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Abstract Prüfer Ideal Theory

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The various characterizations of Prüfer domains in ring theory—by means of identities (between the ideals) which involve intersection, sum, product or residual; as invertibility of finitely generated ideals; and as every localization being a valuation ring—have been largely carried over to the principal abstractions of commutative ideal theory: i.e., to the setting of multiplicative lattices [W, McC, An] and to that of ideal systems [BL]. The equivalence of these abstractions, recently made explicit in [F], permits comparing and unifying these treatments; in the process we make some simplifications, removing unnecessary restrictions and generally tightening and streamlining the description.

To have a sufficiently flexible formulation we adopt as basic setting an integral commutative m-semilattice (or join-semireticulated commutative monoid); explicitly, [B, Chap. XIV], this structure is a join-semilattice-ordered commutative monoid in which multiplication distributes over join—hence is isotone—and whose multiplicative identity e is a greatest element—hence makes products less than or equal to their factors. Thus hereditary (i.e., closed under smaller element) subsemilattices are closed under pairwise join as well as under multiplication by any element of the m-semilattice: they are appropriately called "ideals." As in ring theory, the system of ideals can be equipped with a multiplication: the product of a pair of ideals is the ideal (order) generated by the (updirected set of) pairwise products of their elements; moreover, the smallest hereditary subset containing the subsemilattice of pairwise joins of their elements is also the smallest ideal containing them. These operations make the ideals into an m-semilattice which contains an isomorphic copy of the given m-semilattice as the subsystem of principal ideals. The larger m-semillatice of ideals admits some additional operations: it is closed under arbitrary intersection and admits "ideal quotients" or *residuals*: with any ideals I, J, the set $I:J=\{p: pJ\subset I\}$ is an ideal. Henceforth we consider these operations to be defined also on the original elements, with the

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understanding that they take as values ideals—which will be in the original system insofar as they are principal.

A morphism between m-semilattices should preserve product, join, and e. It extends to a map of ideals: it sends an ideal onto a subsemilattice, the smallest hereditary subset containing which will be designated as its image. This is a morphism on the ideal level (but need not preserve the additional operations). In the sequel there is need for a special type of morphism, related to the formation of fractions, which is determined by the set of elements sent on e as the "universal" one having this set for "kernel." Such a set is a co-hereditary (i.e., closed under larger elements) submonoid; conversely, the weakest multiplicatively compatible order strengthening which makes the elements of such a *filter* M dominate e is the relation $p \geqslant mq$ for some $m \in M$. The equivalence modulo which this becomes anti-symmetric—the *quotient* modulo M, which identifies it with e—preserves not only product and join but any existent residuals and finite meets.[1] More important for our purpose, it preserves an element's cancellability (from inequalities follows by distributivity over join); the weaker property of being "*cancellable modulo its annihilator*"—i.e., of $q'p \leqslant qp$ entailing $q' \leqslant q \vee r$ with $rp = 0$ (more commonly called "weak join principality"); and its *multiplicativity*—i.e., the property of dominating only its multiples (this is the "principality" of [WD], nowadays designated "weak meet principality"). Extended to the ideals, these quotient morphisms—which may be described within the system of ideals by the map from an ideal I to its saturation, the updirected union of the $I : m$ as m runs through the kernel M—preserve not only product and (even infinite) join (as do extensions of arbitrary semilattice morphisms) but also finite meet—i.e., the multiplicative lattice structure—as well as residuation by principal[2] ideals.

In an m-semilattice, the assignment to filters of the quotient congruences modulo them is finite meet preserving (since $p \geqslant mq$ and $\geqslant m'q$ entails $p \geqslant (m \vee m')q$; it is join preserving in any pomonoid). Thus subdirect irreducibility entails that proper filters meet properly; this is implied by e join-irreducible (i.e., not a proper pairwise join); and implies it (in any pomonoid in which pairwise joins are distributive at e): Indeed, $e = p \vee x = p \vee y$ then entails $e = p \vee (p \vee x)y = p \vee py \vee xy = p \vee xy$,

[1] These facts are developed in [McC].

[2] That residuation by a non-principal ideal need not be preserved may be seen by taking the quotient of the negative cone of the lexicographic plane modulo its unique proper filter, the half axis. The saturated ideals are just the non-principal ones and residuation by one of these of a principal ideal yields a saturated ideal strictly less than the residuation of its saturation. This also shows that the extended morphism on the ideals is not a quotient modulo its kernel, the filter generated by (the principal ideals representing) the unextended morphism's kernel.

whence the elements "co-disjoint" from any p constitute a filter; while those in turn co-disjoint from each of its elements constitute another, which contains p, and has only e in common with the former. Subdirectly irreducible m-semilattices are thus "local," which is the name attached to structures with join-irreducible unit. Although this yields a subdirect representation in terms of locals for every m-semilattice on universal algebraic grounds, a tighter representation, using only the projections which are the quotient morphisms modulo the filters sent on join-irreducible e will be needed.

The filter which a morphism sends on a join-irreducible unit can contain a pairwise join only if it contains one of the terms: its complement is closed under pairwise join; and since it is hereditary, it is an ideal—indeed a "prime ideal" since *its* complement is multiplicatively closed. Conversely, of course, the complement of every prime ideal is just such a "join-prime" filter, modulo which the quotient (as image of a join-preserving map) has join-irreducible unit. That the quotient morphisms separate may now be established by noting that maximal ideals are prime (since $(p \vee x)(p \vee y) \leqslant p \vee xy$) and that $p : q$ is a proper ideal whenever $p \ngeqslant q$, hence is contained in some maximal ideal whose complement is then a filter modulo which the failure of this inequality is preserved. Finally, as a tool for what follows, one has the "globalization" property, enabling one to recover every ideal from its "localizations" as the intersection of its saturations: If q belonged to one of the $I : m$ for every prime filter M, then this set of m, being contained in no maximal ideal, would generate an improper ideal and so would have a finite join $Vm = e$—then $q = eq = Vmq \in I$ (cf. [LMcC, 3.13, p. 70]).

Our proposed definition of "Prüfer" is that the "localized" quotients be totally ordered—a local m-semilattice is Prüfer just when it is totally ordered. Total order comes to the same for an m-semilattice as for its ideal lattice. A join-irrducible unit remains such in the ideal lattice—indeed its complement constitutes a unique maximal ideal. Hence when the unit is join-irreducible, Prüfer may be characterized by the identity $x : y \vee y : x = e$ in the ideal lattice—an identity which passes back to the represented semilattice from the totally ordered quotients of any subdirect representation (i.e., by not necessarily all irreducible quotients) and conversely forward to any quotient. Another characterizing identity is $(x \vee y) : z = x : z \vee y : z$—it is clearly necessary (following by globalization from total order of quotients even for x, y ideals) and conversely entails $e = (x \vee y) : (x \vee y) = [x : (x \vee y) \vee y : \ (x \vee y)] \leqslant x : y \vee y : x$; a dual identity in lattices is $z : x \vee z : y = z : (x \wedge y)$—which is also characterizing when formulated for the ideal lattice, since with z the intersection $x \wedge y$ of the principal ideals, it entails $x : y \vee y : x \geqslant (x \wedge y) : x \vee (x \wedge y) : y = (x \wedge y) : (x \wedge y) = e$.

It was noted that multiplicativity (i.e., dominating only multiples) and cancellability (from equalities would entail, by distributivity, from inequalities) resp. modulo annihilators are preserved by localization. The conjunction of these will be referred to as "invertibility" (an appellation which will be justified further on) resp. "modulo annihilators." Like the cancellables, the invertibles are closed under product and taking of factors: $p \geqslant pq \geqslant x = pr$ entails $q \geqslant r = qs$ whence $x = pqs$; and from $q \geqslant x$ follows $pq \geqslant px = pqr$ whence $x = qr$. Invertibles remain such in the ideal lattice. Cancellability modulo annihilators is preserved under removing of non-zero-multiple factors.

A subdirect product of totally ordered lattices with isotone multiplication (such as a Prüfer ideal lattice) satisfies the identities[3] $(x \wedge y)z = xz \wedge yz$ and $(x \vee y)z = xz \vee yz$, which entail $(x \wedge y)(x \vee y) = xy$; in the ideal lattice of an m-semilattice, this would make the invertibles join-closed. Conversely, being join-closed makes the invertibles a multiplicative subsemilattice which is then necessarily Prüfer. Indeed, if invertibles p and q in some localized quotient were incomparable then the multiples of the multiplicative $p \vee q$ which yield each of p and q are non-units; and by its cancellability the join of these non-unit multiples would be e, contradicting the latter's join-irreducibility. (This argument only requires $pq = p$ for p a non-zero join of generators to entail $q = e$ in each localized quotient, rather than full cancellability—thus for such joins in the original semilattice to satisfy: $pq \geqslant pm$ and $0 \notin pM$ only for $q \in M$, the complement of any prime ideal; equivalently for every prime overideal of $0 : p$ to contain with each q also $pq : p$, e.g., for such joins to be cancellable modulo their annihilators. Also it shows that the invertibility of the pairwise joins from any join-generating set of invertibles suffices for Prüfer—or, for that matter, the multiplicativity of pairwise joins from a join-generating set of invertibles modulo annihilators, since these will be localized as elements which absorb only e.)

A commutative monoid will be enlarged to a monoid of fractions on taking as denominators any submonoid of cancellables;[4] if these were cancellable from inequalities for a given pomonoid structure, then the requirement, that they continue to act isotonely and be cancellable from inequalities, fixes a unique pomonoid structure on the fractions which extends the original one. The integral elements in this pomonoid of fractions—i.e., those $\leqslant e$—will not be enlarged beyond the original pomonoid just when the denominators are invertible. (Any element which obtains a

[3] These are definitely not sufficient for Prüfer—adjoining a new unit to a proper direct product will produce an m-lattice satisfying these identities and having a join-irreducible unit but not totally ordered.

[4] Their divisors will also become (all the) denominators.

multiplicative inverse in[5] such a pomonoid of fractions must have been invertible: as a factor of the identity it is cancellable and its inverse times any integral element it dominated is integral, *showing it multiplicative*. Conversely, when the invertible denominators are co-initial, every multiplicative element becomes multiplicatively invertible.) In particular, the pomonoids consisting only of invertibles will have, as full denominator fractions, a pogroup—directed, since e and any fraction have the numerator as common lower bound—of which it is the negative cone; conversely, the negative cone of any pogroup is a pomonoid of invertibles (whose fractions would fill a proper subgroup in the absence of directedness). Thus Prüfer m-semilattices join-generated by invertibles are just the negative cones of l-groups.

To summarize what has been established: That an m-semilattice be Prüfer—i.e., have all its quotients totally ordered—is equivalent to each of the identities (in its ideal lattice) $x : y \vee y : x = e$, $(x \vee y) : z = x : z \vee y : z$, $z : x \vee z : y = z : (x \wedge y)$; when its invertibles modulo annihilators (finitely) join-generate, it is also equivalent to each of: $(x \wedge y)z = xz \wedge yz$, $(x \wedge y)(x \vee y) = xy$, join-closure of invertibles modulo annihilators—even multiplicativity of the pairwise joins from a join-generating subset—and under invertible join-generation to being the negative cone of an l-group.[6]

When comparing these characterizations with the classical ones for integral domains, it must be taken into account that the latter are couched in terms of a specific set of invertible join-generators—the non-zero elements or principal (in the usual sense) ideals; and that the identities may be formulated for all ideals rather than for the finitely generated ones. But, as noted parenthetically during the course of the above development, it would suffice, that an integral domain have its finitely generated ideals invertible, for ideals with two generators to be invertible;[7] and that although the identities for finitely generated ideals suffice for Prüfer, they are actually necessary for all ideals (except for residuation, which should only be done by finitely generated ideals). This enables one to see that a formulation such as [LMcC, VI, 6.6, p. 127] may be specialized from what has been done here. Of the 10 equivalent conditions given there, only (3) and (10) are not included; these would have corresponded (for an invertibly join-generated m-semilattice) to cancellability of every element of

[5] A multiplicative inverse is usually formulated by allowing the inverse to be an (updirected) ideal of fractions; but this comes to the same since a proper updirected ideal could not have an inverse; i.e., an invertible (indeed a muliplicative absorbing only e) is finitely generated.

[6] The Prüferness of these cones shows that commutative l-groups are subdirect products of totally ordered groups.

[7] More generally, the invertibility of pairwise joins from a subset implies that of all its finite joins, as follows from the identity $(x \vee y \vee z)(xy \vee xz \vee yz) = (x \vee y)(x \vee z)(y \vee z)$ and induction [G].

the m-semilattice and distributivity of its ideal lattice. However (unlike what occurs in rings), distributivity does not ensure the total order of semilattices with join-irreducible unit: see [BL, Proposijon 78, p. 57]; while the insufficiency of cancellability is already in [L].

To exhibit the connection with ideal systems, we show that the invertibly join-generated m-semilattices are just the finitary ideals of Lorenzen r-systems. The latter may be described as finitary closure operators on the semigroup ideals of cancellative monoids which commute with translation by elements. The singly generated ideals in such a system are invertible join-generators: these ideals consist of (monoid) multiples of the generator, hence are multiplicative in the sense being used here; and a cancellable monoid element generates a cancellable ideal (see [F, (A), p. 396]). In the other direction, since the invertibles in an m-semilattice are closed for product, they constitute a submonoid which may be equipped with the closure assigning each finite set the invertibles dominated by its join—that this satisfies the Lorenzen requirement of translation invariance follows from multiplicativity of these generators, distributivity of their multiplication over join, and cancellability from inequalities. The lattice ideals correspond to system ideals and the invertible ones to those with a multiplicative inverse in the fractional ideals (as noted parenthetically above). We can conclude that those of our Prüfer characterizations based on invertible join-generation are in essence the same as those of [BL, Kapitel 3]—we have only reformulated and derived them in a different way.

Although [An] and [McC] are already formulated for lattices, their setting is not directly comparable with ours, since they operate in (what for us was) the complete multiplicative lattice of all ideals. However, both postulate join-generation by a submonoid of compact elements; the (still compact) finite joins of these would be a sub-m-semilattice whose ideal lattice coincides with the originally given complete lattice. This enables the comparison to be made.

That our equivalences include those set out in [An, Theorem 3.4] seems fairly certain. A detailed comparison is rendered difficult since the definition of the "r-lattice domain" for which they are asserted to hold has not been furnished in [An]. Presumably it entails cancellability of the join-generating submonoid of "principals"—this notion being more restrictive than "multiplicativity" which we have borrowed from [WD], cancellability would assure join-generation by our invertibles and so the validity of all our equivalences. In any event, the last equivalence in Theorem 3.4—representability as the ideal lattice of a Prüfer domain—is conclusive, if circumstantial, evidence for inclusion in our setting, via the equivalence with the ideal system formulation of the preceding paragraph.

[McC] also postulates join-generation by the more restrictive principals;

and these are to be compact; as a minor deviation from [An] and the above, he does not require compactness of the unit, hence must explicitly postulate that every non-unit be dominated by a proper prime in order to be able to globalize; his specifically Prüfer postulate is that the finite joins of these compact (join-generating) principals be multiplicative. (These postulates are decidedly less stringent than those of [An], whose Prüfer characterizations are thus no more than a weaker form of those already available in the earlier publication.) Principals are however invertible modulo annihilators, so that our argument also furnishes the Prüfer equivalents (not all of which were developed by [McC]) for this setting.

In conclusion, we point out the connection between the more general formation of fractions, using not necessarily cancellable denominators from an arbitrary submonoid—as expounded, e.g., for ideal systems in [A] (there miscalled "localization")—and the passage to the quotient modulo M used above for localizing at prime ideals. The universal monoid morphism making the elements of M cancellable is the surjective map with kernel the pairs equalized by multiplication with some element of M—this is indeed a monoid congruence (by virtue of commutativity). More generally, the weakest order strengthening of a commutative pomonoid which will make the elements of a submonoid M cancellable from inequalities is the antisymmetrization of the pre-ordering relation $mp \geqslant mq$ for some $m \in M$. This strengthening preserves any existent multiplicatively distributive joins: the universal pomonoid morphism induced on an m-semilattice would preserve its structure. The pomonoid, or m-semilattice, of fractions built with these now cancellable elements as denominators is universal for making the elements of M bijective multipliers—i.e., divisors of e; since the quotient modulo M is universal for identifying M with e, it could be obtained from these fractions as the image of the (non-integral commutative pomonoid) universal morphism constant on the units.

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Errata

Volume **111**, Number 2 (1987), in the article "Subdirect Products of Totally Ordered BCK-Algebras," by Isidore Fleischer, pages 384–387: In the Abstract, the name of the algebras was abbreviated because of an editing error. The Abstract should read:

> Subdirect products of totally ordered BCK-algebras are characterized by the condition $y^{-1}x \vee x^{-1}y = 1$.

Volume **115**, Number 2 (1988), in the article "Abstract Prüfer Ideal Theory," by Isidore Fleischer, pages 332–339: In footnote 5, page 336, the words "A multiplicative inverse is" were inserted erroneously. The footnote should read:

[5] Usually formulated by allowing the inverse to be an (updirected) ideal of fractions; but this comes to the same since a proper updirected ideal could not have an inverse, i.e., an invertible (indeed a multiplicative absorbing only e) is finitely generated.

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