Reprinted from JOURNAL OF ALGEBRA
All Rights Reserved by Academic Press, New York and London

Vol. 142, No. 2, October 1, 1991
Printed in Belgium

Almost Bézout Domains

D. D. ANDERSON

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242

AND

M. ZAFRULLAH

Department of Mathematics, Winthrop College, Rock Hill, South Carolina 29733

Communicated by Richard G. Swan

Received February 24, 1989

DEDICATED TO P. M. COHN

An integral domain R is said to be an almost Bézout domain (respectively, almost GCD-domain) if for $x, y \in R - \{0\}$, there exists an n with (x^n, y^n) (respectively, $(x^n, y^n)_v$) principal. In this paper we continue the investigation of AGCD-domains begun by the second author and introduce the notion of an almost Bézout domain. We show that R is an almost Bézout domain if and only if $\bar{R}$, the integral closure of R, is a Prüfer domain with torsion class group and for every $x \in \bar{R}$, there exists an n with $x^n \in R$. © 1991 Academic Press, Inc.

1. INTRODUCTION

In [15], Storch introduced the notion of an almost factorial Krull domain. One characterization of an almost factorial Krull domain R is that R is a Krull domain with the property that given $a, b \in R - \{0\}$, there exists an n (throughout n will represent a natural number) with $a^n R \cap b^n R$ principal. In [16], the second author began a general theory of almost factoriality. One important class of integral domains introduced in [16] was that of almost GCD-domains or AGCD-domains. Here R is an AGCD-domain if for each $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $a^n R \cap b^n R$ principal (or equivalently $(a^n, b^n)_v$ is principal). The purpose of this paper is to continue the study of AGCD-domains begun in [16] and to introduce several new closely related classes of integral domains.

We introduce the notions of almost Bézout domains and almost principal ideal domains. Here R is an almost Bézout domain (AB-domain) if for $a, b \in R - \{0\}$, there is an n with (a^n, b^n) principal. Hence an

0021-8693/91 $3.00

AB-domain is an AGCD-domain. R is an almost principal ideal domain (API-domain) if for any nonempty subset $\{a_\alpha\} \subseteq R - \{0\}$, there exists an n with $(\{a_\alpha^n\})$ principal. It is shown (Corollary 4.8) that R is an AB-domain if and only if the integral closure $\bar{R}$ of R is a Prüfer domain with torsion class group and for each $x \in \bar{R}$ there exists an n with $x^n \in R$. A particularly interesting example of an API-domain from number theory is given. For m square-free, $Z[\sqrt{m}]$ is a non-integrally closed API-domain if $m \equiv 5 \bmod 8$, while for $m \equiv 1 \bmod 8$, $Z[\sqrt{m}]$ is not an API-domain (see Theorem 4.17).

This paper consists of five sections besides the Introduction. Section 2 consists of preliminaries. Several definitions are given and root extensions are investigated. ($R \subseteq S$ is a root extension if for each $s \in S$, there exists an n with $s^n \in R$.) Section 3 begins by reviewing some of the material on AGCD-domains from [16]. Theorem 3.4 states that the t-class group of an AGCD-domain is torsion. Theorem 3.5 states that a flat overring of an AGCD domain is a localization. This answers a question raised in [16].

The longest and most important section is Section 4. Here the definitions of AB-domains and API-domains are introduced and the basic properties of these rings are given. For example, Lemma 4.5 shows that an overring of an AB-domain is an AB-domain. As previously mentioned, Corollary 4.8 gives a satisfactory characterization of AB-domains. An integrally closed domain is shown to be an API-domain if and only if it is Dedekind and has torsion class group. Many examples of AB-domains and API-domains are given. Besides the previously mentioned examples from number theory, we have the interesting result (Example 4.14) that $R = K + XL[X]$ is an AB-domain for any purely inseparable field extension $K \subseteq L$, but that R is an API-domain if and only if there is a bound on the degree of inseparability. However, perhaps the simplest example of a non-integrally closed API-domain is $Z[2i]$ where $i = \sqrt{-1}$.

Section 5 contains some more results on AGCD-domains and AB-domains. Theorem 5.3 shows that a prime ideal P of an AGCD-domain R is a t-ideal if and only if R_P is an AB-domain. An interesting corollary is that an AGCD-domain R is an AB-domain if and only if Spec(R) is treed. We show (Theorem 5.8) that R is an AP-domain (for $a, b \in R - \{0\}$, there exists an n with (a^n, b^n) invertible) if and only if for each maximal ideal M of R, R_M is an AB-domain. We also consider the question of when $\bar{R}$ an AGCD-domain implies that R is an AGCD-domain.

The last section concerns ideals generated by powers of elements. Given an ideal A of a ring R, we define $A_n = (\{a^n \mid a \in A\})$. Then $A_n \subseteq A^n$ is an ideal of R and it is natural to ask when $A_n = A^n$. Perhaps the most interesting result of this paper is Theorem 6.12 which states that if a ring R contains a field of characteristic 0, then $A_n = A^n$ for every ideal A of R. Also of interest is the fact (Corollary 6.4) that for R n-root closed and $A = (\{a_\alpha\})$, $(\{a_\alpha^n\})_t = (A_n)_t = (A^n)_t$. In fact, this property characterizes R

being n-root-closed (Theorem 6.8). An ideal A is said to be nearly principal if A_n is principal for some n. This notion is used to define several other classes of integral domains related to AB-domains and API-domains.

Our general reference for results from multiplicative ideal theory will be [9]. The end of a proof will be designated by ▮.

The first author acknowledges the support services provided by University House, The University of Iowa. We also thank Evan Houston and Joe Mott for several helpful conversations during the preparation of this paper.

2. Preliminaries

Let R be an integral domain with quotient field K. Let $F(R)$ be the set of nonzero fractional ideals of R and $f(R)$ the subset of $F(R)$ consisting of finitely generated fractional ideals. For $I \in F(R)$, $I^{-1} = \{x \in K \mid xI \subseteq R\}$ is again a member of $F(R)$. We will denote $(I^{-1})^{-1}$ by I_v. It is well known that $I_v = \bigcap \{Rz \in F(R) \mid Rz \supseteq I\}$. The operation on $F(R)$ that sends I to I_v is an example of a star operation, namely, the *v-operation.* Recall that a *star operation* is a function $*: F(R) \to F(R)$ that satisfies (1) $(a)^* = (a)$, $(aA)^* = aA^*$; (2) $A \subseteq A^*$, if $A \subseteq B$, then $A^* \subseteq B^*$; and (3) $(A^*)^* = A^*$. We call I a *$*$-ideal* if $I = I^*$ and I is a *finite type $*$-ideal* if $I = J^*$ for some $J \in f(R)$. The reader may consult [9, Sects. 32 and 34] for the basic properties of star operations and the v-operation.

Another star operation that will play an important role in this paper is the *t-operation.* Here $I_t = \bigcup \{J_v \mid J \subseteq I \text{ with } J \in f(R)\}$. In particular, if I is finitely generated, $I_t = I_v$. A fractional ideal I is said to be *t-invertible* if there exists a fractional ideal J with $(IJ)_t = R$. In this case, we may take $J = I^{-1}$. It can be shown that a t-invertible t-ideal has finite type. Let $f_t(R)$ be the set of finite type t-ideals of R. Then $f_t(R)$ forms a semigroup under the "t-product" $I * J = (IJ)_t$. Evidently $f_t(R)$ forms a group if and only if every finite type t-ideal is t-invertible. An integral domain is called a *Prüfer v-multiplication domain* (PVMD) if $f_t(R)$ forms a group. Many other characterizations of PVMDs are known. For example, see [11, 12, 14].

While in general $f_t(R)$ need not form a group, the set $I_t(R)$ of t-invertible t-ideals forms a subgroup of $f_t(R)$, in fact, it is just the group of units of $f_t(R)$. Let $P(R)$ be the subgroup of $I_t(R)$ consisting of principal ideals. The quotient group $I_t(R)/P(R)$ is called the *t-class group of R* and will be denoted by $Cl_t(R)$. Thus $Cl_t(R)$ measures how far away t-invertible t-ideals are from being principal. For R a Krull domain, $Cl_t(R)$ is just the usual divisor class group $Cl(R)$, while for a Prüfer domain R, $Cl_t(R)$ is just the ideal class group $C(R)$ of invertible ideals modulo principal ideals. For results on the t-class group, the reader is referred to [2–4].

Let $R \subseteq S$ be an extension of commutative rings. We will use $\bar{R}$ to denote the integral closure of R in S. If no S is specified, $\bar{R}$ denotes the integral closure of R in its total quotient ring. Of particular interest will be extensions $R \subseteq S$ with the property that for each $s \in S$ there exists a natural number n (depending on s) with $s^n \in R$. An extension $R \subseteq S$ having this property will be called a *root extension*. (Note that we do *not* assume there is some fixed n with $s^n \in R$ for all $s \in S$.) Of course a root extension is an integral extension. One usually thinks of root extensions in characteristic $p > 0$ where $s^{p^m} \in R$ for some m. Such an extension will be called *purely inseparable*. But they also occur in characteristic zero. For example, the extension $Z + 2iZ = Z[2i] \subset Z[i]$ ($i = \sqrt{-1}$) has the property that $x^2 \in Z[2i]$ for each $x \in Z[i]$. We isolate the following result concerning the prime spectra of the rings of a root extension which is usually only stated in the case where S is purely inseparable over R (e.g., [9, pp. 108–109]).

THEOREM 2.1. *Suppose that $R \subseteq S$ is a root extension of commutative rings. The map $\mathcal{Q}$: $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ given by $\mathcal{Q}(Q) = Q \cap R$ is an order isomorphism and a homeomorphism. Its inverse is given by $\mathcal{Q}^{-1}(P) = \sqrt{P} = \{s \in S \mid s^n \in P$ for some $n \geq 1\}$.*

Proof. Since $R \subseteq S$ is integral, the map $\mathcal{Q}$ is surjective. Suppose $Q_1 \cap R = Q_2 \cap R$ for $Q_1, Q_2 \in \operatorname{Spec}(S)$. Then $x \in Q_1 \subseteq S$ has $x^n \in R$ for some n, so $x^n \in Q_1 \cap R = Q_2 \cap R \subseteq Q_2$, and hence $x \in Q_2$. Thus $Q_1 \subseteq Q_2$. Interchanging Q_1 and Q_2 gives that $Q_2 \subseteq Q_1$ and hence $Q_1 = Q_2$; so $\mathcal{Q}$ is injective. Thus $\mathcal{Q}$ is a bijection. It easily follows that $\mathcal{Q}$ is an order isomorphism.

Let $P \in \operatorname{Spec}(R)$ and let Q be the unique prime ideal of S lying over P. If $s \in Q$, then some $s^n \in Q \cap R = P$, so $s \in \sqrt{P}$. On the other hand, if $s \in \sqrt{P}$, then $s^n \in P \subseteq Q$ implies $s \in Q$. Hence $Q = \sqrt{P}$.

It remains to show that $\mathcal{Q}$ is a homeomorphism. To do this it suffices to show that for an ideal I of S, $\mathcal{Q}(V(I)) = V(I \cap R)$. (As usual, $V(I) = \{Q \in \operatorname{Spec}(S) \mid Q \supseteq I\}$.) Now $Q \in V(I)$ implies $Q \supseteq I$, so $Q \cap R \supseteq I \cap R$ and hence $\mathcal{Q}(Q) \in V(I \cap R)$. Suppose that $P \in V(I \cap R)$, so $P \supseteq I \cap R$. Let Q be the unique prime ideal of S lying over P, so $Q = \sqrt{P}$. We need that $Q \supseteq I$. Let $i \in I$. Then some $i^n \in R$, so $i^n \in I \cap R \subseteq P$. Hence $i \in \sqrt{P} = Q$, so $I \subseteq Q$. ∎

We say that $\operatorname{Spec}(R)$ is *treed* if $\operatorname{Spec}(R)$, as a partially ordered set, is a tree or equivalently, if $\operatorname{Spec}(R_M)$ is totally ordered for each maximal (or prime) ideal M of R. The following result is an immediate corollary of Theorem 2.1.

COROLLARY 2.2. *Let $R \subseteq S$ be a root extension of commutative rings. Then* $\operatorname{Spec}(R)$ *is treed if and only if* $\operatorname{Spec}(S)$ *is treed.*

3. AGCD-Domains

The second author [16] introduced a general theory of almost factoriality which subsumed the almost factorial Krull domains of Storch ([15] or [8]). A fundamental definition introduced in [16] is that of an almost GCD-domain (AGCD-domain). An integral domain R is called an AGCD-*domain* if for $x, y \in R - \{0\}$, there exists an $n = n(x, y)$ with $x^n R \cap y^n R$ principal. Observe that $x^n R \cap y^n R$ is principal if and only if $(x^n, y^n)_v$ is principal. It is easily seen (for example, see the paragraph after Lemma 3.3) that R is an AGCD-domain if and only if for $x_1, \ldots, x_s \in R - \{0\}$, there exists an $n = n(x_1, \ldots, x_s)$ with $x_1^n R \cap \cdots \cap x_s^n R$ principal (or equivalently, $(x_1^n, \ldots, x_s^n)_v$ is principal). The following theorem summarizes some of the results from [16] concerning AGCD-domains.

THEOREM 3.1. (1) *Let R be an AGCD-domain. Then $\bar{R}$ is an AGCD-domain and $R \subseteq \bar{R}$ is a root extension.* (2) *Let R be an integrally closed integral domain. Then R is an AGCD-domain if and only if R is a PVMD with torsion t-class group.*

We next give a slight generalization of part of Theorem 3.1 (1) which will be used later.

PROPOSITION 3.2. *Let R be an integral domain and let $a, b \in R - \{0\}$. Suppose that there exists a positive integer n with $a^n R \cap b^n R$ locally principal (e.g., invertible). Then a/b is integral over R if and only if $(a/b)^n \in R$. In particular, if R is an integral domain with the property that for each $a, b \in R - \{0\}$, there exists an $n = n(a, b)$ with $a^n R \cap b^n R$ locally principal, then $x \in \bar{R}$ if and only if $x^m \in R$ for some positive integer m.*

Proof. Suppose that a/b is integral over R. Let M be a maximal ideal of R. Then a/b is integral over R_M and $a^n R_M \cap b^n R_M = (a^n R \cap b^n R)_M$ is principal for some n. It suffices to show that $(a/b)^n \in R_M$, for then $(a/b)^n \in \bigcap R_N = R$ (where the intersection runs over all maximal ideals N of R). Since $a^n R_M \cap b^n R_M$ is principal, we can write $(a/b)^n = a^n/b^n = u/w$ where $u, w \in R_M$ with $((u, w)_M)_v = R_M$. But a/b integral over R_M implies that $a^n/b^n = u/w$ is integral over R_M. Since $((u, w)_M)_v = R_M$ and u/w is integral over R_M, w must be a unit in R_M, so $a^n/b^n \in R_M$. ∎

In the spirit of [1], we can define an integral domain R to be an *almost generalized* GCD-*domain* (AGGCD-*domain*) if for each $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $a^n R \cap b^n R$ invertible. Many of the results concerning AGCD-domains and AP-domains (for $a, b \in R - \{0\}$, (a^n, b^n) is invertible for some n) can be extended to AGGCD-domains. We leave these extensions to the interested reader.

Theorem 3.1 shows that an integrally closed AGCD-domain has torsion t-class group. We can extend this result to arbitrary AGCD-domains, but to do so, we need to generalize the well-known fact (which we will use throughout this paper) that for an invertible ideal $A = (\{a_\alpha\})$ we have $A^n = (\{a_\alpha^n\})$. Section 6 continues the investigation of the relationship between the ideals A^n and $(\{a_\alpha^n\})$.

LEMMA 3.3. *Let R be an integral domain and let $\{a_\alpha\} \subseteq R - \{0\}$. If $(\{a_\alpha\})_t$ is t-invertible, then $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$ and hence $(\{a_\alpha^n\})_t$ is also t-invertible.*

Proof. Since $(\{a_\alpha\})_t$ is t-invertible, $(\{a_\alpha\})_t = (a_1, ..., a_s)_t$ for some finite subset $\{a_1, ..., a_s\} \subseteq \{a_\alpha\}$. Let $A = (a_1, ..., a_s)$. It is easily verified that $A^{s(n-1)+1} = A^{(s-1)(n-1)}(a_1^n, ..., a_s^n)$. (This is the key step in the proof that an invertible ideal $(a_1, ..., a_s)$ satisfies $(a_1, ..., a_s)^n = (a_1^n, ..., a_s^n)$, see [9, Theorem 6.5].) Multiplying both sides by $(A^{-1})^{(s-1)(n-1)}$, applying the t-operation to both sides, and using the fact that $(AA^{-1})_t = R$ yields that $(a_1^n, ..., a_s^n)_t = ((a_1, ..., a_s)^n)_t$. Since $(a_1^n, ..., a_s^n)_t \subseteq (\{a_\alpha^n\})_t \subseteq ((\{a_\alpha\})^n)_t = ((a_1, ..., a_s)^n)_t$, we have the desired equality. ∎

Lemma 3.3 can be used to prove the previously mentioned fact that if R is an AGCD-domain, then for $a_1, ..., a_s \in R - \{0\}$, $(a_1^n, ..., a_s^n)_v$ is principal for some $n = n(a_1, ..., a_s)$. For by induction assume that $(a_1^m, ..., a_{s-1}^m)_v$ is principal for some m, say $(a_1^m, ..., a_{s-1}^m)_v = (d)$. Now for some l, $(d^l, (a_s^m)^l)_v$ is principal. But $(d^l, (a_s^m)^l)_v = ((a_1^m, ..., a_{s-1}^m)_v^l, a_s^{ml})_v = ((a_1^{ml}, ..., a_{s-1}^{ml})_v, a_s^{ml})_v = (a_1^{ml}, ..., a_{s-1}^{ml}, a_s^{ml})_v$.

THEOREM 3.4. *Let R be an AGCD-domain. Then $Cl_t(R)$, the t-class group of R, is torsion.*

Proof. Let A be a t-invertible t-ideal of R. Then $A = (a_1, ..., a_s)_t$ for some $a_1, ..., a_s \in K - \{0\}$. Then $(a_1, ..., a_s)_t = d^{-1}(b_1, ..., b_s)_t$ where $b_1, ..., b_s \in R - \{0\}$. Since R is an AGCD-domain, there exists an n with $(b_1^n, ..., b_s^n)_t$ principal. By Lemma 3.3, $(A^n)_t = d^{-n}(b_1^n, ..., b_s^n)_t$ and hence $(A^n)_t$ is principal. ∎

A Krull domain R has the property that every flat overring is a localization if and only if $Cl(R)$ is torsion, i.e., R is almost factorial. The question was raised in [16] whether a PVMD with torsion t-class group (or equivalently, an integrally closed AGCD-domain) has the property that every flat overring is a localization. We next show that this is indeed the case for any AGCD-domain.

THEOREM 3.5. *Let R be an AGCD-domain. Then every flat overring of R is a localization of R.*

Proof. Let R' be a flat overring of R. Let $a/b \in R'$ where $a, b \in R - \{0\}$. Then there exists an n with $(a^n, b^n)_v = (c)$ for some $c \in R$. Put $a^n = a'c$, $b^n = b'c$, so $a^n/b^n = a'/b'$ where $(a', b')_v = R$. Now $(a', b')_v = R$ implies that $a'R \cap b'R = a'b'R$. Since R' is a flat overring of R, $a'R' \cap b'R' = (a'R \cap b'R)R' = a'b'R'$. But $a'/b' \in R'$, so $a' \in b'R'$ and hence $a'R' = a'R' \cap b'R' = a'b'R'$. This implies that $b'R' = R'$, so b' is a unit in R'. So $a'/b' \in R_S$ where $S = \{s \in R \mid s \text{ is a unit in } R'\}$. But $(a/b)^n = a^n/b^n = a'/b' \in R_S$. This shows that $R_S \subseteq R'$ is an integral extension. Since $R_S \subseteq R'$ is also a flat extension, we must have $R' = R_S$ [13, Theorem 4.15]. ∎

COROLLARY 3.6. *Let R be a PVMD with torsion t-class group. Then every flat overring of R is a localization of R.*

The converse of Corollary 3.6 is not true since even a Prüfer domain with the property that every (necessarily flat) overring is a localization need not have torsion ideal class group. The question of exactly which PVMDs have the property that every flat overring is a localization will be considered in a future paper by Evan Houston and the second author.

We end this section by correcting a remark made in [16]. Remark 3.10 stated that an AGCD-domain of characteristic 0 is integrally closed. This is not true. As we shall see (Theorem 4.17), the domain $Z + 2iZ = Z[2i]$ is an AGCD-domain of characteristic 0 that is not integrally closed.

4. ALMOST BÉZOUT DOMAINS

A GCD-domain R is characterized by the property that for $a, b \in R - \{0\}$, $(a, b)_v$ is principal, while a Bézout domain R is characterized by the stronger property that for $a, b \in R - \{0\}$, (a, b) is principal. Since an AGCD-domain R is defined by the property that for $a, b \in R - \{0\}$, $(a^n, b^n)_v$ is principal for some n, it seems reasonable to make a corresponding extension of the definition of a Bézout domain to define an almost Bézout domain.

DEFINITION 4.1. An integral domain R is an *almost Bézout* (AB-)*domain* if for $a, b \in R - \{0\}$ there exists a positive integer $n = n(a, b)$ such that (a^n, b^n) is principal while R is an *almost Prüfer* (AP-)*domain* if for $a, b \in R - \{0\}$ there exists a positive integer $n = n(a, b)$ such that (a^n, b^n) is invertible.

DEFINITION 4.2. Let R be an integral domain. R is called an *almost* *principal ideal* (API-)*domain* if for any nonempty subset $\{a_\alpha\} \subseteq R - \{0\}$, there exists an $n = n(\{a_\alpha\})$ with $(\{a_\alpha^n\})$ principal. R is called an *AD-domain* if for any nonempty subset $\{a_\alpha\} \subseteq R - \{0\}$, there exists an $n = n(\{a_\alpha\})$ with $(\{a_\alpha^n\})$ invertible.

We have avoided using the term "almost Dedekind" for an AD-domain because the term almost Dedekind is already used to mean an integral domain that is locally Dedekind. The next lemma shows that we could have defined an AB-domain by a formally stronger property.

LEMMA 4.3. *An integral domain R is an AB-domain (respectively, AP-domain) if and only if for $a_1, ..., a_s \in R - \{0\}$, there exists an $n = n(a_1, ..., a_s)$ with $(a_1^n, ..., a_s^n)$ principal (respectively, invertible).*

Proof. (1) *AB-domain case.* ($\Leftarrow$) Clear. ($\Rightarrow$) Assume $s > 2$. By induction there exists an n with $(a_1^n, ..., a_{s-1}^n) = (e)$ and an m with $(a_{s-1}^m, a_s^m) = (f)$ for some $e, f \in R$. Then $(e^m) = (a_1^n, ..., a_{s-1}^n)^m = (a_1^{nm}, ..., a_{s-1}^{nm})$ and $(f^n) = (a_{s-1}^{nm}, a_s^{nm})$. Choose l with $((e^m)^l, (f^n)^l)$ principal. Then $(a_1^{nml}, ..., a_s^{nml}) = (a_1^{nml}, ..., a_{s-1}^{nml}) + (a_{s-1}^{nml}, a_s^{nml}) = (a_1^{nm}, ..., a_{s-1}^{nm})^l + (a_{s-1}^{nm}, a_s^{nm})^l = (e^{ml}, f^{nl})$ is principal.

(2) AP-*domain case.* ($\Leftarrow$) Clear. ($\Rightarrow$) Again the proof is by induction on s. As in the proof of the AB-domain case, there is a t with $A = (a_1^t, ..., a_{s-1}^t)$, $B = (a_2^t, ..., a_s^t)$, and $D = (a_1^t, a_s^t)$ invertible. Put $E = a_1^t A^{-1} D^{-1} + a_s^t B^{-1} D^{-1}$. It is easily seen that $(a_1^t, ..., a_s^t) E = R$, so $(a_1^t, ..., a_s^t)$ is invertible. (This proof is essentially (2) $\Rightarrow$ (1) of [13, Theorem 6.6]. ∎

Certainly an AB-domain is an AGCD-domain and hence has torsion t-class group. The next result gives the exact relationship between AB-domains and AP-domains.

LEMMA 4.4. *Let R be an integral domain. Then the following conditions are equivalent.*

(1) *R is an AB-domain (respectively, API-domain).*

(2) *R is an AP-domain (respectively, AD-domain) with torsion t-class group.*

(3) *R is an AP-domain (respectively, AD-domain) with torsion class group.*

Proof. We only do the AB-domain case. The proof of the API-domain case is similar. (1) $\Rightarrow$ (2). Since an AB-domain is an AGCD-domain, it has torsion t-class group by Theorem 3.4. (2) $\Rightarrow$ (3). Clear. (3) $\Rightarrow$ (1). Let $a, b \in R - \{0\}$. By hypothesis, there is an n with (a^n, b^n) invertible. Since the class group is torsion, there is an l with $(a^n, b^n)^l = (a^{nl}, b^{nl})$ principal. Hence R is AB-domain. ∎

It is well known that an overring of a Bézout domain is a Bézout domain (e.g., [5]). The same is true of AB-domains. (We are thankful to Evan Houston for this observation.)

LEMMA 4.5. *Let R be an AB-domain (respectively, AP-domain) and let S be an overring of R. Then S is an AB-domain (respectively, AP-domain).*

Proof. Suppose that R is an AB-domain. Let $x, y \in S$. Then there exists a $0 \neq r \in R$ with $rx, ry \in R$. Then for some n, $((rx)^n, (ry)^n) R$ is principal. Hence $((rx)^n, (ry)^n) S$ is principal. Since $((rx)^n, (ry)^n) S = r^n(x^n, y^n) S$ is principal, so is $(x^n, y^n) S$. Hence S is an AB-domain.

Suppose that R is an AP-domain. The above proof remains valid if "principal" is replaced by "invertible." ∎

If R is an AGCD-domain, then $\bar{R}$ is an AGCD domain and $R \subseteq \bar{R}$ is a root extension. A similar statement holds for AB-domains; in fact, its converse is true too.

THEOREM 4.6. *Let R be an integral domain and S an overring with $R \subseteq S \subseteq \bar{R}$. Then R is an AB-domain (respectively, AP-domain) if and only if S is an AB-domain (respectively, AP-domain) and for each $s \in S$, there exists an $n = n(s)$ with $s^n \in R$.*

Proof. $(\Rightarrow)$ Suppose that R is either an AB-domain or an AP-domain. It follows from Lemma 4.5 that S is either an AB-domain or an AP-domain. The fact that $R \subseteq S$ is a root extension follows from Proposition 3.2.

$(\Leftarrow)$ Suppose that S is an AP-domain. Let $a, b \in R - \{0\}$. Then there exists an n with $(a^n, b^n) S$ invertible. Hence there is an ideal C of S with $(a^n, b^n) SC = zS$ for some $0 \neq z \in S$. Here C is necessarily invertible and hence is finitely generated, say $C = (c_1, ..., c_k)$ where $c_i \in S$. (In the case where S is an AB-domain, we may take $k = 1$ and $C = (1)$.) Now each $a^n c_i = zd_{1i}$, $b^n c_i = zd_{2i}$ for some $d_{1i}, d_{2i} \in S$, $i = 1, ..., k$. Now there exists an l with $z^l, c_i^l, d_{1i}^l, d_{2i}^l \in R$, $i = 1, ..., k$. Hence $a^{nl} c_i^l = z^l d_{1i}^l$ and $b^{nl} c_i^l = z^l d_{2i}^l$, so $(a^{nl}, b^{nl}) R(c_1^l, ..., c_k^l) R \subseteq z^l R$. Hence $(a^{nl}, b^{nl}) R((c_1^l, ..., c_k^l)) R = z^l A$ for some ideal A of R. Now $z^l AS = (a^{nl}, b^{nl}) S(c_1^l, ..., c_k^l) S = ((a^n, b^n) S)^l ((c_1, ..., c_k) S)^l = z^l S$ since $(a^n, b^n) S$ and $(c_1, ..., c_k) S$ are invertible. Hence $AS = S$. Since $R \subseteq S$ is an integral extension, $A = R$. So $(a^{nl}, b^{nl})(c_1^l, ..., c_k^l) = z^l R$. Thus (a^{nl}, b^{nl}) is invertible. Hence R is an AP-domain. (In the case where S is an AB-domain, taking $C = (1)$ gives $(a^{nl}, b^{nl}) = z^l R$.) ∎

Integrally closed AB-domains are easily characterized. Recall that an integral domain R is said to be root closed if for $x \in K$, the quotient field of R, $x^n \in R$ for some positive integer n implies that $x \in R$.

THEOREM 4.7. *Let R be an integral domain. Then the following statements are equivalent.*

(1) *R is an integrally closed AB-domain (respectively, AP-domain).*

(2) *R is a root closed AB-domain (respectively, AP-domain).*

(3) *R is a Prüfer domain with torsion class group (respectively, Prüfer domain).*

Proof. (1) $\Rightarrow$ (2). Clear. (2) $\Rightarrow$ (3). Let R be an AP-domain that is root closed. Let M be a maximal ideal of R. It suffices to show that R_M is a valuation domain. Now R_M is a quasi-local AP-domain. So for $a, b \in R_M - \{0\}$ some $(a^n, b^n) R_M$ is principal. Hence a^n/b^n or $b^n/a^n \in R_M$ since R_M is quasi-local. Since R_M is root closed, a/b or $b/a \in R_M$. Hence R_M is a valuation domain. If further R is an AB-domain, then R has torsion ideal class group by Lemma 4.4. (3) $\Rightarrow$ (1). Clear. ∎

COROLLARY 4.8. (1) *R is an AB-domain if and only if $\bar{R}$ is a Prüfer domain with torsion class group and $R \subseteq \bar{R}$ is a root extension.*

(2) *R is an AP-domain if and only if $\bar{R}$ is a Prüfer domain and $R \subseteq \bar{R}$ is a root extension.*

Corollary 4.8 gives a satisfactory characterization of AB-domains and AP-domains. As an application of Corollary 4.8, we next give a general method for constructing new AB-domains from old ones.

THEOREM 4.9. *Let D be an integral domain with quotient field K. Then D is an AB-domain (respectively, AP-domain) if and only if $R = D + XK[X]$ is an AB-domain (respectively, AP-domain).*

Proof. $R = D + XK[X]$ has quotient field $K(X)$. Also, $\bar{R} = \bar{D} + XK[X]$ where $\bar{D}$ is the integral closure of D in K. It easily follows that $D \subseteq \bar{D}$ is a root extension if and only if $R \subseteq \bar{R}$ is a root extension. Also, it is well known (for example, see [6]) that $\bar{D}$ is Prüfer if and only if $\bar{D} + XK[X]$ is Prüfer. By Corollary 4.8(2), R is an AP-domain if and only if $R + XK[X]$ is an AP-domain.

Suppose that $\bar{D}$ is a Prüfer domain. Every ideal of $\bar{D} + XK[X]$ has the form $f(X)(A + XK[X])$ for some ideal A of $\bar{D}$ [6]. It easily follows that $C(\bar{D})$ and $C(\bar{R})$ are isomorphic. Hence $C(\bar{D})$ is torsion if and only if $C(\bar{R})$ is torsion; so D is an AB-domain if and only if R is an AB-domain. ∎

Actually, the fact that $D+XK[X]$ is an AB-domain implies that D is an AB-domain follows since D is a homomorphic image of $D+XK[X]$ and our next theorem which states that the homomorphic image of an AB-domain is an AB-domain. This result could have been stated right after Definition 4.2.

THEOREM 4.10. *Let R be an integral domain. Let P be a prime ideal of R. If R is an AB-domain, then R/P is an AB-domain. Similar statements hold for AP-domains, API-domains, and AD-domains.*

Proof. Let $x, y \in R/P$. Then there exist $a, b \in R$ with $\bar{a}=x$ and $\bar{b}=y$. Now R is an AB-domain, so (a^n, b^n) is principal for some n. Hence $(x^n, y^n)=(\bar{a}^n, \bar{b}^n)=\overline{(a^n, b^n)}$ is principal. ∎

We have shown that R is an AB-domain if and only if $\bar{R}$ is an AB-domain and $R \subseteq \bar{R}$ is a root extension. It seems reasonable to conjecture that R is an API-domain if and only if $\bar{R}$ is an API-domain and $R \subseteq \bar{R}$ is a root extension. Unfortunately, this conjecture is false as is seen by Example 4.14. However, the following more restrictive result is true.

THEOREM 4.11. *Let R be an integral domain and S an overring with $R \subseteq S \subseteq \bar{R}$. Suppose that there exists a fixed positive integer n where $s^n \in R$ for each $s \in S$. Then R is an API-domain (respectively, AD-domain) if and only if S is an API-domain (respectively, AD-domain).*

Proof. ($\Rightarrow$) Suppose that R is an API-domain. Let $\{s_\alpha\} \subseteq S-\{0\}$. Then $\{s_\alpha^n\} \subseteq R-\{0\}$. So there is an integer $k>0$ with $\{s_\alpha^{nk}\}\, R$ principal. Hence $\{s_\alpha^{nk}\}\, S$ is also principal. The same proof with "principal" replaced by "invertible" shows that if R is an AD-domain, then S is also an AD-domain.

($\Leftarrow$) Let $\{x_\alpha\} \subseteq R-\{0\}$. Now S is an API-domain, so $(\{x_\alpha^k\})\, S=zS$ for some integer $k>0$ and $z \in S$. Now $x_\alpha^k=s_\alpha z$ for some $s_\alpha \in S$. Hence $x_\alpha^{nk}=s_\alpha^n z^n$ where $s_\alpha^n, z^n \in R$. So $(\{x_\alpha^{kn}\})\, R=(\{z^n s_\alpha^n\})\, R=z^n(\{s_\alpha^n\})\, R$. Now $z^n S=(zS)^n=((\{x_\alpha^k\})\, S)^n=(\{x_\alpha^{nk}\})\, S=z^n(\{s_\alpha^n\})\, S$; so $(\{s_\alpha^n\})\, S=S$. Since $R \subseteq S$ is integral, we must have $(\{s_\alpha^n\})\, R=R$. Hence $(\{x_\alpha^{nk}\})\, R=z^n R$. Thus R is an API-domain. A similar proof shows that if S is an AD-domain, then R is an AD-domain. ∎

While Theorem 4.11 may be viewed as the API-domain analog of Theorem 4.6, it should be noted that in the proof of Theorem 4.11 we have not used the hypothesis that S is contained in the quotient field of R. In fact, the following result is true and its proof follows along the same lines as the proof of Theorem 4.11. Let $R \subseteq S$ be an extension of commutative rings. Suppose that there is a natural number n with the property that $s \in S$

implies that $s^n \in R$. Then R is an API-ring (respectively, AD-ring) if and only if S is an API-ring (respectively, AD-ring). Theorem 4.6 also has a similar extension. Let $R \subseteq S$ be a root extension of commutative rings. Then R is an AB-ring (respectively, AP-ring) if and only if S is an AB-ring (respectively, AP-ring).

The API-domain analog of Theorem 4.7 does carry over.

THEOREM 4.12. *For an integral domain R the following statements are equivalent.*

(1) *R is an integrally closed API-domain (respectively, AD-domain).*

(2) *R is a root closed API-domain (respectively, AD-domain).*

(3) *R is a Dedekind domain with torsion class group (respectively, Dedekind domain).*

Proof. (1) $\Rightarrow$ (2). Clear. (2) $\Rightarrow$ (3). Suppose that R is a root closed API-domain. Let A be an ideal of R, say $A = \{a_\alpha\}$. Then for some n, $(\{a_\alpha^n\})$ is principal. Let M be a maximal ideal of R. Then $(\{a_\alpha^n\})_M$ is principal and is in fact generated by some $a_{\alpha_0}^n$. So for each α, $a_\alpha^n/a_{\alpha_0}^n \in R_M$. Since R_M is root closed, each $a_\alpha/a_{\alpha_0} \in R_M$. Hence $A_M = (\{a_\alpha\})_M = (a_{\alpha_0})_M$ is principal and $A_M^n = (a_{\alpha_0}^n)_M = (\{a_\alpha^n\})_M$. Since the equality $A_M^n = (\{a_\alpha^n\})_M$ holds for each maximal ideal M, we have $A^n = (\{a_\alpha^n\})$ and hence A^n is principal. Thus A is invertible. Hence R is a Dedekind domain with torsion class group. A similar proof shows that if R is an AD-domain, then R is a Dedekind domain. (3) $\Rightarrow$ (1). Suppose that R is a Dedekind domain and let $\{x_\alpha\} \subseteq R - \{0\}$. Then $(\{x_\alpha\})$ is invertible, so R is an AD-domain. Suppose that $C(R)$ is torsion. Then there exists an n with $(\{x_\alpha\})^n$ principal. But then $(\{x_\alpha^n\}) = (\{x_\alpha\})^n$ is principal, so R is an API-domain. ∎

COROLLARY 4.13. *Let R be an integral domain. Suppose that there exists an n so that $x \in \bar{R}$ implies that $x^n \in R$. Then R is an API-domain if and only if $\bar{R}$ is a Dedekind domain with torsion class group. R is an AD-domain if and only if $\bar{R}$ is a Dedekind domain.*

Proof. Combine Theorems 4.11 and 4.12. ∎

EXAMPLE 4.14. Let K be a field of characteristic $p > 0$ and let $K \subseteq L$ be a purely inseparable field extension. Let $R = K + XL[X]$. Then the integral closure $\bar{R}$ of R in its quotient field $L(X)$ is $L[X]$, a PID. For each $f \in L[X]$, there is an $m \geqslant 0$ with $f^{p^m} \in R$. So R is an AB-domain. However, R is an API-domain if and only if there exists a fixed m with $L^{p^m} \subseteq K$. In particular, if $K = Z_p(T)$ and $L = \bigcup_{n=1}^{\infty} Z_p(T^{1/p^n})$, then $R = K + XL[X]$ has the property that $\bar{R}$ is a PID and $R \subseteq \bar{R}$ is a root extension, but R is *not* an API-domain.

Proof. If $L^{p^m} \subseteq K$, then $f^{p^m} \in R$ for each $f \in \bar{R} = L[X]$. So by Theorem 4.11, R is an API-domain. Suppose that R is an API-domain. Consider the set $\{lX \mid l \in L\}$. Then there exists an n with $\{l^n X^n\} R$ finitely generated, say $\{l^n X^n\} R = \{l_1^n X^n, ..., l_s X^n\} R$ for some $l_1, ..., l_s \in L$. But then for each $l \in L$, $l^n X^n = f_1 l_1^n X^n + \cdots + f_s l_s^n X^n$ where $f_1, ..., f_s \in R$. Equating coefficients, gives that $l^n = f_1(0) l_1^n + \cdots + f_s(0) l_s^n \in K(l_1^n, ..., l_s^n)$. But then for some fixed n, $l^n \in F$ for each $l \in L$ where F is a finite field extension of K contained in L. Now for $l \in L$, some $l^{p^t} \in F$ and $l^n \in F$. Let $k = \mathrm{GCD}(n, p^t) = p^{t_1}$ where $0 \leqslant t_1 \leqslant t$. Then $l^k \in F$. Hence if we choose r with $p^r \geqslant n$, then $l^{p^r} \in F$ for each $l \in L$. Since $[F : K] < \infty$, we actually have $l^{p^m} \in K$ for some m. ∎

Here is another example of an API-domain in characteristic $p > 0$. In fact, it was this example which motivated the definition of an AB-domain.

EXAMPLE 4.15. Let F be a field of characteristic $p \geqslant 0$ and let $R = F[[\{X^s \mid s \in S\}]]$ where S is a primitive numerical monoid (i.e., S is an additive submonoid of the nonnegative integers under addition and $\mathrm{GCD}\{S\} = 1$). Hence there exists an n with $m \in S$ for $m \geqslant n$ [10, Theorem 2.4]. So $R \supseteq F[[X^n, X^{n+1}, ...]]$ and hence $\bar{R} = F[[X]]$ is a PID. Suppose that char $F = p > 0$. Let $f = \sum_{i=0}^{\infty} a_i X^i \in F[[X]]$. Then $f^{p^n} = \sum_{i=0}^{\infty} a_i^{p^n} X^{ip^n} \in F[[X^n, X^{n+1}, ...]] \subseteq R$ because $p^n \geqslant n$. By Corollary 4.13, R is an API-domain.

On the other hand, suppose that char $F = 0$. If R is an AB-domain, then there exists an m with $(1+X)^m \in R$. But $1 + mX + \cdots = (1+X)^m \in R$ implies that $1 \in S$, so $R = F[[\{X^s \mid x \in S\}]] = F[[X]]$.

In [16] it was incorrectly stated that an AGCD-domain of characteristic 0 is integrally closed. So far, all our non-integrally closed examples have been in characteristic $p > 0$. The next example is perhaps the simplest example of a non-integrally closed API-domain of characteristic 0. While Example 4.16 is essentially a special case of Theorem 4.17, we have included it due to its simplicity.

EXAMPLE 4.16. Let $R_n = Z + 2^n Zi$ where $i = \sqrt{-1}$. Then for $a + bi \in Z[i]$, $(a+bi)^{2^n} \in R_n$ since $2^n \mid \binom{2^n}{j}$ for j odd. Since $Z[i]$ is a PID, R_n is an API-domain. Note that char $R_n = 0$ and that for $n \geqslant 1$, R_n is *not* integrally closed.

In the preceding example, $Z[2i] = Z + 2iZ$ is a non-integrally closed API-domain while $Z[i]$ is of course a PID. This raises the interesting question of when $Z[2\sqrt{m}]$ or $Z[\sqrt{m}]$ is an API-domain. The next theorem completely answers this question.

THEOREM 4.17. *Let m be a square-free integer.*

(1) *If* $m \equiv 2, 3 \bmod 4$, $Z[\sqrt{m}]$ *is a Dedekind domain with finite class group and hence is an API-domain.*

(2) *If* $m \equiv 5 \bmod 8$, $Z[\sqrt{m}]$ *is not integrally closed, but* $\overline{Z[\sqrt{m}]} = Z + [(1+\sqrt{m})/2]\,Z$ *is a Dedekind domain with finite class group and* $x^3 \in Z[\sqrt{m}]$ *for each* $x \in \overline{Z[\sqrt{m}]}$. *So* $Z[\sqrt{m}]$ *is an API-domain.*

(3) *If* $m \equiv 1 \bmod 8$, $\overline{Z[\sqrt{m}]} = Z + [(1+\sqrt{m})/2]\,Z$ *is a Dedekind domain with finite class group, but* $[(1+\sqrt{m})/2]^n \notin Z[\sqrt{m}]$ *for all* $n \geqslant 1$, *so* $Z[\sqrt{m}]$ *is not an API-domain (in fact, not even an AP-domain).*

(4) $Z[2\sqrt{m}] = Z + 2\sqrt{m}Z$ *is* not *integrally closed, but is an API-domain if and only if* $m \not\equiv 1 \bmod 8$.

Proof. (1) It is well known that $\overline{Z[\sqrt{m}]}$ is a Dedekind domain with finite class group and hence is an API-domain by Theorem 4.12. It is also well known that for $m \equiv 2, 3 \bmod 4$, $\overline{Z[\sqrt{m}]} = Z[\sqrt{m}]$ while for $m \equiv 1 \bmod 4$, $\overline{Z[\sqrt{m}]} = Z + [(1+\sqrt{m})/2]\,Z$. So (1) follows.

(2) Suppose that $m \equiv 5 \bmod 8$, so $\overline{Z[\sqrt{m}]} = Z + [(1+\sqrt{m})/2]\,Z$. Hence $Z[\sqrt{m}]$ is not integrally closed. Let $x = a + b[(1+\sqrt{m})/2] \in \overline{Z[\sqrt{m}]}$. Then $x^3 = (a^3 + 3ab(2a + b(1+m))/4 + b^3(1+3m)/8) + ((3a^2b + 3ab^2)/2 + b^3(3+m)/8)\sqrt{m}$. Now $1+m$ is even, so $2a + b(1+m) = 2(a + b((m+1)/2))$ where $(m+1)/2$ is odd. If a or b is even, the product is divisible by 2 while if a and b are both odd, $a + b(m+1)/2$ is even. In either case, $ab(2a + b(1+m))$ is divisible by 4, so $3ab(2a + b(1+m))/4 \in Z$. Since $m \equiv 5 \bmod 8$, $1 + 3m \equiv 0 \bmod 8$, so $b^3(1+3m)/8 \in Z$. So the first quantity in parentheses is in Z. Now $(3a^2b + 3ab^2)/2 = 3ab((a+b)/2) \in Z$ since if a and b are both odd, then $a+b$ is even. Also, $m + 3 \equiv 5 + 3 \equiv 0 \bmod 8$, so $b^3(3+m)/8 \in Z$. Hence $x^3 \in Z[\sqrt{m}]$. By Corollary 4.13, $Z[\sqrt{m}]$ is an API-domain.

(3) We show that for $m \equiv 1 \bmod 8$, $Z[\sqrt{m}] \subsetneq \overline{Z[\sqrt{m}]}$ is *not* a root extension. To do this, it suffices to show that $[(1+\sqrt{m})/2]^n \notin Z[\sqrt{m}]$ for all $n \geqslant 1$. Put $[(1+\sqrt{m})/2]^n = (a_n + b_n\sqrt{m})/2$ where $a_n, b_n \in Z$. It suffices to prove the following.

CLAIM. $a_n \equiv b_n \equiv 1$ *or* $3 \bmod 4$.

Proof by induction on n; the case $n = 1$ being certainly true. Now

$$\frac{a_{n+1} + b_{n+1}\sqrt{m}}{2} = \left[\frac{1+\sqrt{m}}{2}\right]\left[\frac{a_n + b_n\sqrt{m}}{2}\right]$$
$$= \frac{[(a_n + mb_n)/2 + [(a_n + b_n)/2]\sqrt{m}}{2}$$

(by induction, $a_n + mb_n$ and $a_n + b_n$ are both even!). So $a_{n+1} = (a_n + mb_n)/2$ and $b_{n+1} = (a_n + b_n)/2$. Since $m \equiv 1 \mod 8$, $m = 8k+1$, so $a_{n+1} = (a_n + (8k+1)\, b_n)/2 = (a_n + b_n)/2 + 4kb_n \equiv (a_n + b_n)/2 \equiv b_{n+1} \mod 4$. Suppose that $b_{n+1} \equiv 0$ or $2 \mod 4$. Then $(a_n + b_n)/2 \equiv 0, 2 \mod 4$, so $a_n + b_n \equiv 0 \mod 4$. But this is impossible, since $a_n \equiv b_n \equiv 1$ or $3 \mod 4$.

(4) $Z[2\sqrt{m}] = Z + 2\sqrt{m}\,Z$ is *not* integrally closed since $\sqrt{m} \notin Z[2\sqrt{m}]$ but $(\sqrt{m})^2 = m \in Z[2\sqrt{m}]$. Suppose that $Z[2\sqrt{m}]$ is an AB-domain, then the overring $Z[\sqrt{m}]$ is also an AB-domain, so $m \not\equiv 1 \mod 8$. Conversely, suppose that $m \not\equiv 1 \mod 8$. Then $Z[\sqrt{m}]$ is an API-domain. However, for $x \in Z[\sqrt{m}]$, $x^2 \in Z[2\sqrt{m}]$. Thus by Theorem 4.11, $Z[2\sqrt{m}]$ is also an API-domain. ∎

We have studied AB-domains and AP-domains via the invertibility of certain ideals. Prüfer domains are also characterized by the property that they are locally valuation domains. The analog of this characterization of AP-domains is given in the next section (Theorem 5.8). Some more results on AB-domains are also given in the next section. There are many other questions concerning AB-domains and API-domains that we have left unanswered.

5. More on AGCD-Domains and AB-Domains

In this section we will show that a prime ideal P of an AGCD-domain R is a t-ideal if and only if R_P is an AB-domain. We show that an AGCD-domain R is an AB-domain if and only if Spec(R) is treed. We also show that R is an AP-domain if and only if R_P is an AV-domain for each maximal ideal P of R. Here by an AV-domain we mean an integral domain R with the property that for $a, b \in R - \{0\}$, $a^n \mid b^n$ or $b^n \mid a^n$ for some n. We end this section by considering the question of when $\bar{R}$ an AGCD-domain implies that R is an AGCD-domain.

An integral domain R will be called *t-local* if R has a unique maximal t-ideal. It is easily seen that R is t-local if and only if R has a unique maximal ideal M and M is a t-ideal. The *t-dimension* of an integral domain R is the length of the longest chain of prime t-ideals. It is easily seen that an integral domain R has t-dimension one if and only if every prime t-ideal is minimal. Of course, a minimal prime ideal is always a t-ideal.

Lemma 5.1. *Let R be a t-local AGCD-domain. Then R is an AB-domain.*

Proof. Let $x, y \in R - \{0\}$. Then for some n, $(x^n, y^n)_v = (d)$ where $d \in R$. Hence $(x^n/d, y^n/d)_v = R$. Since R is t-local, either x^n/d or y^n/d must be a unit. But this amounts to $x^n \mid y^n$ or $y^n \mid x^n$, so (x^n, y^n) is principal. ∎

In [17, 18] it was shown that if P is a prime t-ideal of an integral domain R it is not necessary that P_P should also be a prime t-ideal of R_P. However, this can not happen for an AGCD-domain as shown by the next lemma.

LEMMA 5.2. *Let R be an AGCD-domain. Let P be a prime t-ideal of R. Then P_P is a prime t-ideal of R_P.*

Proof. Suppose that P_P is *not* a t-ideal of R_P. Then there exist $x_1, ..., x_n \in P - \{0\}$ with $((x_1, ..., x_n)_P)_v = R_P$. Hence $((x_1^m, ..., x_n^m)_P)_v = R_P$ for all $m \geqslant 1$ by Lemma 3.3. But since R is an AGCD-domain, there is an m with $(x_1^m, ..., x_n^m)_v = (d)$. Since $x_1^m, ..., x_n^m \in P$ and P is a t-ideal, we must have $d \in P$. But then $(x_1^m, ..., x_n^m)_P \subseteq (d)_P \subseteq P_P$. Hence $R_P = ((x_1^m, ..., x_n^m)_P)_v \subseteq P_P$, a contradiction. ∎

THEOREM 5.3. *Let R be an AGCD-domain. Let P be a nonzero prime ideal of R. Then P is a t-ideal if and only if R_P is an AB-domain.*

Proof. $(\Rightarrow)$ Suppose that P is a prime t-ideal. By Lemma 5.2, P_P is a t-ideal of R_P. But (R_P, P_P) is a t-local AGCD-domain and hence by Lemma 5.1 is an AB-domain.

$(\Leftarrow)$ Suppose that R_P is an AB-domain. Then $\bar{R}_P$ is a Prüfer domain and hence Spec$(\bar{R}_P)$ is treed. By Corollary 2.2, Spec(R_P) is also treed. Let $0 \neq x \in P$. Shrink P to a prime ideal P_x minimal over (x). Then P_x is a t-ideal. Hence $P = \bigcup P_x$ is also a t-ideal. ∎

Theorem 5.3 generalizes the well-known result that a prime ideal P of a GCD-domain R is a t-ideal if and only if R_P is a valuation domain.

COROLLARY 5.4. *Let R be an AGCD-domain. Then the following statements are equivalent.*

(1) *R is an AB-domain.*

(2) *Every prime ideal of R is a t-ideal.*

(3) *Every maximal ideal of R is a t-ideal.*

(4) Spec(R) *is treed.*

Proof. $(1) \Rightarrow (2)$. Theorem 5.3. $(2) \Rightarrow (3)$. Clear. $(3) \Rightarrow (4)$. Let M be a maximal ideal of R. Then M is a t-ideal, so by Theorem 5.3, R_M is an AB-domain. Hence as in the proof of $(\Leftarrow)$ of Theorem 5.3, Spec(R_M) is treed. Hence Spec(R) itself is treed. $(4) \Rightarrow (1)$. Suppose that Spec(R) is treed. But then Spec$(\bar{R})$ is also treed by Corollary 2.2. So $\bar{R}$ is a PVMD with Spec(R) treed, hence $\bar{R}$ is a Prüfer domain and $\bar{R}$ has torsion $(t-)$ class group. By Corollary 4.8, R is an AB-domain.

P. M. Cohn [5] called an integral domain R pre-Bézout if in R coprime elements are comaximal. He showed that R is Bézout if and only if R is a GCD-domain and is pre-Bézout. Let us call R a v-pre-Bézout domain if for $x, y \in R - \{0\}$, $(x, y)_v = R$ implies that $(x, y) = R$. Then R is an AB-domain if and only if R is an AGCD-domain and is v-pre-Bézout. The proof is straightforward. Thus while the v-pre-Bézout condition is much weaker than the pre-Bézout condition, it sometimes still implies the same results. However, while any t-local Noetherian domain is v-pre-Bézout, it was established in [14] that a pre-Bézout Noetherian domain must be a PID.

One of the many characterizations of Prüfer domains is that Prüfer domains are locally valuation domains. Similarly almost Prüfer domains may be characterizing using almost valuation domains.

DEFINITION 5.5. Let R be an integral domain. R is an *almost valuation domain* (AV-*domain*) if for $a, b \in R - \{0\}$, there exists an $n = n(a, b)$ with a^n/b^n or b^n/a^n.

Equivalently, R is an almost valuation domain if and only if for each $x \in K - \{0\}$, there exists an $n = n(x) \geqslant 1$ with x^n or $x^{-n} \in R$. It is easily seen that if R is an AV-domain and S is an overring of R, then S is also an AV-domain. Also, given a root extension $R \subseteq S$ (where S need not be contained in the quotient field of R), R is an AV-domain if and only if S is an AV-domain. If R is an AV-domain with quotient field K and L is a subfield of K, then $R \cap L$ is an AV-domain with quotient field L. The next theorem gives several characterizations of AV-domains.

THEOREM 5.6. *For an integral domain R the following conditions are equivalent.*

(1) *R is an AV-domain.*

(2) *$\bar{R}$ is a valuation domain and $R \subseteq \bar{R}$ is a root extension.*

(3) *R is a t-local AGCD-domain.*

(4) *R is a quasi-local AB-domain.*

Proof. (1) $\Rightarrow$ (2). Now R is an AB-domain, so $R \subseteq \bar{R}$ is a root extension by Theorem 4.6. Let $x \in K - \{0\}$. Then there exists an n with $x^n \in R$ or $x^{-n} \in R$. Hence x^n or $x^{-n} \in \bar{R}$. Since $\bar{R}$ is integrally closed, x or $x^{-1} \in \bar{R}$. Hence $\bar{R}$ is a valuation domain. (2) $\Rightarrow$ (3). By Corollary 4.8, R is an AB-domain and hence an AGCD-domain. Since $R \subseteq \bar{R}$ is a root extension, Spec(R) and Spec($\bar{R}$) are homeomorphic (Theorem 2.1). Hence R is quasi-local. By Theorem 5.3, the maximal ideal of R is a t-ideal. Hence R is t-local. (3) $\Rightarrow$ (4). By Lemma 5.1, R is an AB-domain. And certainly a t-local domain is quasi-local. (4) $\Rightarrow$ (1). Let $a, b \in R - \{0\}$. Then there exists an n with (a^n, b^n) principal. Since R is quasi-local, $(a^n, b^n) = (a^n)$ or $(a^n, b^n) = (b^n)$. So $a^n \mid b^n$ or $b^n \mid a^n$. Thus R is an AV-domain. ∎

The following technical lemma is needed to show that a locally AV-domain is an AP-domain.

LEMMA 5.7. *Let R be an integral domain. Let $x, y \in R - \{0\}$. Suppose that for each maximal ideal M of R, there exists a natural number n_M with $(x^{n_M}, y^{n_M})_M$ principal. Then there exists an N with (x^N, y^N) invertible.*

Proof. Let M be a maximal ideal of R. Now $(x^{n_M}, y^{n_M})_M$ is principal, so $(x^{n_M}, y^{n_M})_M = aR_M$ for some $a_M \in (x^{n_M}, y^{n_M})$. In fact, we can take $a_M = x^{n_M}$ or y^{n_M}. So there exists an $f_M \in R - M$ with $f_M(x^{n_M}, y^{n_M}) \subseteq a_M R$. Hence for $k \geqslant 1$, $f_M^k(x^{n_M k}, y^{n_M k}) \subseteq f_M^k(x^{n_M}, y^{n_M})^k \subseteq a_M^k R$. So $(x^{n_M k}, y^{n_M k}) R_{f_M} = a_M^k R_{f_M}$ for all $k \geqslant 1$. Since $R = (\{f_M \mid M$ is a maximal ideal of $R\})$, we have $R = (f_1, ..., f_m)$ for some finite set of maximal ideals $\{M_1, ..., M_m\}$ with $f_{M_i} = f_i \in R - M_i$. Put $N = n_{M_1} \cdots n_{M_m}$. Then $(x^N, y^N) R_{f_i} = a_{M_i}^{N/n_{M_i}} R_{f_i}$. Since for each maximal ideal M of R, we have some $f_i \notin M$, it follows that $(x^N, y^N)_M$ is principal. Thus (x^N, y^N) is finitely generated and locally principal, hence invertible. ∎

THEOREM 5.8. *Let R be an integral domain. Then R is an AP-domain if and only if for each maximal ideal M of R, R_M is an AV-domain.*

Proof. $(\Rightarrow)$ Suppose that R is an AP-domain. Let M be a maximal ideal of R. Then R_M is a quasi-local AP-domain, hence a quasi-local AB-domain and hence an AV-domain. $(\Leftarrow)$ Suppose that for each maximal ideal M of R, R_M is an AV-domain. Let $x, y \in R - \{0\}$. Then for each maximal ideal M of R, there is a natural number n_M with $(x^{n_M}, y^{n_M})_M$ principal. By Lemma 5.7, there is a natural number N with (x^N, y^N) invertible. Hence R is an AP-domain. ∎

We have seen that R is an AB-domain if and only if $\bar{R}$ is an AB-domain and $R \subseteq \bar{R}$ is a root extension. If R is an AGCD-domain, then $\bar{R}$ is an AGCD-domain and $R \subseteq \bar{R}$ is a root extension. We have been unable to prove the converse: if $R \subseteq \bar{R}$ is a root extension and $\bar{R}$ is an AGCD-domain, then R is an AGCD-domain. However, we do have the following result.

THEOREM 5.9. *An integral domain R is an AGCD-domain if and only if* (i) *$\bar{R}$ is an AGCD-domain,* (ii) *$R \subseteq \bar{R}$ is a root extension, and* (iii) *if $x_1, ..., x_n \in R - \{0\}$ with $((x_1, ..., x_n)\bar{R})_v = \bar{R}$, then $((x_1, ..., x_n)R)_v = R$.*

Proof. $(\Rightarrow)$ Suppose that R is an AGCD-domain. Then (i) and (ii) hold by Theorem 3.1. Suppose that $x_1, ..., x_n \in R - \{0\}$ with

$((x_1, ..., x_n)\, R)_v \neq R$. Then for some m $((x_1^m, ..., x_n^m)\, R)_v = (d)$ where $d \in R$ is necessarily a nonunit. Then $(x_1^m, ..., x_n^m)\, \bar{R} \subseteq d\bar{R} \neq \bar{R}$ where the last inequality follows since $R \subseteq \bar{R}$ is integral. So $((x_1^m, ..., x_n^m)\, \bar{R})_v \neq \bar{R}$. It follows from Lemma 3.3 that $((x_1, ..., x_n)\, \bar{R})_v \neq \bar{R}$. ($\Leftarrow$) Suppose that (i), (ii), and (iii) hold. Let $x, y \in R - \{0\}$. Then there exists an $n \geqslant 1$ and a $d \in \bar{R}$ with $((x^n, y^n)\, \bar{R})_v = d\bar{R}$. Now x^n/d, $y^n/d \in \bar{R}$, so there exists an l with x^{nl}/d^l, $y^{nl}/d^l \in R$. But then $((x^{nl}/d^l, y^{nl}/d^l)\, \bar{R})_v = \bar{R}$. By (iii), $(x^{nl}/d^l, y^{nl}/d^l)_v = R$. Hence $(x^{nl}, y^{nl})_v = d^l R$. It follows that R is an AGCD-domain. ∎

We give condition (iii) a name.

DEFINITION 5.10. Let R be an integral domain and S an overring of R. We say that R is *t-linked under* S if whenever $x_1, ..., x_n \in R - \{0\}$ with $((x_1, ..., x_n)\, S)_v = S$, then $((x_1, ..., x_n)\, R)_v = R$.

Note that R t-linked under S is equivalent to if $x_1, ..., x_n \in R$ do not share a maximal t-ideal in $\bar{R}$, then $x_1, ..., x_n$ do not share a maximal t-ideal in R. This is the converse of the notion of S being a t-linked overring of R that was introduced in [7]. There an overring S of R was defined to be t-linked if for each $A \in f(R)$ with $A_v = R$, then $(AS)_v = S$. It was shown that S is t-linked over R if and only if for each prime t-ideal P of S, $(P \cap R)_t \neq R$. It is not known whether the integral closure $\bar{R}$ of a general integral domain R is t-linked over R. However, it is easily shown that if $R \subseteq \bar{R}$ is a root extension and $\bar{R}$ is an AGCD-domain, then $\bar{R}$ is t-linked over R. We end this section with the following result.

THEOREM 5.11. *Let R be an integral domain of t-dimension one. If $\bar{R}$ is an AGCD-domain and $R \subseteq \bar{R}$ is a root extension, then R is an AGCD-domain.*

Proof. We first show that if R is an integral domain of t-dimension one, then R is t-linked under $\bar{R}$. Let $x_1, ..., x_n \in R$ such that $((x_1, ..., x_n)\, \bar{R})_v = \bar{R}$. Suppose that $(x_1, ..., x_n)_v \neq R$. Then $x_1, .., x_n$ belong to a maximal t-ideal P of R. Let P' be a prime ideal of $\bar{R}$ lying over P. Since R has t-dimension one, rank $P = 1$. Hence rank $P' = 1$, so P' is a t-ideal. But then $((x_1, ..., x_n)\, \bar{R})_v \subseteq P' \neq \bar{R}$, a contradiction. Thus (i), (ii), and (iii) of Theorem 5.9 hold, so R is an AGCD-domain. ∎

6. IDEALS GENERATED BY POWERS OF ELEMENTS

In the previous sections we have been given elements of a ring R and have considered the ideal generated by powers of these elements. In this section we change our point of view slightly. Given an ideal I, instead of just a set of elements, we consider the ideal generated by all nth powers of elements of the ideal I.

DEFINITION 6.1. Let I be an ideal of a commutative ring R. For $n \geqslant 1$, define $I_n = (\{i^n \mid i \in I\})$.

I is *nearly principal* (respectively, *nearly invertible*) if for some $n \geqslant 1$, I_n is principal (respectively, invertible).

Certainly I_n is an ideal of R with $I_n \subseteq I^n$. If $I = (\{a_\alpha\})$, then $(\{a_\alpha^n\}) \subseteq I_n \subseteq I^n$. If I is locally principal, then $I^n = (\{a_\alpha^n\})$ and hence $I^n = I_n = (\{a_\alpha^n\})$. Of course, $I_1 = I^1$. As we shall see (Theorem 6.12), if R contains a field of characteristic 0, then $I^n = I_n$ for all $n \geqslant 1$. However, in $R = Z[X, Y]$, we have $(X^n, Y^n) \subsetneq (X, Y)_n \subsetneq (X, Y)^n$ for $n > 1$. In fact for $n = 2$, we have $(X^2, Y^2) \subsetneq (X^2, 2XY, Y^2) = (X, Y)_2 \subsetneq (X, Y)^2 = (X^2, XY, Y^2)$. For any ring R, it is easily seen that $(a, b)_2 = (a^2, 2ab, b^2)$.

Clearly if I is principal (respectively, invertible), then I is nearly principal (respectively, nearly invertible). Also, for I invertible, $I_n = I^n$ for $n \geqslant 1$, so I is nearly principal if and only if I^n is principal for some n, i.e., I is torsion in $C(R)$.

Using Definition 6.1 we can define some classes of integral domains closely related to the AB-domains and API-domains previously defined.

DEFINITION 6.2. Let R be an integral domain. Then R is *nearly Bézout* (respectively, *nearly Prüfer*) if for each finitely generated nonzero ideal I of R, I_n is principal (respectively, invertible) for some $n = n(I)$.

R is *nearly* PID (respectively, *nearly Dedekind*) if for each nonzero ideal I of R, I_n is principal (respectively, invertible) for some $n = n(I)$.

Alternative, R is nearly Bézout if every finitely generated ideal is nearly principal while R is nearly PID if every ideal is nearly principal.

We will show that for R root closed, the notions of almost Bézout and nearly Bézout coincide. Similar statements will hold for almost Prüfer domains, API-domains and AD-domains. This will follow from our investigation of $(I_n)_t$. Recall that an integral domain R is *n-root closed* if for $x \in K$, the quotient field of R, $x^n \in R$ implies $x \in R$. Certainly an integrally closed domain is n-root closed for each $n \geqslant 1$ (i.e., root closed).

LEMMA 6.3. *Let R be an integral domain that is n-root closed. Let $\phi \neq \{a_\alpha\} \subseteq R - \{0\}$. Then $(\{a_\alpha^n\})_v = ((\{a_\alpha\})^n)_v$.*

Proof. It suffices to show that $\{a_\alpha^n\} \subseteq d/bR$ implies that $(\{a_\alpha\})^n \subseteq d/bR$. For then $(\{a_\alpha\})^n \subseteq (\{a_\alpha^n\})_v$ and hence $(\{a_\alpha^n\})_v = ((\{a_\alpha\})^n)_v$. Now consider $a_{\alpha_1}, \ldots, a_{\alpha_r} \in \{a_\alpha\}$ and let $n = n_1 + \cdots + n_r$. Since $\{a_\alpha^n\} \subseteq d/bR$, each $ba_{\alpha_i}^n \in dR$, so $d \mid ba_{\alpha_i}^n$. So $d^{n_i} \mid b^{n_i} a_{\alpha_i}^{nn_i}$. Hence $d^n = d^{\Sigma n_i} \mid b^{\Sigma n_i} \prod (a_{\alpha_i}^n)^{n_i} =$

$b^n(a_{\alpha_1}^{n_1} \cdots a_{\alpha_r}^{n_r})^n$. So $(ba_{\alpha_1}^{n_1} \cdots a_{\alpha_r}^{n_r}/d)^n \in R$. Thus $ba_{\alpha_1}^{n_1} \cdots a_{\alpha_r}^{n_r}/d \in R$ since R is n-root closed. Hence $a_{\alpha_1}^{n_1} \cdots a_{\alpha_r}^{n_r} \in d/bR$, so that $(\{a_\alpha\})^n \subseteq d/bR$. ∎

COROLLARY 6.4. *Let R be an n-root closed integral domain. Then $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$ for any nonempty collection $\{a_\alpha\} \subseteq R - \{0\}$.*

Proof. Let $x \in ((\{a_\alpha\})^n)_t$. Then $x \in ((a_{\alpha_1}, ..., a_{\alpha_r})^n)_v$ for some $\{a_{\alpha_1}, ..., a_{\alpha_r}\} \subseteq \{a_\alpha\}$. By Lemma 6.3, $((a_{\alpha_1}, ..., a_{\alpha_r})^n)_v = (a_{\alpha_1}^n, ..., a_{\alpha_r}^n)_v \subseteq (\{a_\alpha^n\})_t$. Since the other containment is always true, we have equality. ∎

THEOREM 6.5. *Let R be an n-root closed integal domain and let $A = (\{a_\alpha\})$ be an ideal of R. Then $(\{a_\alpha^n\})_t = (A_n)_t = (A^n)_t$. Hence $(\{a_\alpha^n\})_v = (A_n)_v = (A^n)_v$.*

Proof. By Corollary 6.4, $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$. Since $A = (\{a_\alpha^n\})$, we have $(\{a_\alpha^n\})_t \subseteq (A_n)_t \subseteq (A^n)_t = ((\{a_\alpha\})^n)_t = (\{a_\alpha^n\})_t$ and the desired equality follows. ∎

COROLLARY 6.6. *Let R be an n-root closed integral domain and let $a_1, ..., a_r \in R - \{0\}$. Then $(a_1^n, ..., a_r^n)_t = ((a_1, ..., a_r)_n)_t = ((a_1, ..., a_r)^n)_t$.*

COROLLARY 6.7. *Suppose that R is an n-root closed integral domain. Let $A = (\{a_\alpha\})$. If $(\{a_\alpha^n\})$ is a t-ideal (e.g., $(\{a_\alpha^n\})$ is locally principal), then $(\{a_\alpha^n\}) = A_n = A^n$. If A_n is a t-ideal, then $A_n = A^n$.*

Proof. Suppose that $(\{a_\alpha^n\})$ is a t-ideal. Then $A^n \supseteq (\{a_\alpha^n\}) = (\{a_\alpha^n\})_t = (A^n)_t \supseteq A^n$, so $(\{a_\alpha^n\}) = A_n = A^n$. The second statement is proved in a similar manner. ∎

Actually, the converse of Corollary 6.4 is also true. This gives an interesting characterization of R being n-root closed.

THEOREM 6.8. *For an integral domain R and positive integer n, the following statements are equivalent.*

(1) *R is n-root closed.*

(2) *For any $\{a_\alpha\} \subseteq R - \{0\}$, $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$.*

(3) *For any $\{a_\alpha\} \subseteq R - \{0\}$, $(\{a_\alpha^n\})_v = ((\{a_\alpha\})^n)_v$.*

(4) *For any $a_1, ..., a_s \in R - \{0\}$, $(a_1^n, ..., a_s^n)_v = ((a_1, ..., a_s)^n)_v$.*

(5) *For any $a, b \in R - \{0\}$, $(a^n, b^n)_v = ((a, b)^n)_v$.*

Proof. $(1) \Rightarrow (2)$. Corollary 6.4. $(2) \Rightarrow (3)$. This follows since for any ideal A, $(A_t)_v = A_v$. $(3) \Rightarrow (4) \Rightarrow (5)$. Obvious. $(5) \Rightarrow (1)$. Suppose that $(a/b)^n \in R$ where $a, b \in R - \{0\}$. Now $a^n/b^n \in R$ implies that $(a^n, b^n) = (b^n)$.

Hence $(b^n) = (b^n)_v = (a^n, b^n)_v = ((a, b)^n)_v$. So $R = b^{-1}((a, b)^n)_v = (b^{-n}(a, b)^n)_v = ((a/b, 1)^n)_v$. Hence $a/b \in (a/b, 1)^n \subseteq R$. ∎

Of course, in (4) and (5) instead of the v-operation, we could have used the t-operation. Also, it follows from Theorem 6.8, that an integral domain R is root closed if and only if $(a^n, b^n)_v = ((a, b)^n)_v$ for all $n \geqslant 1$ and for all $a, b \in R - \{0\}$.

Let R be an integral domain. In Section 4 (Lemma 4.4) we observed that R is an AB-domain (respectively, API-domain) if and only if R is an AP-domain (respectively, AD-domain) and $C(R)$ is torsion. Similarly, R is nearly Bézout (respectively, nearly PID) if and only if R is nearly Prüfer (respectively, nearly Dedekind) and $C(R)$ is torsion. Also, in each case, the condition that $C(R)$ be torsion can be replaced by the condition that $Cl_t(R)$ be torsion. Finally, it is easily seen that an API-domain is a nearly PID.

While the exact relationship between almost Bézout domains and nearly Bézout domains remains somewhat of a mystery, the two notions coincide for integrally closed domains.

THEOREM 6.9. *For an integral domain R, the following conditions are equivalent.*

(1) *R is an AB-domain (respectively, API-domain) and R is root closed.*

(2) *R is an AB-domain (respectively, API-domain) and R is integrally closed.*

(3) *R is nearly Bézout (respectively, nearly PID) and R is root closed.*

(4) *R is nearly Bézout (respectively, nearly PID) and R is integrally closed.*

(5) *R is Prüfer (respectively, Dedekind) with $C(R)$ torsion.*

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(5). Theorem 4.7 (respectively, Theorem 4.12). Certainly (5)$\Rightarrow$(4)$\Rightarrow$(3). (3)$\Rightarrow$(5). Suppose that R is root closed and nearly Bézout. Let I be a nonzero finitely generated ideal of R. Since R is nearly Bézout, there is an n with I_n principal. By Corollary 6.7, $I_n = I^n$. Since I^n is principal, I is invertible and I is torsion in $C(R)$. Hence R is Prüfer and $C(R)$ is torsion. The proof for R nearly PID is similar. ∎

A similar theorem holds for nearly Prüfer domains.

THEOREM 6.10. *For an integral domain R, the following conditions are equivalent.*

(1) *R is an AP-domain (respectively, AD-domain) and R is root closed.*

(2) *R is an AP-domain (respectively, AD-domain) and R is integrally closed.*

(3) *R is nearly Prüfer (respectively, nearly Dedekind) and R is root closed.*

(4) *R is nearly Prüfer (respectively, nearly Dedekind) and R is integrally closed.*

(5) *R is Prüfer (respectively, Dedekind).*

Remark 6.11. In the statement of Theorem 6.9 we can add the following condition: (3a) For $a, b \in R - \{0\}$, (a, b) is nearly principal (respectively, nearly invertible) and R is root closed. For $(a, b)_n$ principal implies that $(a, b)^n = (a, b)_n$ and hence that (a, b) is invertible. Since every ideal of R generated by two elements is invertible, it is well-know that R is Prüfer. Moreover, $C(R)$ is torsion. For let $I = (a_1, ..., a_s)$. Now for each n, $I^n = (a_1^n, ..., a_s^n)$. By induction on s, some power $(a_1^n, ..., a_{s-1}^n) = (a_1, ..., a_{s-1})^n$ is principal, say $= (c)$. Then $I^n = (a_1^n, ..., a_{s-1}^n, a_s^n) = (c, a_s^n)$. So there is an m with $I^{nm} = (c, a_s^n)^m$ principal.

We next look at conditions under which $A_n = A^n$ for an ideal A of R. Our main result along these lines is the next theorem which states that $A_n = A^n$ for all ideals A of R if R contains a copy of the rational numbers Q.

THEOREM 6.12. *Let R be a commutative ring with identity containing a field of characteristic zero. Let I be an ideal of R. Then $I^n = I_n \equiv (\{a^n \mid a \in I\})$ for all natural numbers n.*

Proof. The proof is by induction on n; the case $n = 1$ being trivial. Suppose by induction that $I_{n-1} = I^{n-1}$.

Consider the set T of all polynomials $\theta(X, Y) \in Q[X, Y]$ of the form $\theta(X, Y) = X^{n-1}Y + l_2 X^{n-2} Y^2 + \cdots + l_{n-1} XY^{n-1}$. Put $l(\theta(X, Y)) =$ the number of nonzero l_i. So $l(\theta(X, Y)) \geqslant 0$ with $l(\theta(X, Y)) = 0$ if and only if $\theta(X, Y) = X^{n-1} Y$. Let $S = \{\theta(X, Y) \in T \mid \theta(a, b) \in I_n$ for all $a, b \in I\}$. Since $(1/n)((a+b)^n - a^n - b^n) = a^{n-1}b + \cdots \in I_n$ for all $a, b \in I$, we have $\theta(X, Y) = (1/n)((X+Y)^n - X^n - Y^n) \in S$. So $S \neq \phi$. Choose $\theta(X, Y) \in S$ with $l(\theta)$ minimal. If $l(\theta) = 0$, then $\theta(X, Y) = X^{n-1} Y$, so $a^{n-1} b \in I_n$ for all $a, b \in I$. Hence for fixed $b \in I$, $I_{n-1} b \subseteq I_n$. But by induction $I_{n-1} = I^{n-1}$, so $I^{n-1} b \subseteq I_n$ for all $b \in I$. Hence $I^n = I^{n-1} I \subseteq I_n$, so $I^n = I_n$. So we may suppose that $l(\theta) > 0$. Let $\theta(X, Y) = X^{n-1} Y + l_2 X^{n-2} Y^2 + \cdots + l_i X^{n-i} Y^i$ where $l_i \neq 0$. So $i \geqslant 2$. Now $2^i\theta(X, Y) - \theta(X, 2Y) = (2^i - 2) X^{n-1} Y + (2^i - 4) \quad l_2 X^{n-2} Y^2 + \cdots + (2^i - 2^i) l_i X^{n-i} Y^i$. So $\theta'(X, Y) = (2^i - 2)^{-1} (\theta(X, Y) - \theta(X, 2Y)) \in S$ and has $l(\theta'(X, Y)) < l(\theta(X, Y))$. This contradiction shows that $l(\theta) = 0$. ▮

COROLLARY 6.13. *Let R be an integral domain containing a field F of characteristic zero.*

(1) *R is nearly Prüfer if and only if R is Prüfer.*

(2) *R is nearly Bézout if and only if R is Prüfer and $C(R)$ is torsion.*

(3) *R is nearly Dedekind if and only if R is Dedekind.*

(4) *R is nearly PID if and only if R is Dedekind and $C(R)$ is torsion.*

Suppose that R contains a field F with $\operatorname{char} F = p > 0$. The proof of Theorem 6.12 breaks down in two places. First, we need $1/n \in F$. But this is easily handled by assuming that $(n, p) = 1$. In fact, this assumption is necessary. For let F be any field with $\operatorname{char} F = p > 0$ and take $R = F[[X^2, X^3]]$. Then $(X^2, X^3)^{p^l} = (X^{2p^l}, X^{2p^l+1})$ while $(X^2, X^3)_{p^l} = (X^{2p^l})$. The second place where the proof breaks down is finding a $k \in F$ with $k^i - k$ a unit in F (or just in R). Also, to apply the induction hypothesis, we need that $I_{n-1} = I^{n-1}$. Suppose that we take n with $1 \leqslant n < p$. Then $(n, p) = 1$ and since $2 \leqslant i \leqslant n-1 < p-1$, $X^i - X = 0$ has at most i solutions in Z_p, so there is a $k \in Z_p$ with $k^i - k$ a unit in Z_p and all $i < n-1$ have these properties. Hence for n with $1 \leqslant n < p$, $I_n = I^n$. We state this result as the next theorem.

THEOREM 6.14. *Let R be a commutative ring with identity containing a field of characteristic $p > 0$. Let I be an ideal of R. Then $I^n = I_n \equiv (\{a^n \mid a \in I\})$ for all natural numbers n with $1 \leqslant n < p$.*

In $Z[X, Y]$, $(X, Y)^n \supsetneqq (X, Y)_n$ for all $n > 1$. So if R does not contain a field, we may have $I_n \subsetneqq I^n$ for all $n > 1$. So suppose R contains a field F. If $\operatorname{char} F = 0$, then $I_n = I^n$ for all $n \geqslant 1$ by Theorem 6.12. If $\operatorname{char} F = p > 0$, then $I_n = I^n$ for all n with $1 \leqslant n < p$ by Theorem 6.14. But we may have $I_p \subsetneqq I^p$. (For in $F[X, Y]$, $(X, Y)^{p^n} = (X^{p^n}, X^{p^n-1}Y, ..., Y^{p^n}) \supsetneqq (X^{p^n}, Y^{p^n}) = (X, Y)_{p^n}$.) It seems reasonable to conjecture that if $(p, n) = 1$, then $I^n = I_n$. But even this need not be true. For in $Z_2[X, Y]$ we have $(X, Y)^3 = (X^3, X^2Y, XY^2, Y^3)$, while it may be verified that $(X, Y)_3 = (X^3, X^2Y + XY^2, Y^3)$.

However, if $\operatorname{char} F = 2$ and F has more than two elements then $(X, Y)_3 = (X, Y)^3$ in $F[X, Y]$. For then there is an element $l \in F$ with $l^2 \neq l$. Then $l^2X^2Y + l^2XY^2$ and $lX^2Y + l^2XY^2 = X^2(lX) + X(lY)^2 \in (X, Y)_3$. So $(l^2 - l)X^2Y \in (X, Y)_3$ and hence $X^2Y \in (X, Y)_3$, so $(X, Y)^3 = (X, Y)_3$. This shows that $(a, b)^3 = (a, b)_3$ for any ring R containing a field F with either $\operatorname{char} F > 3$ or $\operatorname{char} F = 2$ and $F \neq Z_2$.

Note added in proof. Two of the first author's students, Rebecca Lewin and Kent Knopp have continued the investigations of this paper. Lewin studied AGGCD-domains and nearly GCD-domains and Knopp investigated the ideal I_n and showed that nearly Bézout implies almost Bézout, but not conversely.

REFERENCES

1. D. D. ANDERSON AND D. F. ANDERSON, Generalized GCD domains, *Comment. Math. Univ. St. Paul.* **28** (1979), 215–221.
2. D. F. ANDERSON, A general theory of class groups, *Comm. Algebra* **16**, No. 4 (1988), 805–847.
3. A. BOUVIER, Le groupe des classes d'un anneau intégré, *in* "107ᵉ Congrès National des Sociétés Savantes, Brest, 1982," sciences, Fasc. IV, pp. 85–92.
4. A. BOUVIER AND M. ZAFRULLAH, On some class groups of an integral domain, *Bull. Soc. Math. Grèce (N.S.)* **29** (1988), 45–49.
5. P. M. COHN, Bézout rings and their subrings, *Proc. Cambridge Philos. Soc.* **64** (1968), 251–264.
6. D. COSTA, J. MOTT, AND M. ZAFRULLAH, The construction $D+XD_S[X]$, *J. Algebra* **53** (1978), 423–439.
7. D. DOBBS, E. HOUSTON, T. LUCAS, AND M. ZAFRULLAH, t-linked overrings and Prüfer v-multiplication domains, *Comm. Algebra* **17**, No. 8 (1989), 2835–2852.
8. R. FOSSUM, "The Divisor Class Group of a Krull Domain," Springer-Verlag, New York, 1973.
9. R. GILMER, "Multiplicative Ideal Theory," Dekker, New York, 1972.
10. R. GILMER, "Commutative Semigroup Rings," Univ. of Chicago Press, Chicago, 1984.
11. M. GRIFFIN, Some results on v-multiplication rings, *Canad. J. Math.* **19** (1967), 710–722.
12. B. G. KANG, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, *J. Algebra* **123** (1989), 151–170.
13. M. LARSEN AND P. J. MCCARTHY, "Multiplicative Theory of Ideals," Academic Press, New York/London, 1971.
14. J. MOTT AND M. ZAFRULLAH, On Prüfer v-multiplication domains, *Manuscripta Math.* **35** (1981), 1–26.
15. U. STORCH, Fastfaktorielle rings, *in* "Schriftenreihe Math. Inst. Univ. Munster," Vol. 36, University Münster, Münster, 1967.
16. M. ZAFRULLAH, A general theory of almost factoriality, *Manuscripta Math.* **51** (1985), 29–62.
17. M. ZAFRULLAH, The $D+XD_S[X]$ construction from GCD-domains, *J. Pure Appl. Algebra* **50** (1988), 93–107.
18. M. ZAFRULLAH, Well-behaved prime t-ideals, *J. Pure Appl. Algebra* **65** (1990), 199–207.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium