NAGATA-LIKE THEOREMS FOR INTEGRAL DOMAINS OF FINITE CHARACTER AND FINITE t-CHARACTER

D.D. ANDERSON, GYU WHAN CHANG, AND MUHAMMAD ZAFRULLAH

Abstract. An integral domain D is said to be *of finite character* (resp., *finite t-character*) if every nonzero nonunit of D belongs to at most a finite number of maximal ideals (resp., maximal t-ideals) of D. Let S be a multiplicative set of D. In this paper we study when D_S being of finite character (resp., finite t-character) implies that D is of finite character (resp., finite t-character).

Introduction

An integral domain D is called a domain *of finite character* if every nonzero nonunit of D belongs to at most a finite number of maximal ideals of D. Domains of finite character include semi-quasi-local domains and one-dimensional Noetherian domains. Likewise, D is said to be *of finite t-character* if every nonzero nonunit of D belongs to at most a finite number of maximal t-ideals. Integral domains of finite t-character include UFD's, Krull domains, Noetherian domains, and Prüfer domains of finite character. Necessary definitions will be provided in the following. In [3, Proposition 3.1], where the emphasis was on domains of finite t-character, a statement dubbed as a "Nagata-like theorem" was proved. The statement goes as:

Proposition 1. *Let S be a saturated multiplicative set of D, and consider the following two conditions: (i) each nonzero nonunit $x \in D$ belongs to only finitely many maximal t-ideals intersecting S and (ii) every maximal t-ideal P of D with $P \cap S = \emptyset$ is contracted from a maximal t-ideal of D_S: If D_S is of finite t-character, then the following are equivalent.*

1. *D is of finite t-character.*
2. *The condition (i) holds and D is conditionally well behaved.*
3. *The conditions (i) and (ii) hold.*

Here D is said to be *conditionally well behaved* if for each maximal t-ideal M of D we have MD_M a t-ideal. We know that if D is of finite t-character, then D is conditionally well behaved [3, Lemma 1.1]. A working introduction to the notions involved will be provided in due course. The aim of this note is to provide some further applications of Proposition 1 (in Section 1) and to give a non t-operation version of this result (in Section 2) as: *Let S be a saturated multiplicative set of D such that every nonzero nonunit of D belongs to at most a finite number of maximal*

Date: August 9, 2014.
2010 Mathematics Subject Classification: 13A15, 13G05.
Key Words and Phrases: finite (t-) character, Nagata-like theorem, conditionally well behaved.

ideals M of D with $M \cap S \neq \emptyset$. If D_S is of finite character, then so is D. We shall also look at the consequences of this statement.

Let D be an integral domain with quotient field K and let $F(D)$ denote the set of nonzero fractional ideals of D. For $A \in F(D)$, the set $A^{-1} = \{x \in K \mid xA \subseteq D\}$ is a fractional ideal of D. Denote the fractional ideal $(A^{-1})^{-1}$ by A_v and define $A_t = \cup F_v$, where F ranges over finitely generated nonzero subideals of A. The functions on $F(D)$ defined by $A \mapsto A_v$ and by $A \mapsto A_t$ are examples of the so-called star operations. For a review of star operations, the reader may look up Sections 32 and 34 of [13]. If $*$ represents the v- or t-operation, then A is called a *$*$-ideal* if $A = A_*$ and a *$*$-ideal of finite type* if $A = B_*$ for a finitely generated ideal B. A $*$-ideal of D is called a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals of D. Let t-Max(D) be the set of all maximal t-ideals of D. It is well known that each proper t-ideal is contained in a maximal t-ideal; each maximal t-ideal is a prime ideal; and $D = \cap_{P \in t\text{-Max}(D)} D_P$.

1. Domains of finite t-character

Let D be an integral domain, S be a multiplicative set of D, and $N(S) = \{x \in D \mid (x, s)_v = D$ for all $s \in S\}$. So $N(S)$ is a saturated multiplicative set. It is easy to see that $D = D_S \cap D_{N(S)}$, and hence $I_t = (ID_S)_t \cap (ID_{N(S)})_t$ for all $I \in F(D)$ [1, Theorem 2].

Proposition 2. *Let S be a multiplicative set of D, and assume that for a maximal t-ideal P of D, $P \cap S = \emptyset$ if and only if $P \cap N(S) \neq \emptyset$.*

(1) *Every maximal t-ideal P of D with $P \cap S = \emptyset$ is contracted from a maximal t-ideal of D_S.*
(2) *Each nonzero nonunit $x \in D$ belongs to only finitely many maximal t-ideals intersecting S if and only if $D_{N(S)}$ is of finite t-character.*

Proof. (1) Let P be a maximal t-ideal of D with $P \cap S = \emptyset$. Then $P \cap N(S) \neq \emptyset$ by assumption, and hence $P = P_t = (PD_S)_t \cap (PD_{N(S)})_t = (PD_S)_t \cap D_{N(S)}$. Thus, $(PD_S)_t = PD_S$. Note that the contraction of a prime t-ideal of D_S is a prime t-ideal of D. Hence PD_S is a maximal t-ideal because P is a maximal t-ideal.

(2) This result follows from the observation that every maximal t-ideal P of D with $P \cap S \neq \emptyset$ is contracted from a maximal t-ideal of $D_{N(S)}$ (this can be proved using the assumption and an argument similar to the proof of (1) above). □

Corollary 3. *Let S be a multiplicative set of D, and assume that for a maximal t-ideal P of D, $P \cap S = \emptyset$ if and only if $P \cap N(S) \neq \emptyset$. Then D is of finite t-character if and only if D_S and $D_{N(S)}$ are both of finite t-character.*

Proof. This follows directly from Propositions 1 and 2. □

A saturated multiplicative set S of D is said to be *splitting* if each nonzero nonunit $d \in D$ can be written as $d = st$, where $s \in S$ and $t \in N(S)$. Clearly, if S is splitting, then $N(S)$ is also a splitting set of D. Also, it can be easily verified that if M is a maximal t-ideal of D, then $M \cap S = \emptyset$ if and only if $M \cap N(S) \neq \emptyset$. Thus, MD_S is a maximal t-ideal of D_S when $M \cap S = \emptyset$ [4, Lemma 2] (or, see the proof of Proposition 2(1)). Hence, by Proposition 2(2) and Corollary 3, we have

Corollary 4. (1) *Let S be a splitting set of D such that every nonzero nonunit of D belongs to at most a finite number of maximal t-ideals of D intersecting S. Then D_S is of finite t-character if and only if D is of finite t-character.*

(2) *Let S be a splitting set generated by principal primes. Then D_S is of finite t-character if and only if D is.*

Let $D[X]$ be the polynomial ring over D. For any $f \in D[X]$, let $c(f)$ be the ideal of D generated by the coefficients of f. It is known that if we let $S = D \setminus \{0\}$, then $N(S) = \{f \in D[X] \mid c(f)_v = D\}$ (cf. [5, Proposition 3.6]). Hence, if Q is a maximal t-ideal of $D[X]$, then $Q \cap S = \emptyset$, i.e., $Q \cap D = (0)$, if and only if $Q \cap N(S) \neq \emptyset$ [15, Proposition 1.2 and Theorem 1.4]. Also, since $D[X]_S$ is a UFD, by Corollary 3, we have

Corollary 5. *If $S = D \setminus \{0\}$, then $D[X]$ is of finite t-character if and only if $D[X]_{N(S)}$ is of finite t-character.*

We next give another example of multiplicative sets S with property that $P \cap S = \emptyset$ if and only if $P \cap N(S) \neq \emptyset$ for all maximal t-ideals P of D. Let Γ be a torsionless grading monoid, and let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be an integral domain graded by Γ. That is, each nonzero $x \in R_\alpha$ has degree α, i.e., $\deg(x) = \alpha$, and thus each nonzero $f \in R$ can be written as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. Let H be the saturated multiplicative set of nonzero homogeneous elements of R. Then R_H, called the homogeneous quotient field of R, is a graded integral domain whose nonzero homogeneous elements are units. For $f \in R_H$, let $C(f)$ denote the (homogeneous) fractional ideal of R generated by the homogeneous components of f. For a fractional ideal I of R with $I \subseteq R_H$, let $C(I) = \sum_{f \in I} C(f)$; so $C(I)$ is a homogeneous fractional ideal of R. It is known that $N(H) = \{0 \neq f \in R \mid C(f)_v = R\}$ and $R = R_H \cap R_{N(H)}$ [6, Lemma 1.2]. As in [6], we say that R satisfies property $(\#)$ if, for any nonzero ideal I of R, $C(I)_t = R$ implies that $I \cap N(H) \neq \emptyset$, i.e., there is an $f \in I$ such that $C(f)_v = R$. If $R = D[\Gamma]$, where $\Gamma \neq (0)$, or if R contains a unit of nonzero degree, then R satisfies $(\#)$ [6, Example 1.6].

Lemma 6. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property $(\#)$. If Q is a maximal t-ideal of R, then $Q \cap H = \emptyset$ if and only if $Q \cap N(H) \neq \emptyset$.*

Proof. If $Q \cap H = \emptyset$, then $Q \subsetneq C(Q)_t \subseteq R$, and since Q is a maximal t-ideal, we have $C(Q)_t = R$. Thus, $Q \cap N(H) \neq \emptyset$ because R satisfies property $(\#)$. Conversely, if $Q \cap H \neq \emptyset$, then $QR_{N(H)}$ is a maximal ideal of $R_{N(H)}$ [6, Proposition 1.4(2)], and thus $Q \cap N(H) = \emptyset$. □

Corollary 7. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with property $(\#)$. Then R is of finite t-character if and only if R_H and $R_{N(H)}$ are both of finite t-character.*

Proof. This follows directly from Corollary 3 and Lemma 6. □

An upper to zero in $D[X]$ is a nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$, while D is called an *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal. It is well known that D is an integrally closed UMT-domain if and only if D is a Prüfer v-multiplication domain (PvMD) (that is, a domain in which each nonzero finitely generated ideal I of D is t-invertible, i.e., $(II^{-1})_t = D$). Also, if

M is a maximal t-ideal of a UMT-domain D, then MD_M is a t-ideal of D_M [11, Theorem 1.5]; hence a UMT-domain is conditionally well behaved.

Proposition 8. (1) *Let D be conditionally well behaved and S be a saturated multiplicative set generated by primes such that every nonzero nonunit of D is divisible by at most a finite number of nonassociated primes from S. If D_S is of finite t-character, then so is D.*

(2) *Let D be a UMT-domain (e.g., PvMD) and S be a saturated multiplicative set generated by primes such that every nonzero nonunit of D is divisible by at most a finite number of nonassociated primes from S. If D_S is of finite t-character, then so is D.*

Proof. (1) This holds because the hypothesis meets the requirements of Proposition 1 once we note that every prime ideal generated by a principal prime is a maximal t-ideal and that every prime t-ideal intersecting S contains a prime p from S and hence is a maximal t-ideal generated by p and so being divisible by a prime from S is equivalent to belonging to the prime ideal generated by that prime.

(2) This is an immediate consequence of (1) because a UMT-domain is conditionally well behaved. □

In [3], the authors studied when the ring $D + XD_S[X]$ is a PvMD of finite t-character, where S is a multiplicative set of D. We end this section with some necessary conditions for $D + X^nD_S[X]$ to be of finite t-character. Recall from [10, Lemma 2.5] that the map $\varphi : t\text{-Max}(D+XD_S[X]) \to t\text{-Max}(D+X^nD_S[X])$, given by $\varphi(Q) = Q \cap (D + X^nD_S[X])$, is bijective.

Proposition 9. *Let S be a multiplicative set of D, and let $R_n = D + X^nD_S[X]$ for an integer $n \geq 1$. Then the following statements are equivalent.*

(1) *R_1 is of finite t-character.*
(2) *R_n is of finite t-character for every integer $n \geq 1$.*
(3) *R_n is of finite t-character for some integer $n \geq 1$.*

Proof. (1) ⇒ (2) Let $f \in R_n$ be a nonzero nonunit. Then $f \in R_1$, and since R_1 is of finite t-character, f is contained in at most a finite number of maximal t-ideals of R_1. Hence, by [10, Lemma 2.5], f is contained in only a finite number of maximal t-ideals of R_n. (2) ⇒ (3) Clear. (3) ⇒ (1) Let $g \in R_1$ be a nonzero nonunit, and assume that R_n is of finite t-character for some integer $n \geq 1$. Note that $X^ng \in R_n$; so by assumption, X^ng is contained in finitely many maximal t-ideals of R_n. So, again by [10, Lemma 2.5], X^ng, and hence g is contained in at most a finite number of maximal t-ideals of R_1. □

Corollary 10. (cf. [3, Corollary 3.8]) *Let K be the quotient field of D. If D is semi-quasi-local whose maximal ideals are t-ideals, then $D + X^nK[X]$ is of finite t-character for every integer $n \geq 1$.*

Proof. This is an immediate consequence of Proposition 9, because $D + XK[X]$ is of finite t-character if and only if D is semi-quasi-local whose maximal ideals are t-ideals [3, Corollary 3.8]. □

A multiplicative set S of D is called a *t-splitting set* if, for each nonzero $d \in D$, we have $dD = (AB)_t$, where A and B are ideals of D with $A_t \cap sD = sA_t$ for all

$s \in S$ and $B_t \cap S \neq \emptyset$. The notion of t-splitting sets was introduced in [2] in order to study when $D + XD_S[X]$ is a PvMD. It is known that $D + XD_S[X]$ is a PvMD of finite t-character if and only if D is a PvMD of finite t-character, S is a t-splitting set, and the set of maximal t-ideals of D that intersect S is finite [3, Theorem 2.5]. Thus, by Proposition 9, we have

Corollary 11. (cf. [3, Theorem 2.5]) *Let S be a multiplicative set of a PvMD D and n be a positive integer. Then $D + X^n D_S[X]$ is of finite t-character when D is of finite t-character, S is a t-splitting set, and the set of maximal t-ideals of D that intersect S is finite.*

Corollary 12. (cf. [3, Corollary 2.6]) *Let S be a multiplicative set of a Krull domain D and n be a positive integer. Then $D + X^n D_S[X]$ is of finite t-character if the set of maximal t-ideals of D that intersect S is finite.*

Proof. This follows directly from Corollary 11 because every multiplicative set of a Krull domain is a t-splitting set [2, p. 8]. □

2. Domains of finite character

Notice that in the statement of Proposition 1 we needed to make sure that every maximal t-ideal of an integral domain D that is disjoint with S extends to a maximal t-ideal of D_S. Yet if we were to consider a statement involving maximal ideals we would have no problem of that kind. Thus, we have the following result.

Proposition 13. *Let S be a multiplicative set of D such that every nonzero nonunit of D belongs to at most a finite number of maximal ideals M of D with $M \cap S \neq \emptyset$. If D_S is of finite character, then so is D.*

Proof. Let x be a nonzero nonunit of D. If $x \in S$, then x belongs to at most a finite number of maximal ideals intersecting S and to no maximal ideals of D_S. If $x \notin S$ then $x \in M_i$ for $i = 1, 2, \ldots, r$ where $M_i \cap S \neq \emptyset$ by the condition. Now let $x \in M_\alpha$ where $M_\alpha \cap S = \emptyset$. So $x \in M_\alpha D_S$ and as D_S is of finite character there are at most a finite number of distinct $M_\alpha D_S$ containing xD_S. Thus $\{M_\alpha \mid x \in M_\alpha$ and $M_\alpha \cap S = \emptyset\}$ is finite. Thus x belongs to only a finite number of maximal ideals of D in either case. □

Corollary 14. *Let S be a saturated multiplicative set of D generated by principal primes that generate maximal ideals such that every nonzero nonunit of D is divisible by at most a finite number of nonassociated primes from S. If D_S is of finite character, then so is D.*

Proof. Note that a nonzero nonunit x being divisible by a prime from S is equivalent to x belonging to a maximal ideal intersecting S, (here, by hypothesis, the primes generate maximal ideals). So, by Proposition 13, D is of finite character. □

Corollary 15. *Let S be a saturated multiplicative set generated by principal primes of a Prüfer domain D such that every nonzero nonunit of D is divisible by at most a finite number of nonassociated primes from S. If D_S is of finite character, then D is of finite character.*

Proof. This follows from the fact that every nonzero principal prime of a Prüfer domain generates a maximal ideal. □

Let D be an integral domain with quotient field K. It is known that $R = D+XK[X]$ is of finite t-character if and only if D is semi-quasi-local whose maximal ideals are t-ideals [3, Corollary 3.8].

Corollary 16. *Let D be an integral domain with quotient field K, L an extension field of K, and X an indeterminate over L. Then $R = D + XL[X]$ is of finite character (resp., finite t-character) if and only if D has only finitely many maximal ideals (resp., is semi-quasi-local whose maximal ideals are t-ideals).*

Before we prove Corollary 16 it seems pertinent to give a brief, practical, introduction to the ring $D + XL[X]$. With D, X and L as in the statement of Corollary 16, $D + XL[X]$ is the set of all the polynomials of $L[X]$ with constant terms from D. It is not too hard to show that $R = D + XL[X]$ is indeed an integral domain, which is a special case of the general $D + M$ construction of [9]. It can be shown that every ideal I of R with $I \cap D \neq (0)$ is of the form $IR = I \cap D + XL[X]$ (see [16, page 107] for details and further references). From [16, Theorem 3] we conclude that for an ideal P of D, $P + XL[X]$ is a prime (maximal) ideal of R if and only if P is. Also, from [16, Lemma 2 and Theorem 4], every prime ideal of R that is incomparable to $XL[X]$ is a maximal ideal of the form $(1 + Xg(X))R$ and is of height one and every element of the form $1 + Xf(X) \in R$ is a product of powers of primes that generate maximal ideals. Let's also note that if $I \neq (0)$ is an ideal of D, then it is easy to show that $(I + XL[X])^{-1} = I^{-1} + XL[X]$ and so $(I + XL[X])_v = I_v + XL[X]$ (see e.g. [7, Proposition 2.4]). Using this information one can easily establish that if P is a nonzero ideal of D, then $P + XL[X]$ is a t-ideal (prime t-ideal, maximal t-ideal) if and only if P is. Thus we have the following picture: If D is an integral domain that is not a field, $D + XL[X]$ is described as above, $A = \{M + XL[X] \mid M \in \mathrm{Max}(D)\}$, $B = \{(1 + Xf(X))R \mid 1 + Xf(X)$ generates a maximal ideal of $R\}$, and $C = \{M + XL[X] \mid M \in t\text{-}\mathrm{Max}(D)\}$, then $\mathrm{Max}(R) = A \cup B$ and $t\text{-}\mathrm{Max}(R) = B \cup C$. Now note that in $L[X]$ every prime of the form $r + Xf(X)$, with $r \neq 0$, is an associate in $L[X]$ of the prime $1 + \frac{X}{r}f(X)$ and $1 + \frac{X}{r}f(X) \in D + XL[X]$. As there are infinitely many primes of the form $1 + \frac{X}{r}f(X)$ in $L[X]$ there are infinitely many primes of the form $1 + Xg(X)$. Since each prime of the form $1 + Xg(X)$ generates a maximal ideal in $D + XL[X]$ we conclude that $D + XL[X]$ is not semi-local.

Proof. of Corollary 16. If D has infinitely many maximal ideals then $D+XL[X]$ is not of finite character because X belongs to all maximal ideals $M+XL[X]$ where M ranges over all the maximal ideals of D. So for R to be of finite character D must be semi-local. Conversely let D be semi-local with maximal ideals $M_1, M_2, ..., M_r$ and let $S = D\backslash\{0\}$. Then $M_i+XL[X]$ are precisely the maximal ideals of R intersecting S and so every nonzero nonunit of R can be in at most a finite number of maximal ideals intersecting S. Next $(D+XL[X])_S = K+XL[X]$ which is a one-dimensional Mori domain [12] and hence of finite character. So, by Proposition 13, $D+XL[X]$ is of finite character.

A similar argument also shows the case of finite t-character. □

Next, and it is important to note that while D being of finite t-character forces $D[X]$ to be of finite t-character, as was verified in [3], we do not have this facility with D being of finite character. For if D is not a field then $D[X]$ is never of finite character. This can be established as follows.

Proposition 17. *Let D be an integral domain that is not a field. Then $D[X]$ is not of finite character.*

Proof. Suppose D is not a field, so D has a nonzero maximal ideal M. Then $D[X]/M[X] \simeq (D/M)[X]$ a ring of polynomials over a field and hence has infinitely many maximal ideals. Thus $M[X]$ is contained in infinitely many maximal ideals of $D[X]$ and so $D[X]$ is not of finite character. □

In Corollary 16 we established that if L is a field containing a domain D then $D+XL[X]$ is of finite character if and only if D is semi-quasi-local. A variation on Proposition 17 lets us establish a more satisfying result on when the $A+XB[X]$ construction is of finite character.

Proposition 18. *Let $A \subseteq B$ be an extension of integral domains and let X be an indeterminate and consider the ring $R = A+XB[X]$. If B is not a field, then R is not of finite character.*

Proof. Suppose that B is not a field, so no maximal ideal of B is zero. Let M be a maximal ideal of B. Let $M \cap A = P$. Then, of course $M[X] \cap A = P$. Suppose that $M[X] \cap R$ is contained in only a finite number of maximal ideals of R. But then $R/(M[X] \cap R) \simeq A/P + X(B/M)[X]$ must be semi-quasi-local. The contradiction comes from the fact that $A/P + X(B/M)[X]$ is the $D + XL[X]$ construction discussed prior to the proof of Corollary 16 and $D + XL[X]$ is never semi-quasi-local. □

Recall that in [8] the authors showed that if A is a field in the extension of integral domains $A \subseteq B$ then $A+XB[X]$ is atomic. Now we know that if B is not a field then $R = A+XB[X]$ can never be of finite character. But of course we can do much better using Proposition 18.

Corollary 19. *Let $A \subseteq B$ be an extension of integral domains and let X be an indeterminate, then the ring $R = A+XB[X]$ is of finite character if and only if B is a field and A is a semi-quasi-local ring.*

Proof. If R is of finite character, then by Proposition 18, B must be a field. But then by Corollary 16 R is of finite character if and only if A is semi-qusai-local. The converse again falls to Corollary 16. □

In [3] we used the Nagata-type theorem to prove that if D is a domain of finite t-character, then so is $D[X]$. Why haven't we given the reason as to why that trick fails in the finite character case? Well, the reason is amply provided by Proposition 17. As the proof indicates, there are elements of $D[X]$, coming from $D\backslash\{0\}$ that belong to infinitely many maximal ideals of $D[X]$. So, while every nonzero nonunit of D may belong to only a finite number of maximal ideals of D it belongs to infinitely many maximal ideals of $D[X]$. Thus the Nagata-type theorem for finite character provided by Proposition 13 is not applicable.

So before making a decision about finite character or about the possibility of using our Nagata-type theorem it is best to use the oldest trick in the book and check to see if there is an easy way to find a nonzero nonunit that is in infinitely many maximal ideals. Thus if D has infinitely many maximal ideals, then $D[[X]]$, the power series ring over D, is not of finite character, because then $D[[X]]$ would have infinitely many maximal ideals each containing X. However, if D is semi-quasi-local, then so is $D[[X]]$ and hence of finite character.

We next give two general constructions for (Noetherian) domains of finite character. The first is modeled after Nagata's celebrated example of a Noetherian integral domains of infinite Krull dimension [17, Example 1, p 203]. The second construction is due to Gulliksen [14] where he constructs Noetherian integral domains of arbitrary Krull ordinal.

Example 20. Let K be a field and Λ a nonempty set. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a set of indeterminates over K indexed by Λ and let $D = K[\{X_\alpha\}_{\alpha\in\Lambda}]$. Let $\{A_\beta\}_{\beta\in\Gamma}$ be a partition of Λ. For each A_β, let M_β be the prime ideal of D generated by $\{X_\alpha\}_{\alpha\in A_\beta}$. Let $S = D\backslash\bigcup_{\beta\in\Gamma} M_\beta$ and put $R = D_S$.

(1) $\{(M_\beta)_S\}_{\beta\in\Gamma}$ is the set of maximal ideals of R.
(2) Each $f \in R\backslash\{0\}$ is contained in only finitely many $(M_\beta)_S$, so R is of finite character.
(3)
$$ht(M_\beta)_S = \dim(R_{(M_\beta)_S}) = \dim(D_{M_\beta}) = \begin{cases} |A_\beta| & \textit{if } |A_\beta| < \infty \\ \infty & \textit{otherwise.} \end{cases}$$
(4)
$$\dim(R) = \begin{cases} \sup|A_\beta| & \textit{if } |A_\beta| < \infty \textit{ for all } \beta\in\Gamma \\ \infty & \textit{otherwise.} \end{cases}$$
(5) R is Noetherian if and only if each A_β is finite.

Proof. The proof is pretty much the same as that of [17]. First note that, because a polynomial involves only finitely many indeterminates every $f \in D\backslash\{0\}$ belongs to at most a finite number of M_β, because of the partition. Then show that if a prime ideal P is contained in $\bigcup M_\beta$ then P is contained in some M_{β_0}. Then (1) and (2) follow from this observation, (3) and (4) are clear and (5) follows since a domain of finite character is Noetherian if and only if it is locally Noetherian and $R_{(M_\beta)_S} = D_{M_\beta}$ is Noetherian if and only if each A_β is finite. □

For Nagata's example, we can take $\Lambda = \mathbb{N}$ and $\Gamma = \{\{1\}, \{2,3\}, \{4,5,6\}, \{7,8,9,10\}, ...\}$ where the nth subset of Γ has n elements. Then R is an infinite dimensional regular Noetherian domain of finite character with exactly one maximal ideal of height n for each $n \geq 1$. For a second example, take $\Lambda = \mathbb{N}$ and $\Gamma = \{\{1,2\}, \{3\}, \{4\}, \{5\}, ...\}$. Then R is a two-dimensional regular Noetherian domain of finite character where every localization at a maximal ideal save one is a DVR and that localization is a two-dimensional regular local ring.

Note that the domains R of finite character obtained from Example 20 are all independent in the sense that each nonzero prime ideal P of R is contained in a unique maximal ideal, i.e., R/P is quasi-local for each nonzero prime ideal P of

R. The next construction due to Gulliksen [14] constructs Noetherian domains of finite character that need not be independent.

Example 21. Let A be any integral domain, not necessarily Noetherian, and let $X = \{X_\alpha\}$ be a set of indeterminates over A. Let $\mathcal{M}$ be a family of finite subsets of X and let $\mathcal{P}(\mathcal{M})$ be the family consisting of all the subsets of all the members of $\mathcal{M}$. For simplicity, we make the additional assumption that the members of $\mathcal{M}$ are mutually incomparable. Let $MA[X]$ be the ideal generated in $A[X]$, by each $M \in \mathcal{M}$. Let $S = A[X] \backslash \bigcup_{M \in \mathcal{M}} MA[X]$ and put $R = A[X]_S$. If the set $\mathcal{P}(\mathcal{M})$ is Noetherian (i.e., satisfies ACC w.r.t. inclusion), then a prime ideal P of $A[X]$ is contained in $\bigcup_{M \in \mathcal{M}} MA[X]$ if and only if P is contained in some $MA[X]$. So, in this case, $\{MA[X]_S\}_{M \in \mathcal{M}}$ is the set of maximal ideals of R. It is easily checked that R has finite character if and only if each X_α is contained in only a finite number of M in $\mathcal{M}$. If $\mathcal{P}(\mathcal{M})$ is Noetherian, then R is Noetherian of finite character.

For a non-independent example, take A to be a field K, Let $X = \{X_n\}_{n=1}^{\infty}$ and take $\mathcal{M} = \{\{X_1,\ X_2\},\ \{X_1, X_3\}, \{X_4\}, ...\}$, so $\mathcal{P}(\mathcal{M})$ is obviously a Noetherian partially ordered set. Here $S = K[\{X_\alpha\}] \backslash \bigcup \{(X_1, X_2), (X_1, X_3), (X_4), ...\}$ and so the maximal ideals of $R = K[\{X_n\}]_S$ are $(X_1, X_2)_S, (X_1, X_3)_S, (X_4)_S, ...$ Here R is a two-dimensional Noetherian domain of finite character, but the intersection is non-independent, as $R/(X_1)_S$ is a PID with two maximal ideals.

References

[1] D.D. Anderson, *Star-operations induced by overrings*, Comm. Algebra 16 (1988), 2535-2553.
[2] D.D. Anderson, D.F. Anderson and M. Zafrullah, *The ring $D + XD_S[X]$ and t-splitting sets*, Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001), 3-16.
[3] D.D. Anderson, G.W. Chang and M. Zafrullah, *Integral domains of finite t-character*, J. Algebra 396 (2013), 169-183.
[4] D.D. Anderson and M. Zafrullah, *Splitting sets and weakly Matlis domains*, Commutative algebra and its applications, pp. 1-8, Walter de Gruyter, Berlin, 2009.
[5] D.F. Anderson and G.W. Chang, *The m-complement of a multiplicative set*, Lecture Notes in Pure Appl. Math., Chapman & Hall, 241 (2005), 180-187.
[6] D.F. Anderson and G.W. Chang, *Graded integral domains and Nagata rings*, J. Algebra 387 (2013), 169-184.
[7] D.F. Anderson and A. Ryckaert, *The class group of $D + M$*, J. Pure Appl. Algebra 52 (1988), 199-212.
[8] V. Barucci, L. Izelgue and S. Kabbaj, *Some factorization properties of A+XB[X] domains*, Lecture Notes in Pure and Applied Mathematics Dekker 185 (1997) 69-78.
[9] J.W. Brewer and E.A. Rutter, *$D + M$ constructions with general overrings*, Michigan Math. J. 23 (1976), 33-42.
[10] G.W. Chang and J.W. Lim, *Almost Prüfer v-multiplication domains and related domains of the form $D + D_S[\Gamma^*]$*, Comm. Algebra 41 (2013), 2650-2664.
[11] M. Fontana, S. Gabelli and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra 26 (1998), 1017-1039.
[12] S. Gabelli and E. Houston, *Coherentlike conditions in pullbacks*, Michigan Math. J. 44 (1997), 99-123.
[13] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
[14] T.H. Gulliksen, The Krull ordinal, coprof and Noetherian localizations of large polynomial rings, Amer. J. Math. 96 (1974), 324-339.
[15] E. Houston and M. Zafrullah, *On t-invertibility II*, Comm. Algebra 17(1989), 1955-1969.
[16] J. Mott and M. Zafrullah, *Unruly Hilbert domains*, Canad Math. Bull. 33 (1990), 106-109.
[17] M. Nagata, Local Rings, Interscience, New York, 1962.

(Anderson) Department of Mathematics, The University of Iowa, Iowa City, IA 52242, USA

E-mail address: dan-anderson@uiowa.edu

(Chang) Department of Mathematics Education, Incheon National University, Incheon 406-772, Republic of Korea

E-mail address: whan@incheon.ac.kr

(Zafrullah) Department of Mathematics, Idaho State University, Pocatello, ID 83209, USA

E-mail address: mzafrullah@usa.net, http://www.lohar.com