime blocks of an integral domain R is an equivalence relation.*

PROOF. Reflexivity and symmetry of the relation follow from Proposition 1.5. For transitivity we proceed as follows. Let $\sim$ denote similarity and let $p \sim q$ and $q \sim r$ where p, q and r are prime blocks. Then by Proposition 1.5 there exist m, n such that $p^m \mid q^m$ or $q^m \mid p^m$ and $q^n \mid r^n$ or

$r^n \mid q^n$. This gives $p^{mn} \mid q^{mn}$ or $q^{mn} \mid p^{mn}$ and $q^{mn} \mid r^{mn}$ or $r^{mn} \mid q^{mn}$. From this arise the following four cases.
(i) $p^{mn} \mid q^{mn}$ and $q^{mn} \mid r^{mn}$, (ii) $p^{mn} \mid q^{mn}$ and $r^{mn} \mid q^{mn}$, (iii) $q^{mn} \mid p^{mn}$ and $q^{mn} \mid r^{mn}$, (iv) $q^{mn} \mid p^{mn}$ and $r^{mn} \mid q^{mn}$.
Cases (i) and (iv) are similar and in both $p \sim r$ follows from Proposition 1.5. In case (ii) by Proposition 1.3 there exists k such that $p^{mnk} \mid r^{mnk}$ or $r^{mnk} \mid p^{mnk}$ and by Proposition 1.5 $p \sim r$. In case (iii), $(p^{mn}, r^{mn})_{\underline{v}} \neq 1$ which by Lemma 1.1 means that $(p,r)_{\underline{v}} \neq 1$. Hence in all the cases $p \sim r$ as required.

<u>PROPOSITION 1.7. Every non-unit factor of a power of a prime block is again a prime block.</u>

<u>PROOF.</u> Let p be a prime block and let for some $n \in N$, $x \mid p^n$ where x is a non-unit. Then $xR \cap aR \neq xaR$ and $xR \cap bR \neq xbR$ implies $p^nR \cap aR \neq p^naR$ and $p^nR \cap bR \neq p^nbR$ which means that $pR \cap aR \neq paR$ and $pR \cap bR \neq pbR$. Now by the definition there exist $n \in N$ and $d \in R$ such that $a^n/d, b^n/d \in R$, and $(a^n/d,p)_{\underline{v}} = 1$ or $(b^n/d,p)_{\underline{v}} = 1$. Now $(h,p)_{\underline{v}} = 1$ implies $(h,p^n)_{\underline{v}} = 1$, by (4) of Lemma 1.1, and this implies $(h,x)_{\underline{v}} = 1$ by (3) of Lemma 1.1. Putting $h = a^n/d$ or $h = b^n/d$ completes the proof.

<u>COROLLARY 1.8. The product of two similar prime blocks is again a prime block.</u>

<u>PROOF.</u> Let b_1, b_2 be two similar prime blocks. Then by Proposition 1.7 there exists n such that $b_1^n \mid b_2^n$ or $b_2^n \mid b_1^n$. In each case $(b_1b_2)^n \mid b_i^{2n}$ where i=1 or 2.

<u>THEOREM 1.9. Let $x \in R$ be expressible as a finite product of mutually v-coprime prime blocks then this expression is unique up to associates.</u>

PROOF. Let $x = x_1x_2\ldots x_n$ where x_i are mutually v-coprime prime blocks and suppose that $x = y_1y_2\ldots y_m$ also where y_j are mutually v-coprime prime blocks. Then $y_1y_2\ldots y_m = x_1x_2\ldots x_n$. Now $y_1 \mid x_1\ldots x_n$. Since x_i are mutually v-coprime and since y_1 is a prime block it cannot be similar to two different x_i (cf. Corollary 1.6.). So $y_1R \cap x_jR \neq y_1x_jR$ for at most one j. By (2) of Lemma 1.1 $y_1 \mid x_j$ for some j. Using similar reasoning we show that $x_j \mid y_1$. So y_1 and x_j are associates.The remainder of the proof is routine.

DEFINITION 1.10. An integral domain R is an almost unique factorization domain (AUFD) if for every non-zero non-unit $x \in R$ there is $n \in N$ such that x^n is expressible as a product of finitely many mutually v-coprime prime blocks.

REMARK 1.11. In view of Corollary 1.8 it is sufficient to say that R is an AUFD if every non-zero non-unit of R is expressible as a product of finitely many prime blocks.

2. ALMOST UFD's

In this section we study the ideal theory of AUFD's . Here we study the prime t-ideals and maximal t-ideals of AUFD's and their connection with prime blocks. We show that in an AUFD every maximal t-ideal is of the type $\mathcal{P}(b) = \{ x \in R \mid (x,b)_v \neq 1 \}$ where b is a prime block. We also show that an AGCD-domain R is an AUFD if and only if (1) every non-zero non-unit of R belongs to only a finite number of maximal t-ideals of R and (2) no two maximal t-ideals of R contain a common non-zero prime ideal. Thus we establish, without stating it, that in an AUFD a maximal t-ideal has the same status as a principal prime has in a UFD. Then in the end we use a simple $D + M$ construction to construct an example of an AUFD which is not integrally closed.

Recall from the introduction that R is an almost GCD-domain if for all $x, y \in R$ there exists $n \in N$ such that $x^nR \cap y^nR$ is principal. Clearly a GCD-domain is an AGCD-domain. We show via the following proposition that an AUFD is an AGCD-domain.

PROPOSITION 2.1. *Let $x, y \in R$ be such that each of x, y is expressible as a finite product of prime blocks. Then there exists $n \in N$ such that $x^nR \cap y^nR$ is principal.*

PROOF. By Corollary 1.8 we can write x, y as products of mutually v-coprime prime blocks. If $(x,y)_v = 1$ then $xR \cap yR = xyR$. So let $(x,y)_v \neq 1$ and let $x = p_1p_2\cdots p_rq_1\cdots q_s$; $y = p_1'p_2'\cdots p_r'a_1a_2\cdots a_t$ where p_i, p_i' are similar prime blocks and p_i, q_j and a_m are all mutually v-coprime. Now for some $n_i, p_i^{n_i} \mid p_i'^{n_i}$ or $p_i'^{n_i} \mid p_i^{n_i}$. Therefore if $n = n_1n_2\cdots n_r$ we have $x^nR \cap y^nR = P_1\cdots P_rq_1^n\cdots a_t^n$ where $P_i = p_i^n$ or $p_i'^n$; according as $p_i'^n \mid p_i^n$ or $p_i^n \mid p_i'^n$.

THEOREM 2.2. *An integral domain R is an AUFD if and only if every non-zero prime ideal of R contains a prime block.*

To prove this theorem we need the following lemma.

LEMMA 2.3. *If $x \in R$ divides a product of finitely many prime blocks then for some $n \in N$ x^n is expressible as a product of finitely many prime blocks.*

PROOF. Let y be a product of finitely many prime blocks. The y can be expressed as a product of mutually v-coprime prime blocks. Now let $y = p_1 p_2 \cdots p_r$ where p_i are mutually v-coprime and let $x \mid p_1 p_2 \cdots p_r$. Without loss of generality we may assume that $(x, p_i)_v \neq 1$ for $i = 1, 2, \ldots r$. Now as $xR \cap p_1 R \neq xp_1 R$ and $p_1 R \cap p_1 R \neq p_1^2 R$ there exist $n_1 \in N$ and $g_1 \in R$ such that $(x^{n_1}, p_1^{n_1}) \subseteq g_1 R$ such that x^{n_1}/g_1 or $p_1^{n_1}/g_1$ is v-coprime to p_1. Because $x \mid y$ we have x^{n_1}/g_1 v-coprime to p_1. Thus $x^{n_1} = h_1 g_1$ where g_1 is a prime block and $(h_1, g_1)_v = 1$ because $(h_1, p_1)_v = 1$.

Now as $h_1 \mid x^{n_1}$ which divides y^{n_1} and $h_1 R \cap p_2 R \neq h_1 p_2 R$ we use the above procedure to get $h_1^{n_2} = h_2 g_2$ where g_2 is a prime block similar to p_2. This gives $x^{n_1 n_2} = h_2 g_2 (g_1)^{n_2}$. Proceeding in this way we ultimately get
$x^{n_1 n_2 \ldots n_r} = q_1 q_2 \cdots q_r$ where q_i are prime blocks.

COROLLARY 2.4. *A set $S \subseteq R$ generated multiplicatively by elements whose powers are products of prime blocks is saturated.*

PROOF OF THEOREM 2.2. Suppose that every prime ideal of R contains a prime block. If R is not a field then there is at least one prime block in R. Let S be multiplicatively generated by elements x such that for some $n \in N$, x^n is a product of prime blocks. Now all the prime blocks belong to S and by Corollary 2.4 S is saturated. We claim that $S = R - \{0\}$. For if not then $R - S$ contains a prime ideal P which must contain a prime block, a contradiction.

Conversely if R is an AUFD and $x \neq 0$ belongs to a prime P then for some $n \in N, x^n = p_1 p_2 \cdots p_r$ where p_i are prime blocks and of these at least one must belong to P.

We now study prime ideals of multiplicative importance in AUFD's. These are prime t-ideals, maximal t-ideals and associated primes of principal ideals; all of which reduce to principal primes in a UFD. For the sake of completeness we define these notions.

An ideal $A \in F(R)$ is said to be a t-ideal if $A = \bigcup B_v$ where B ranges over finitely generated R-submodules of A. A prime ideal which is also a t-ideal is called a prime t-ideal and an integral ideal maximal w.r.t. being a t-ideal of R is called a maximal t-ideal. According to [3] a maximal t-ideal is a prime ideal. Finally a prime ideal P minimal over any ideal of the type $(a):(b)$ $(\neq R)$ is called an associated prime of a principal ideal or simply an associated prime of R. Using Lemmas 4 and 6 of [13] it can be established that an associated prime is a t-ideal.

THEOREM 2.5. *Let $b \in R$ be a prime block and let $\mathcal{P}(b) = \{ x \in R \mid xR \cap bR \neq xbR \}$. Then $\mathcal{P}(b)$ is a maximal t-ideal and b belongs to no other maximal t-ideal.*

PROOF. If $x, y \in \mathcal{P}(b)$ then by definition there exist $n \in N$ and $d \in R$ such that $x^n/d, y^n/d \in R$, and $(x^n/d, b)_v = 1$ or $(y^n/d, b)_v = 1$. Since both x, y are non v-coprime to b we conclude that $(d, b)_v \neq 1$ that is $d \in \mathcal{P}(b)$. From this it follows that $(x + y)^{2n} \in \mathcal{P}(b)$. By (5) of Lemma 1.1 $((x+y)^{2n}, b)_v \neq 1$ implies that $((x+y), b)_v \neq 1$. Thus $x, y \in \mathcal{P}(b)$ implies $x+y \in \mathcal{P}(b)$. Further, obviously, for all $r \in R$ and $x \in \mathcal{P}(b)$, $rx \in \mathcal{P}(b)$.

Now suppose that $(b,x)_v = 1 = (b,y)_v$. Then
$R = ((b,x)_v(b,y)_v)_v = ((b,x)(b,y))_v = (b^2, by, bx, xy)_v$
$= (b(b,y)_v, bx, xy)_v = (b, bx, xy)_v = (b, xy)_v$. That is $(b,x)_v = 1$ and $(b,y)_v = 1$ implies that $(b, xy)_v = 1$ and so

$xy \in \mathcal{G}(b)$ implies that $x \in \mathcal{G}(b)$ or $y \in \mathcal{G}(b)$. So $\mathcal{G}(b)$ is a union of prime ideals. Therefore it is itself prime as it is an ideal.

Further let $x_1, \ldots, x_r \in \mathcal{G}(b)$ then as $bR \cap x_iR \neq bx_iR$ there exist, by the definition of a prime block, $n_i \in N$ and $d_i \in R$ such that $(x_i^{n_i}, b^{n_i}) \subseteq d_iR \subseteq \mathcal{G}(b)$. Here $d_i | b^{n_i}$ and if we select $m = \max\{n_i\}$ then for all i, $d_i | b^m$. Now by Proposition 1.5 there exist u_{ij} such that $d_i^{u_{ij}} | d_j^{u_{ij}}$ or $d_j^{u_{ij}} | d_i^{u_{ij}}$. If $p = \prod u_{ij}$ then $d_i^p | d_j^p$ or $d_j^p | d_i^p$ for all $i, j = 1, 2, \ldots, r$. From this it follows that there exists a d_k such that $d_k^p | d_i^p$ for all $i = 1, 2, \ldots r$. Consequently $d_k^p | x_i^{n_i p}$ (because $d_i | x^{n_i}$) and as $m = \max\{n_i\}$ $d_k^p | x_i^{mp}$ for all i. From this it is easy to work out that $(x_1, x_2 \ldots x_r)^{rpm} \subseteq (d_k) \subseteq \mathcal{G}(b)$. But then $((x_1, \ldots, x_r)^{rpm})_v \subseteq (d_k) \subseteq \mathcal{G}(b)$ and from this it follows that $((x_1, \ldots, x_r)_v)^{rmp} \subseteq ((x_1, \ldots, x_r)^{rmp})_v \subseteq \mathcal{G}(b)$. But as $\mathcal{G}(b)$ is a prime ideal we conclude that $(x_1, \ldots, x_r)_v \subseteq \mathcal{G}(b)$. From this it follows that $\mathcal{G}(b)$ is a t-ideal. That $\mathcal{G}(b)$ is a maximal t-ideal follows, now, from its definition and the same comment about b belonging to no other maximal t-ideal.

DEFINITION 2.6. *For b a prime block we call* $\mathcal{G}(b) = \{x \in R \mid xR \cap bR \neq xbR\}$ *the prime ideal associated with* b.

THEOREM 2.7. *Let P be a maximal t-ideal of an AUFD. Then there exists a prime block b such that P is associated with* b.

PROOF. Let P be a maximal t-ideal. Then by Theorem 2.2 P contains a prime block b say. Because P is a t-ideal, for all $x, y \in P$ $(x, y)_v \neq R$ and so for every element $x \in P$,

$(x,b)_v \neq 1$. Whence it follows that $P = \mathscr{C}(b)$.

COROLLARY 2.8. *In an AUFD, R every prime t-ideal is contained in a unique maximal t-ideal.*

PROOF. Let P be a prime t-ideal. Then P is contained in a maximal t-ideal P_1. By Theorem 2.7, $P_1 = \mathscr{C}(b_1)$ where b_1 is a prime block. Suppose that P is also contained in a maximal t-ideal $P_2 = \mathscr{C}(b_2)$. Then if $P_1 \neq P_2$, b_1 and b_2 must be v-coprime. Now P being a prime ideal contains, by Theorem 2.2, a prime block c. As $P \subseteq \mathscr{C}(b_1)$, c is similar to b_1 and as $P \subseteq \mathscr{C}(b_2)$, c is similar to b_2. Now by Corollary 1.6 b_1 is similar to b_2 and so $P_1 = P_2$.

From the above result it follows that if R is an AUFD then (1) every maximal t-ideal is of the type $\mathscr{C}(b)$ where b is a prime block, (2) every non-zero non-unit of R belongs to only a finite number of maximal t-ideals, (3) no two maximal t-ideals contain a common non-zero prime ideal and (4) R is an almost GCD-domain. As usual in Mathematics we ask, " If R satisfies (1) - (4) is it an AUFD ?" The answer is indeed yes but to support it we need the following lemmas.

LEMMA 2.9. *Let R be an AGCD-domain and let P be a prime t-ideal of R. Then for all $x, y \in P$ there exist $d \in P$ and $n \in N$ such that $(x^n, y^n)_v = (d)$.*

PROOF. Let P be a prime t-ideal and let $x, y \in P$. Then as R is AGCD there exists n such that $x^nR \cap y^nR$ is principal. This implies that $(x^n, y^n)_v = (d)$ for some $d \in R$. From this it follows that $(x^n/d, y^n/d)_v = R$ and so at least one of $x^n/d, y^n/d$ is not in P; because P is a t-ideal. This implies that $d \in P$ as required.

LEMMA 2.10. *Let R be an AGCD-domain and let P be a maximal t-ideal of R. If $x \in R$ such that $x \in P$ and x belongs to no other maximal t-ideal of R then x is a prime block.*

PROOF. Let x and P be as given then for all n and for all non-units $a | x^n$ the element a belongs to P and to no other maximal t-ideal of R. Now because P is a t-ideal no element of P is v-coprime to x. Further if $(h,x)_v \neq 1$ then as R is AGCD, there exist $d \in P$ and $n \in N$ such that $(h^n, x^n)_v = (d)$ (cf. Lemma 2.9). Thus $h \in P$ if and only if $(h,x)_v \neq 1$. Now let z,y be such that $zR \cap xR \neq zxR$ and $yR \cap xR \neq yxR$. Then $z,y \in P$ and there are $n \in N$ and $d \in R$ with $(z^n, y^n)_v = dR$ which gives $z^n/d \notin P$ or $y^n/d \notin P$; meaning that $(z^n/d, x)_v = 1$ or $(y^n/d, x)_v = 1$. This indeed gives that x is a prime block.

LEMMA 2.11. *Let R be an AGCD-domain such that every non-zero non-unit of R belongs to at most a finite number of maximal t-ideals. Then every maximal t-ideal P of R is of the form $\mathcal{P}(b)$ where b is a prime block.*

PROOF. Let P be a maximal t-ideal of R and let $0 \neq x \in P$. Further let $\{P_1, \dots, P_r\}$ be the set of all other maximal t-ideals containing x. Consider $x_1 \in P - P_1$ then there exists n such that for some $d \in P$ $(x^n, x_1^n)_v = dR$. Clearly $d \in P - P_1$. Now let $\{Q_1, \dots, Q_s\}$ be the set of maximal t-ideals, other than P, which contain d. Then $\{Q_1, \dots, Q_s\} \subseteq \{P_2, \dots, P_r\}$. Following this procedure we get a $d_1 \in P - Q_1$ which divides a power of d, and hence of x. Continuing this process of exclusion of primes we reach a non-zero $b \in P$ such that b divides a power of x and $b \notin P_1, \dots, P_r$. Clearly b belongs to P and to no other maximal t-ideal. Now using Lemma 2.10 we conclude that b is a prime block and that

$P = \mathscr{P}(b)$.

<u>THEOREM 2.12. Let R be an AGCD-domain. Then R is an AUFD if and only if the following hold.</u>

(1) <u>Every non-zero non-unit of R belongs to at most a finite number of maximal t-ideals.</u>

(2) <u>If two maximal t-ideals P_1 and P_2 have a non-zero prime ideal in common then they are equal.</u>

<u>PROOF.</u> The necessity is clear from earlier remarks. For sufficiency we show that every non-zero prime ideal of R contains a prime block. We note that by (1) and Lemma 2.11, every maximal t-ideal contains a prime block. Now let $\underset{\sim}{p}$ be a prime t-ideal. Then by (2) $\underset{\sim}{p}$ is contained in a unique maximal t-ideal P. Now let $x \in \underset{\sim}{p}$. Then by (1) there are at the most a finite number of maximal t-ideals $P_1, \dots, P_r$ such that $x \in \underset{\sim}{p}, P_1, \dots, P_r$. Since $\underset{\sim}{p}$ is a t-ideal and $\underset{\sim}{p} \not\subseteq P_i$ we can use a modification of the procedure used in [12] and in Lemma 2.11 to show that there is a factor d of a power of x such that $d \in \underset{\sim}{p} - P_i$ $(i = 1, \dots, r)$. But then by Lemma 2.10, d is a prime block. Now because a minimal prime of a principal ideal is an associated prime and hence a t-ideal we conclude that every prime ideal of R contains a prime block.

Theorem 2.12 gives a criterion which helps in recognizing an AUFD. Clearly a UFD is an AUFD and considering that fastfaktoriell rings of Storch [11] are AGCD-Krull domains we conclude that non-GCD fastfaktoriell rings are a good example of AUFD's which are not UFD's. Both of these examples are integrally closed. In the following we construct an example of a not integrally closed AUFD.

EXAMPLE 2.13. <u>An</u> AUFD <u>which is not integrally closed.</u>

Let K be a field with characteristic $p \neq 0$ and let L be a purely inseparable extension of K such that $L^p \subseteq K$ (cf.[5] Theorem 100 for a simple example with $p = 2$). Construct $R = K + XL[X] = \{ a_0 + \sum_1^n a_i X^i \mid a_0 \in K \text{ and } a_i \in L \}$. We note that $K[X] \subseteq R$ and that if $f(X) \in R$ then $f^p = a_0^p + \sum_1^n a_i^p X^{ip} \in K[X]$. Since $K[X]$ is a PID, for every two $h(X), k(X) \in R$ there exists $d(X) \in K[X]$ such that $(h(X))^p K[X] + (k(X))^p K[X] = d(X)K[X]$. From this it follows that $(h(X))^p R + (k(X))^p R = dR$ and this implies that $(h(X))^p R \cap (k(X))^p R$ is principal. Thus R is an AGCD-domain. Further let $f(X) \in R$. Then $f^p \in K[X]$ and hence is a product of prime powers in $K[X]$. Thus $f^p = g_1, \ldots, g_r$ where each of g_i is a prime power in $K[X]$. We show that each of g_i is a prime block in R. For this let $h, k \in R$ be such that for some i, $hR \cap g_i R \neq hg_i R$ and $kR \cap g_i R \neq kg_i R$. Then $h^p, k^p \in K[X]$ and clearly h^p and k^p are non coprime with the prime power g_i (in $K[X]$). So we have $dK[X] = h^p K[X] + k^p K[X]$. Now as this equation extends to R we have $dR = h^p R + k^p R$ and this gives $d \in R$ such that h^p/d and k^p/d belong to R and one of them is v-coprime to g_i because one of them is coprime to g_i in $K[X]$. Thus we conclude that each of g_i is a prime block. That R is not integrally closed is obvious.

3. ALMOST GCD-DOMAINS

The existence of a not integrally closed AGCD-domain raises questions like (1) what is the relation between an AGCD-domain and its integral closure ? (2) what are the integrally closed AGCD-domains like ? In this section we provide the following answers to these questions. We show that if R is an AGCD-domain then so is its integral closure R' and that an integrally closed AGCD-domain is a PVMD.

THEOREM 3.1. *Let R be an AGCD-domain with quotient field K. Then* $x \in K - \{0\}$ *is integral over R if and only if* $x^m \in R$ *for some* $m \in N$.

PROOF. Let $x = u/v \in K$. If $(u/v)^m \in R$ then u/v is integral over R. Conversely suppose that u/v is integral over R and let n be such that $u^n R \cap v^n R = dR$ where $d \in R$. Then there is $t \in R$ such that $(u^n, v^n)_v = tR$ and so $(u/v)^n = u^n/v^n = h/k$ where h and k are v-coprime. Now as u/v is integral over R, so is $(u/v)^n = h/k$. But as h and k are v-coprime h/k is integral over R if and only if k is a unit. Whence it follows that if u/v is integral over R then for some $n \in N$ $(u/v)^n \in R$.

COROLLARY 3.2. *Let R be integrally closed and let* $x, y \in R$ *such that for some* $n \in N$, $x^n | y^n$. *Then* $x | y$ *in* R.

PROOF. If $x^n | y^n$ then $(y/x)^n \in R$ which means that x/y is integral over R. But R is integrally closed.

COROLLARY 3.3. *Let R be integrally closed and let* $d, a_1, \ldots, a_r \in R$ *be such that* $d | a_i^n$ *for all* $i = 1, \ldots, r$. *Then* $d \mid a_1^{n_1} a_2^{n_2} \ldots a_r^{n_r}$ *where* $n_1 + n_2 + \ldots + n_r = n$.

PROOF. Note that $(a_1^{n_1} a_2^{n_2} \ldots a_r^{n_r})^n = (a_1^n)^{n_1} \ldots (a_r^n)^{n_r}$. Since $d | a_i^n$ we have $d^{n_i} | (a_i^n)^{n_i}$ or $(d)^{\sum n_i} | \prod (a_i^n)^{n_i}$ or $d^n | (\prod (a_i^{n_i})^n$ or, by Corollary 3.2, $d \mid \prod a_i^{n_i}$.

We proceed next to indicate the relation between an AGCD-domain R and its integral closure R'.

THEOREM 3.4. *If R is an AGCD-domain and R' its integral closure then R' is also an AGCD-domain.*

To facilitate the proof of this theorem we include the following two lemmas.

LEMMA 3.5. *Let* R *be an* AGCD-*domain and let* a,b *be* v-*coprime in* R. *Then* a,b *are* v-*coprime in the integral closure* R' *of* R.

PROOF. Since $(a,b)_v = 1$ in R we have $aR \cap bR = abR$. Now suppose that $x \in aR' \cap bR'$ then $x = ar = bs$ where $r,s \in R'$. Because R' is the integral closure of the AGCD-domain R there exists $n \in N$ such that $x^n = a^n r^n = b^n s^n \in R$. But then for some $m \in N$, $x^m \in a^m R \cap b^m R = a^m b^m R$. Thus $x^m/(a^m b^m) \in R \subseteq R'$. But then $ab \mid x$ in R'. Hence $aR' \cap bR' \subseteq abR'$ and from this it follows that $aR' \cap bR' = abR'$.

LEMMA 3.6. *If in an* AGCD-*domain* R, $x^n R \cap y^n R = dR$ *then* $x^{mn}R \cap y^{mn}R = d^m R$.

PROOF. From $x^n R \cap y^n R = dR$ we conclude that there is h in R such that $(x^n, y^n)_v = hR$. Clearly h is the GCD of x^n and y^n. But then $x^n y^n = dh$. Raising this equation to the power m and noting that $(x^n/h, y^n/h)_v = 1$, and hence $(x^{mn}/h^m, y^{mn}/h^m)_v = 1$, we get the result.

PROOF OF THEOREM 3.4. Let $x,y \in R'$. Then there are p,q in N such that $x^p, y^q \in R$. So there exists $m = pq$ such that $x^m, y^m \in R$. Now as R is AGCD there are $n \in N$ and $d \in R$ such that $x^{mn}R \cap y^{mn}R = dR$. Now let $a \in x^{mn}R' \cap y^{mn}R'$. Then $a = x^{mn}r = y^{mn}s$; $r,s \in R'$. If $r^w, s^z \in R$ then there is $k = wz$ such that $a^k = x^{mnk}r^k = y^{mnk}s^k \in R$. But then $a^k \in x^{mnk}R \cap y^{mnk}R$. According to Lemma 3.5 this gives $a^k/d^k \in R \subseteq R'$. That is a/d is integral over R and $d \mid a$ in R'. But then $x^{mn}R' \cap y^{mn}R' = dR'$.

In the remainder of this section we establish that an integrally closed AGCD-domain is a Prüfer v-multiplication domain of a special type. For this we recall that an ideal $A \in F(R)$ is t-invertible if there exists $B \in F(R)$ such that $(AB)_t = R$. A t-invertible v-ideal is a v-ideal of finite type (cf. Griffin [3]). If both A and B are of finite type it is easy to establish that $(AB)_t = (AB)_v$. According to [3] R is a PVMD if and only if every two-generated ideal of R is t-invertible. We use this result in the sequel to establish that an integrally closed AGCD-domain is a PVMD.

THEOREM 3.7. *Let R be an integrally closed AGCD-domain. Then for* $a, b \in R$ *there exists* $n \in N$ *such that* $((a,b)^n)_v$ *is principal.*

PROOF. Let $a, b \in R$. Then there exists $n \in N$ such that $a^nR \cap b^nR$ is principal. Or equivalently $(a^n, b^n)_v = dR$ for some $d \in R$. Now as $d \mid a^n, b^n$, by Corollary 3.3, $d \mid a^{n-r}b^r$ for all integral r from 0 to n and so $(a,b)^n \subseteq dR$. Now $(a^n, b^n) \subseteq (a,b)^n \subseteq dR$. But then $dR = (a^n, b^n)_v \subseteq ((a,b)^n)_v \subseteq dR$.

COROLLARY 3.8. *Let R be an AGCD-domain. Then the following are equivalent.*

(1) *R is integrally closed,*

(2) *for all* $x, y \in R$, $x^n \mid y^n$ *implies* $x \mid y$,

(3) *for all* $a, b \in R$, $((a,b)^n)_v$ *is principal for some* $n \in N$,

(4) *R is a PVMD.*

PROOF. We note that (1) $\Rightarrow$ (2) follows from Corollary 3.2 and (2) is used to prove Crollary 3.3 which in turn is used to prove Theorem 3.7 that gives (3). Moreover (4)$\Rightarrow$(1) because it is well known that a PVMD is integrally closed. So to prove the equivalence it remains to show that (3) $\Rightarrow$ (4). For this let $a, b \in R$ and suppose that for some $n \in N$ and for some $d \in R$, $((a,b)^n)_v = dR$. Then

$((a,b)(a,b)^{n-1}/d)_v = R$. If $n = 1$ then $(a,b)_v = dR$ and $R = ((a,b)R/d)_v = ((a,b)R/d)_t$. Further if $n > 1$ then $(a,b)^{n-1}$ is finitely generated and so $R = ((a,b)\frac{(a,b)^{n-1}}{d})_v = ((a,b)(a,b)^{n-1}/d)_t$. Thus every two-generated ideal of h is t-invertible and this establishes the implication.

It is easy to note that $(3) \Rightarrow (4)$ does not need the assumption that R is an integrally closed AGCD-domain. In fact, as we shall see below, (3) implies that R is a PVMD and an AGCD-domain.

<u>THEOREM 3.9. In a commutative integral domain</u> R <u>the following are equivalent.</u>

(1) R <u>is an integrally closed</u> AGCD-<u>domain</u>,

(2) R <u>is integrally closed and for all</u> $a, b \in R$ <u>there is</u> $n(a,b) \in N$ <u>such that</u> $(a^n, b^n)_v$ <u>is principal</u>,

(3) <u>for all</u> $a, b \in R$ <u>there is</u> $n \in N$ <u>such that</u> $((a,b)^n)_v$ <u>is principal</u>,

(4) R <u>is integrally closed and for all</u> $x_1, x_2, \ldots, x_r \in R$ <u>there is</u> $n(x_1, \ldots, x_r) \in N$ <u>such that</u> $(x_1^n, \ldots, x_r^n)_v$ <u>is principal</u>,

(5) <u>for all</u> $x_1, \ldots, x_r \in R$ <u>there is</u> $n \in N$ <u>such that</u> $((x_1, \ldots, x_r)^n)_v$ <u>is principal</u>,

(6) R <u>is integrally closed and for all</u> $x_1, \ldots, x_r \in R$ <u>there is</u> $n \in N$ <u>such that</u> $x_1^n R \cap x_2^n R \cap \ldots \cap x_r^n R$ <u>is principal</u>.

<u>PROOF.</u> We first establish $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ and $(4) \Leftrightarrow (5)$ then we show that $(1), (2), (3) \Rightarrow ((4) \Leftrightarrow (5)) \Rightarrow (6) \Rightarrow (1)$.

Now $(1) \Leftrightarrow (2)$ is obvious and $(2) \Rightarrow (3)$ is Theorem 3.7. For $(3) \Rightarrow (2)$ we note that (3) makes R a PVMD (cf. Corollary 3.8). Now according to [1] (Theorem 24.3) and according to the rule for translating results from Prüfer domains to

PVMD's given in [15] (Theorem 3.1), in a PVMD $((x_1,\ldots,x_r)^n)_v = (x_1^n,\ldots,x_r^n)_v$ and putting $r = 2$ we get (2).
(4)$\Rightarrow$(5). Fixing $r = 2$, R is an AGCD-domain. By Corollary 3.8 R is a PVMD and as in (3)$\Rightarrow$(2) $(x_1^n,\ldots,x_r^n)_v = dR$ implies $((x_1,\ldots,x_r)^n)_v = dR$.
(5)$\Rightarrow$(4). Fixing $r = 2$, R is a PVMD by the proof of (3)$\Rightarrow$(4) of Corollary 3.8 and as in the proof of (3)$\Rightarrow$(2) $((x_1,\ldots,x_r)^n)_v = dR$ implies $(x_1^n,\ldots,x_r^n)_v = dR$.

Further, (6)$\Rightarrow$(1) is obvious and to complete the cycle we need to prove that (3)$\Rightarrow$(4) and (5)$\Rightarrow$(6).
(3)$\Rightarrow$(4). Let $x_1,\ldots,x_r \in R$. Then there exists $n_1(x_1,x_2)$ in N such that $((x_1,x_2)^{n_1})_v = (x_1^{n_1},x_2^{n_2})_v = d_1R$. Further, there exists $n_2(x_3,d_1)$ such that $((d_1,x_3^{n_1})^{n_2})_v=(d_1^{n_2},x_3^{n_1n_2})_v$ $= d_2R$. Now $(d_1^{n_2})= (d_1^{n_2})_v = ((((x_1,x_2)^{n_1})_v)^{n_2})_v=((x_1,x_2)^{n_1n_2})_v$ $= (x_1^{n_1n_2},x_2^{n_1n_2})_v$. So $(d_2) = (x_1^{n_1n_2},x_2^{n_1n_2},x_3^{n_1n_2})_v$. Further since by(3), R is a PVMD we have $((x_1,x_2,x_3)^{n_1n_2})_v = (d_2)$. This establishes an induction procedure, on the basis of which we conclude that for $x_1,x_2,\ldots,x_r \in R$ there exist $n \in N$ and $d \in R$ such that $((x_1,\ldots,x_r)^n)_v= (x_1^n,\ldots,x_r^n)_v = (d)$.
(5)$\Rightarrow$(6). Let $x_1,\ldots,x_r \in R$ and let $Y_i = \prod_{i\neq j} x_j$. Then there exists n such that $(Y_1^n,\ldots,Y_r^n)_v = (d)$. Or $(1/x_1^n\ldots x_r^n)(Y_1^n,\ldots,Y_r^n)_v = (d/x_1^n\ldots x_r^n)$ which gives that $(1/x_1^n,\ldots,1/x_r^n)_v$ is principal. But then $(1/x_1^n,\ldots,1/x_r^n)^{-1}$ is principal which gives the result.

REMARK 3.10. Let R be an AGCD-domain and let R' be the integral closure of R then for every $x \in R'$ there is $n(x) \in N$ such that $x^n \in R$. If $R \neq R'$ then according to Nagata [8] R must have characteristic $p \neq 0$ or R'(and hence R) should

False see next page.

be algebraic over a finite field. So if char R= 0 and R an AGCD-domain which is not algebraic over a finite field then R = R'. False: See page 291 of "Almost Bezout domains" (J. Algebra 142(2)(1991) 285-309) by Anderson and Zafrullah

4. INTEGRALLY CLOSED AUFD's

In this section we show that an integrally closed AUFD is a ring of finite character of a special type (rings of finite character are defined below). After this we discuss the connection of integrally closed AUFD's with fast-faktoriell rings of Storch. That is we state the conditions under which an atomic AUFD is a Krull domain.

Rings of finite character are defined as follows. Let $F = \{V_i\}_{i \in I}$ be a family of valuation overrings (rings between R and its field of fractions K) such that $R = \bigcap_{i \in I} V_i$. Then R is a ring of finite character if every non-zero non-unit of R is a non-unit in at most a finite number of incomparable members of F. If each V_i is essential, i.e. a quotient ring of R, then the ring of finite character is called a ring of Krull type. Thus if R has a family $\{P_i\}$ of prime ideals such that (1) $R = \bigcap R_{P_i}$, (2) For each i, R_{P_i} is a valuation ring and (3) every non-zero non-unit of R belongs to a finite number of distinct P_i then R is a ring of Krull type. According to Griffin [2] a ring of Krull type is a PVMD and the family of primes may be selected to be maximal t-ideals. It may be recalled (cf. [3]) that R is a PVMD if and only if for every maximal t-ideal P, R_P is a valuation domain. Finally a ring of Krull type defined by a family $\{P_i\}_{i \in I}$ of maximal t-ideals is called an independent ring of Krull type (IKT) if $P_i \neq P_j$ implies that

$P_i \cap P_j$ does not contain a non-zero prime ideal. With this introduction we prove the following theorem.

THEOREM 4.1. *An integrally closed AUFD is an IKT.*

PROOF. We note that an integrally closed AUFD is a PVMD. Now Theorem 2.12 and related definitions make the proof a simple matter.

The simplest form of an independent ring of Krull type is of course a valuation ring. The following result may be of interest.

PROPOSITION 4.2. *An integrally closed integral domain R is a valuation ring if and only if for all x in the quotient field K there is n such that* $x^n \in R$ *or* $1/x^n \in R$.

We recall that $r \in R$ is a *rigid* element if for all $x,y | r$; $x|y$ or $y|x$. In a valuation domain every non-zero non unit is rigid. In [14] it was shown that a GCD-domain is an independent ring of Krull type if and only if every non zero non-unit of it is a product of finitely many rigid elements. This gives rise to the question," Do the rigid elements have significance for the AUFD's?" In the following we show that they do. Indeed in an integrally closed AUFD some power of every non-zero non-unit is a product of finitely many rigid elements. We establish this result via the following Propositions.

LEMMA 4.3. *In an integrally closed integral domain R a prime block is a rigid element.*

PROOF. Let b be a prime block, let $m \in N$ and let $x,y | b^m$. Then by Proposition 1.5 there is $n \in N$ such that $x^n | y^n$ or $y^n | x^n$. Now as R is integrally closed, by Corollary 3.2, $x|y$ or $y|x$. Putting $m = 1$ the result follows.

PROPOSITION 4.4. *Let R be an integrally closed AUFD. Then for every non-zero non-unit x of R there is $n \in N$ such that x^n is semirigid i.e. a product of finitely many rigid elements.*

PROOF. The proof is immediate from Lemma 4.3.

A Krull domain is obviously an independent ring of Krull type and as fastfaktoriell rings of Storch (loc cit) are Krull AGCD-domains we conclude that the theory of AUFD's is a generalization of that of fastfaktoriell rings. Now we proceed to establish the necessary and sufficient conditions under which an atomic AUFD is a Krull domain. Here, an irreducible element is called an atom and an integral domain whose non-zero non-units are expressible as products of atoms is called atomic. To give a better idea of the notions involved we first note a negative result.

LEMMA 4.5. *Let R be an integrally closed AUFD and let P be a prime t-ideal of rank more than one in R. Than R is not atomic.*

PROOF. Let $P_1 \subsetneq P_2$ be two non-zero prime ideals contained in P. Then as R is an AUFD there exists a prime block $r_1 \in P_1$. Now by selecting a suitable $x \in P_2 - P_1$ and $n \in N$, we can have $(r_1^n, x^n)_v = r_2 \in P_2 - P_1$. Since r_1 and r_2 are clearly similar there is $m \in N$ such that $r_1^m | r_2^m$ or $r_2^m | r_1^m$ which gives $r_2^m | r_1^m$; because $r_2 \in P_2 - P_1$. Now by Corollary 3.2, $r_2 | r_1$. But then r_2^2 is a prime block which is similar to r_1 and by the above reasoning $r_2^2 | r_1$. This contradicts the assumption that r_1 is an atom. Hence if R contains a prime t-ideal of rank more than one R cannot be atomic.

From the above proof it follows that if R is an integrally closed atomic AUFD then every maximal t-ideal of R is of rank one and it contains a prime block p which is an atom. Because every prime block in an integrally closed AUFD is rigid , every prime block, similar to p, in the atomic case, will be a power of p. From these considerations follows the proof of the lemma below.

LEMMA 4.6. *Let R be an integrally closed atomic AUFD and let p be the atomic prime block of a given maximal t-ideal P. Then for all* $x \in P - \{0\}$ *there exist* $n, r \in N$ *such that* $x^n = ap^r$ *where* $(a,p)_v = 1$.

LEMMA 4.7. *Let R, P and p be as in Lemma 4.6 and let* v_P *be the valuation of the quotient field K of R centered at P. Then for* $x \in P - \{0\}$ $p | x$ *if and only if* $v_P(x) \geqslant v_P(p)$.

PROOF. By Lemma 4.6 $x^n = ap^r$ where $(a,p)_v = 1$. Now $v_P(x) \geqslant v_P(p)$ if and only if $nv_P(x) \geqslant nv_P(p)$ i.e. if and only if $rv_P(p) \geqslant nv_P(p)$ i.e. if and only if $r \geqslant n$. But if $r \geqslant n$ then $p^n | x^n$. Now R being integrally closed this gives $p | x$.

THEOREM 4.8. *An integrally closed atomic AUFD R is a Krull domain if and only if for every maximal t-ideal P with atomic rigid element p and associated valuation* v_P *there exists* $x \in P$ *with* $p \nmid x$ *such that for every y with* $p \nmid y$, $v_P(y) \leqslant v_P(x)$.

PROOF. Let R be a Krull domain then for $p \in P$, as defined above, $v_P(p) = r$ say and there exists $z \in R$ such that $v_P(z) = 1$ and so $x = z^{r-1}$ is the required element. Conversely suppose that R is an integrally closed AUFD which is atomic and suppose that the given condition holds. Consider the ideal $A = (p):(x) = \{ r \in R \mid rx \in (p) \}$, where x has the property indicated in the statement. We claim that A is a prime ideal. Because if $uv \in A$ then $p \mid uvx$ and so $v_P(p) \leqslant v_P(uvx)$. Now suppose that $p \nmid vx$ then by the condition $v_P(vx) \leqslant v_P(x)$. From this it follows that $v_P(v) = 0$; which means that $(v,p)_v = 1$. Hence by (2) of Lemma 1.1 $p \mid ux$. Thus A is a prime ideal. Being a quotient of two ideals A is a v- and hence a t-ideal and it is easy to see that $A = P$. Further as R_P is a valuation ring and as $PR_P = ((p):(x))R_P = pR_P:xR_P$ is principal we conclude that for every maximal t-ideal P, R_P is a discrete rank one valuation ring and this is sufficient to show that R is a Krull domain.

5. USUAL EXTENSIONS OF AGCD-DOMAINS

We shall restrict our attention to quotient rings and to polynomial ring formation. For quotient rings we note that $(f^n)\bigcap(g^n)$ extends nicely to the ring of fractions and so the quotient rings of an AGCD-domain are AGCD-domains. For polynomial rings we go slightly comprehensive and include a concept which will be of use in future studies of AGCD-domains and of AUFD's. This concept is an analogue of the divisor class group, of a Krull domain, for PVMD's. This concept is the t-class group which has been treated properly in the introduction.

In this section we show, using simple techniques, that if R is integrally closed then R is a PVMD with torsion $C_T(R)$ if and only if $R[X]$ has this property. We recall from the introduction that if R is a Krull domain $C_T(R)$ is just the divisor class group of R. Further, because an invertible ideal is t-invertible if R is Prufer then $C_T(R)$ is just the class group of R. A study of some aspects of the t-class group has been carried out in collaboration with Alain Bouvier and will appear in due course of time. For the purposes of the present article we note that for a PVMD R $C_T(R)$ is torsion if and only if R is an AGCD-domain (cf. 3.7 - 3.9). So we can refer to PVMD's which are AGCD; as PVMD's with torsion t-class groups.

We proceed to show that if R is a PVMD with torsion t-class group and X is an indeterminate over R then R[X] also has torsion t-class group. For this we note that according to [10] Lemma 1, if R is integrally closed and $f(X), g(X) \in K[X]$ then $(A_{fg})_v = (A_f A_g)_v$ where A_f denotes the content of $f(X)$.

<u>THEOREM 5.1.</u> <u>An integrally closed integral domain</u> R <u>is an</u> AGCD-<u>domain if and only if for all</u> $f(X) \in R[X]$, $(f(X))^n = dH(X)$ <u>for some</u> $n \in N, d \in R$ <u>and</u> $H(X) \in R[X]$ <u>where</u> $(A_H)_v = R$.

<u>PROOF.</u> <u>Sufficiency</u>. Suppose that for all $f(X) \in R[X]$ $(f(X))^n = dH(X)$ with f,n,d,H as described in the hypothesis. Then for every finitely generated ideal $A = (a_0, ..., a_n)$ we can find $f(X)$ such that $A_f = A$ and so for some $n \in N$ $(A^n)_v = ((A_f)^n)_v = (A_{f^n})_v = (A_{dH})_v = d(A_H)_v = (d)$. That is, by Theorem 3.9, R is an AGCD-domain.

<u>Necessity.</u> Let R be an AGCD-domain and let $f(X) \in R[X] - \{0\}$. Then as R is also integrally closed, for some $n \in N$ $((A_f)^n)_v = (d)$. Or $(d) = (A_{f^n})_v$. From this it follows that every coefficient of $(f(X))^n$ is divisible by d. Dividing out by d we get $(f(X))^n = dH(X)$. It is easy to show that $(A_H)_v = R$.

Now if R is a PVMD with torsion t-class group then $R[X]$ is at least a PVMD and so to show that $R[X]$ has a torsion t-class group we need only show that for $f(X), g(X)$ in $R[X]$ there is $n \in N$ such that $(f^n) \cap (g^n)$ is principal. For this we prove the following lemma.

LEMMA 5.2. *Let R be a PVMD with torsion t-class group and let X be an indeterminate over R. If $f(X) \in R[X]$ with $(A_f)_v = R$ then there exists $n \in N$ such that $(f(X))^n = p_1(X) \ldots p_r(X)$ where each $p_i(X)$ is a prime block such that $\sqrt{p_i(X)} = P_i$ is a prime ideal with $P_i \cap R = (0)$.*

PROOF. Let $f(X)$ be as given in the statement. Then every prime ideal minimal over $(f(X))$ is a t-ideal. Because R is a PVMD, according to [14] (Proposition 4) every minimal prime P_i of $(f(X))$ is such that $P_i \cap R = (0)$. Since $K[X]$ is a UFD, $f(X)K[X] = (f_1(X))^{n_1} \ldots (f_r(X))^{n_r}K[X]$, where it can be assumed that $f_i(X) \in R[X]$ and that $f_i(X)$ are mutually non-associated primes in $K[X]$.

Now $f(X) = (a/b)(f_1(X))^{n_1} \ldots (f_r(X))^{n_r}$ for some $a, b \in R$. Because R is a PVMD with torsion t-class group, by Theorem 5.1, for each i there exists m_i such that $((f_i(X))^{n_i})^{m_i} = d_i H_i(X)$ where $(A_{H_i})_v = R$. Now let $M = m_1 \ldots m_r$ and $M_i = M/m_i$. Then $(f(X))^M = (a/b)^M (\prod_i (f_i(X))^{n_i})^M = (a/b)^M \prod_i (((f_i(X))^{n_i})^{m_i})^{M_i}$ $= (a/b)^M \prod d_i^{M_i} (H_i(X))^{M_i}$. Further, as $(A_{H_i})_v = R$ and $(A_f)_v = R$, we have $(a/b)^M (\prod d_i^{M_i}) = 1$. Consequently $(f(X))^M$ $= \prod (H_i(X))^{M_i}$ where for each i there is a prime ideal P_i such that $(H_i(X))^{M_i}K[X] = P_i^{r_i}K[X]$.

Now each of $(H_i(X))$ belongs to P_i and to no other prime t-ideal. This is because every prime t-ideal P in $R[X]$ containing $(H_i(X))$ has the property that $P \cap R = (0)$ (cf. [14]) and $(H_i(X))^{M_i}K[X]$ is a power of $P_iK[X]$. Now putting $p_i(X) = (H_i(X))^{M_i}$ we get the result.

DEFINITION 5.3. *If $p(X)$ is a polynomial in $R[X]$ such that $(A_p)_v = R$ and $p(X)$ has a single minimal prime P with $P \cap R = (0)$ we call $p(X)$ a primary polynomial.*

LEMMA 5.4. *Let R be any integral domain and let $a, b \in R$. Then a and b are v-coprime if and only if they do not share any associated prime.*

PROOF. We note that a and b are v-coprime if and only if $(a,b)_v = R$ (cf. (1) of Lemma 1.1). Now using Lemma 6 of [14] we get the result.

COROLLARY 5.5. *Let R be a PVMD and let $R[X]$ be a polynomial ring over R then the following hold.*

(1) *If $f(X) \in R[X]$ such that $(A_f)_v = R$ then for all $a \in R - \{0\}$, $(a,f)_v = R[X]$,*

(2) *If $f(X)$ and $g(X)$ are two primary polynomials then $f \mid g$ or $g \mid f$ or f and g are v-coprime.*

PROOF. The proof of (1) follows from the fact that for a PVMD R all the associated primes of $R[X]$ that contain f intersect R trivially (cf. [14] Proposition 4) and all those which contain a intersect trivially with the set $S = \{f(X) \in R[X] \mid (A_f)_v = R\}$ because they are of the type $P[X]$ where P is an associated prime of R. From this it follows that a and f do not share any associated prime of $R[X]$ and hence they are v-coprime. For (2) we note that either f and g belong to the same associated prime or they belong to two different associated primes. In the latter case the proof of part (1) applies whereas in the former it is sufficient to note that they belong to a unique prime P which has the property that $(R[X])_P$ is a valuation domain.

THEOREM 5.6. *Let R be an integrally closed integral domain and let X be an indeterminate over R. Then R is a PVMD with torsion t-class group if and only if R[X] has the same property.*

PROOF. Let R be a PVMD with torsion t-class group. Then, it is well known that, R[X] is a PVMD. To show that R[X] has torsion t-class group we have to show that for all $f,g \in R[X]$ there is $n \in N$ such that $(f^n) \cap (g^n)$ is principal. For this we note from Theorem 5.1 and Lemma 5.2 that there is $n_1 \in N$ such that $f^{n_1} = dF$ where $d \in R$, $(A_F)_v = R$ and F is a product of primary polynomials. Similarly there is $n_2 \in N$ such that $g^{n_2} = eG$ where $e \in R$, $(A_G)_v = R$ and G is a product of primary polynomials. Now if for some n_3, $d^{n_3}R \cap e^{n_3}R$ is principal then so is $d^{n_3}R[X] \cap e^{n_3}R[X]$.

Now let $f^{n_1} = dF$ and $g^{n_2} = eG$ where d, e, F and G are described above. Then $f^{n_1n_2} = d^{n_2}F^{n_2}$ and $g^{n_1n_2} = e^{n_1}G^{n_1}$. Because R has torsion t-class group there exists n_4 such that $(d^{n_2})^{n_4} \cap (e^{n_1})^{n_4}$ is principal in R and hence in R[X]. So $(f^{n_1n_2n_4}) \cap (g^{n_1n_2n_4}) = (d^{n_2n_4}F^{n_2n_4}) \cap (e^{n_1n_4}G^{n_1n_4})$
$= ((d^{n_2n_4}) \cap (F^{n_2n_4})) \cap ((e^{n_1n_4}) \cap (G^{n_1n_4}))$ (because $(d,F)_v = 1 = (e,G)_v$)
$= ((d^{n_2n_4}) \cap (e^{n_1n_4})) \cap ((F^{n_2n_4}) \cap (G^{n_1n_4}))$. Noting that $((F^{n_2n_4}) \cap (G^{n_1n_4}))$ is principal, because products of primary polynomials are involved, and that $(d^{n_2n_4}) \cap (e^{n_1n_4})$ is principal and finally that the above two ideals are

are v-coprime we conclude that $(f^n) \cap (g^n)$ is principal.

Conversely, suppose that $R[X]$ is a PVMD with a torsion t-class group. Then it is easy to establish that R is a PVMD. Moreover for any finitely generated ideal B of $R[X]$ $(B^n)_v$ is principal. So if $B = A[X]$ where A is a finitely generated ideal of R then $B^n = (A^n)[X]$ and according to [9] $(B^n)_v = (A^n)_v[X]$. Whence it follows that if $R[X]$ is a PVMD with torsion t-class group then for every finitely generated ideal A of R $(A^n)_v$ is principal for some $n \in N$.

We have already noted that in a Prüfer domain the t-class group coincides with the ideal class group. The following corollary adds to this information in an interesting way.

COROLLARY 5.7. *Let R be a Prüfer domain with torsion ideal class group then $R[X]$ is a PVMD with torsion t-class group.*

This corollary highlights the similarity between the PVMD's and the Prüfer domains, and at the same time it highlights the connection of the t-class groups with the ideal class groups. This leads us to look into Prüfer domains with torsion ideal class groups once again. These integral domains have been extensively studied because of their special property that their overrings are quotient rings. This property is called the QR property. In Krull domains we can find an analogue of the QR property; which states that a Krull domain R has torsion divisor class group if every flat overring of R is a quotient ring. It would be interesting to prove a similar result for PVMD's. It may be noted that if R has the QR property then R is a

Prufer domain but may not have torsion class group. So the possible result mentioned above would read as: If R is a PVMD with torsion ideal class group then every flat overring of R is a quotient ring.

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Al-Faateh University
Department of Mathematics
P.O.Box 13211-Tripoli
LIBYAN ARAB JAMAHIRIYA

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