LOCALLY GCD DOMAINS AND THE RING $D + XD_S[X]$

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Abstract. An integral domain D is called a *locally GCD domain* if D_M is a GCD domain for every maximal ideal M of D. We study some ring-theoretic properties of locally GCD domains. E.g., we show that D is a locally GCD domain if and only if $aD \cap bD$ is locally principal for all $0 \neq a, b \in D$, and flat overrings of a locally GCD domain are locally GCD. We also show that the t-class group of a locally GCD domain is just its Picard group. We study when a locally GCD domain is Prüfer or a generalized GCD domain. We also characterize locally factorial domains as domains D whose minimal primes of nonzero principal ideals are locally principal and discuss conditions that make them Krull. We use the $D + XD_S[X]$ construction to give some interesting examples of locally GCD domains that are not GCD domains.

Introduction

0.1. **Motivation.** Let D be an integral domain and $D^* = D \setminus \{0\}$. An integral domain D is called a *GCD domain* if for each pair $a, b \in D^*$, $GCD(a, b)$ exists. We say that D is a *locally GCD domain* if D_M is a GCD domain for every maximal ideal M of D. Since D is a Prüfer domain (an integral domain whose nonzero finitely generated ideals are invertible) if and only if D_M is a valuation domain for all maximal ideals M of D, Prüfer domains can serve as celebrated examples of locally GCD domains. Also, since a GCD Prüfer domain is Bezout and there are non Bezout Prüfer domains, there are locally GCD domains that are not GCD. Another example of a locally GCD domain is a locally factorial domain (an integral domain D such that D_M is factorial for each maximal ideal M of D). Let S be a multiplicative subset of D, X be an indeterminate over D, and $D^{(S)} = D + XD_S[X] = \{f \in D_S[X] \mid f(0) \in D\}$. The last-named author of this paper showed that $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and S is a splitting set [36, Corollary 1.5] and proved some results about $D^{(S)}$ being locally GCD. In this paper, we study some ring-theoretic properties of locally GCD domains. We also characterize locally factorial domains and indicate conditions under which a locally factorial domains become locally factorial Krull domains. We use the $D + XD_S[X]$ construction to give some interesting examples of locally GCD domains that are not GCD domains.

We show, in Section 1, that D is a locally GCD domain if and only if D_P is a GCD domain for every prime ideal P of D, if and only if $aD \cap bD$ is locally principal for all $a, b \in D^*$. We also give some necessary and sufficient conditions for a locally GCD domain to be a generalized GCD domain (i.e., $aD \cap bD$ is invertible for all

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$a, b \in D^*$) or a Prüfer domain. In Section 2, we study and characterize locally factorial Krull domains. We show that D is a locally factorial Krull domain if and only if every minimal prime of a nonzero principal ideal is locally principal and a v-ideal of finite type, if and only if every minimal prime of a nonzero principal ideal is invertible. Finally, in section 3, we study when $D + XD_S[X]$ is a locally GCD domain. Among other things, we show that $D + XD_S[X]$ is a locally GCD domain if and only if D is a locally GCD domain and the saturation of S in D_P is a splitting set of D_P for every maximal ideal P of D with $P \cap S \neq \emptyset$. As corollaries, we have several interesting results: (i) if D is a locally factorial domain or a locally GCD domain with $\dim D = 1$, where $\dim D$ denotes the (Krull) dimension of D, then $D + XD_S[X]$ is a locally GCD domain for each multiplicative set S of D; and (ii) if S is a saturated multiplicative set of E, the ring of entire functions, generated by infinitely many prime elements, then $E + XE_S[X]$ is a locally GCD domain but not a GCD domain.

0.2. **Definitions and related results.** As our work will take us into the territory of the so-called star operations it seems pertinent to give the reader an idea of some of the notions. Let D be an integral domain, K be the quotient field of D, and $F(D)$ (resp., $f(D)$) be the set of nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of D.

A *star operation* $*$ on D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(i) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(ii) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(iii) $(A^*)^* = A^*$.

A fractional ideal $A \in F(D)$ is called a *$*$-ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ for some $B \in f(D)$. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid B \subseteq A \text{ and } B \in f(D)\}$. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{J_v \mid J \subseteq I \text{ and } J \in f(D)\}$, and $I_w = \{x \in K \mid xJ \subseteq I$ for some $J \in f(D)$ with $J_v = D\}$. The functions defined by $I \mapsto I_v$, $I \mapsto I_t$ and $I \mapsto I_w$ are all examples of star operations. For a finite character star operation $*$, a *maximal $*$-ideal* is an integral $*$-ideal maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. Continuing with star operation $*$ of finite character, it is well known that a maximal $*$-ideal is a prime ideal; every integral $*$-ideal is contained in a maximal $*$-ideal; $*$-Max$(D) \neq \emptyset$ if D is not a field. The t-operation is of finite character and so is the w-operation. Moreover, t-Max$(D) = w$-Max(D); $I_w = \cap_{P \in t\text{-Max}(D)} ID_P$ for all $I \in F(D)$; and $I_w D_P = ID_P$ for all $I \in F(D)$ and $P \in t$-Max(D). An $I \in F(D)$ is said to be *$*$-invertible* if $(II^{-1})^* = D$. We say that D is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal I of D is t-invertible, i.e., $(II^{-1})_t = D$. A reader in need of more introduction may consult [39], [20, Sections 32 and 34], or [22].

Let $T(D)$ be the group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$, and let $Prin(D)$ be its subgroup of nonzero principal fractional ideals of D. Then $Cl_t(D) = T(D)/Prin(D)$, called the *t-class group* of D, is an abelian group, which was defined by Bouvier in [12], at the suggestion of the last named author. Unlike the divisor class group which is defined only for completely integrally closed integral domains, the t-class group is defined for a general integral domain. However, it is interesting to note that for a Krull domain, the t-class group is exactly the divisor class group and for a Prüfer domain or an integral domain of dimension one, the t-class group is precisely the ideal class group. Here, the ideal class group of D is $Inv(D)/Prin(D)$, where $Inv(D)$ is the group of invertible ideals of D under the ordinary ideal multiplication. The ideal class group is also called the *Picard group*, so $Pic(D) = Inv(D)/Prin(D)$. Clearly, $Pic(D)$ is a subgroup of $Cl_t(D)$ because an invertible ideal is t-invertible.

An integral domain D is a GCD domain if and only if for every pair of elements $a, b \in D^*$, we have that $aD \cap bD$ is principal. It is well known that D is a GCD domain if and only if D is a PvMD with $Cl_t(D) = 0$ [12, Proposition 2]. A saturated multiplicative subset S of D is called a *splitting set* of D if for every $d \in D^*$, $d = st$ for some $s \in S$ and $t \in N(S)$, where $N(S) = \{x \in D^* \mid (x, s)_v = D \text{ for all } s \in S\}$. We say that a multiplicative set S of D is a *t-splitting set* if for each $d \in D^*$, we have $dD = (AB)_t$ for some integral ideals A and B of D with $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. Clearly, a splitting set is t-splitting, and if $Cl_t(D) = 0$, then a t-splitting set is splitting. It is known that D is a weakly factorial domain if and only if every saturated multiplicative subset of D is a splitting set [9, Theorem]. (A nonzero element $x \in D$ is said to be *primary* if xD is a primary ideal, and a *weakly factorial domain* (WFD) is an integral domain in which every nonzero nonunit can be written as a finite product of primary elements.) Following [36], we say that D is a *generalized UFD* (GUFD) if D is a GCD domain whose nonzero nonunits can be expressed as a product of finitely many rigid elements r with the following property: for each nonunit $h|r$, there is an integer $n \geq 0$ such that $r|h^n$. Here $r \in D^*$ is a rigid element if for all $x, y|r$ we have $x|y$ or $y|x$. Later, in [5, Theorem 10], it was shown that D is a GUFD if and only if D is a weakly factorial GCD domain, if and only if $D[X]$ is a WFD.

1. Locally GCD domains

We say that an ideal A of D is *locally principal* if AD_M is principal for each maximal ideal M of D. It is known that D is a GCD domain if and only if every nonzero prime ideal of D contains an element $a \in D^*$ with $aD \cap xD$ principal for all $x \in D$ [10, Theorem 2], if and only if $aD \cap bD$ is principal for all $a, b \in D^*$. Our first result is a locally GCD domain analogue of these two characterizations of GCD domains.

Theorem 1.1. *The following are equivalent for an integral domain D.*

(1) *D is a locally GCD domain.*
(2) *D_P is a GCD domain for every prime ideal P of D.*
(3) *$aD \cap bD$ is locally principal for all $a, b \in D^*$.*
(4) *Every nonzero prime ideal of D contains an element $a \in D^*$ such that $aD \cap xD$ is locally principal for all $x \in D$.*

Proof. (1) $\Rightarrow$ (2) This follows from the fact that if S is a multiplicative subset of a GCD domain D, then D_S is a GCD domain.

(2) $\Rightarrow$ (3) Let M be a maximal ideal of D. Then D_M is a GCD domain, and since $(aD \cap bD)D_M = aD_M \cap bD_M$, $(aD \cap bD)D_M$ is principal.

(3) $\Rightarrow$ (1) Let M be a maximal ideal of D, and let $0 \neq \alpha, \beta \in D_M$. Then $\alpha = \frac{a}{s}$ and $\beta = \frac{b}{t}$ for some $a, b \in D$ and $s, t \in D \backslash M$, and hence $\alpha D_M = aD_M$ and $\beta D_M = bD_M$. Also, as $(aD \cap bD)D_M = aD_M \cap bD_M = \alpha D_M \cap \beta D_M$, we conclude that $\alpha D_M \cap \beta D_M$ is principal.

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (3) Let S be the set of elements $a \in D^*$ such that $aD \cap xD$ is locally principal for all $x \in D$. To prove the implication, it suffices to show that $S = D^*$. Let $b, c \in S$, and let $x \in D$. If M is a maximal ideal, then $(bD \cap xD)D_M = byD_M$ for some $y \in D$. Hence, $(bcD \cap xD)D_M = bcD_M \cap (bD_M \cap xD_M) = bcD_M \cap byD_M = b(cD_M \cap yD_M)$ is principal. Next, assume that $d, e \in D^*$ with $d \notin S$. Then there is a $z \in D$ such that $dD \cap zD$ is not locally principal. Hence, $deD \cap ezD$ is not locally principal, and so $de \notin S$. Thus, S is a saturated multiplicative set of D. If $S \neq D^*$, then as S is saturated, there is a nonunit $\alpha \in D^* \setminus S$, so $\alpha D \cap S = \emptyset$. Again, since S is saturated, we can enlarge αD to a prime ideal P of D such that $P \cap S = \emptyset$ which is contrary to the assumption. □

Corollary 1.2. *Let D be a locally GCD domain and let R be a flat overring of D (e.g. a fraction ring of D). Then R is a locally GCD domain.*

Proof. Let Q be a maximal ideal of R, and let $P = Q \cap D$. Then $R_Q = D_P$, and hence R_Q is a GCD domain by Theorem 1.1 because D is a locally GCD domain. Thus, R is a locally GCD domain. □

Clearly, a GCD domain is a locally GCD domain, while a locally GCD domain need not be a GCD domain. To give some examples, recall that D is a Prüfer domain if and only if D_M is a valuation domain for every maximal ideal M of D (see, for example, [20, Theorem 22.1]), and a valuation domain is a GCD domain. Hence, a Prüfer domain is a locally GCD domain. A good example of a Prüfer domain is a Dedekind domain which can be characterized as a Noetherian Prüfer domain. It is well known that $\mathbb{Z}[\sqrt{-5}]$ is a non-PID Dedekind domain. Also, a Dedekind domain that is a GCD domain is a PID. Thus, $\mathbb{Z}[\sqrt{-5}]$ is a locally GCD domain which is not a GCD domain.

Recall that D is an *essential domain* if there is a family $\{P_\alpha\}_{\alpha \in \Lambda}$ of prime ideals of D such that $D = \cap D_{P_\alpha}$ and D_{P_α} is a valuation domain for each $\alpha \in \Lambda$. The family $\{P_\alpha\}_{\alpha \in \Lambda}$ may be called a defining family of valued primes of the essential domain D. A PvMD D is essential because D_M is a valuation domain for each maximal t-ideal M of D and $D = \cap_{M \in t\text{-Max}(D)} D_M$. Since a GCD domain is a PvMD, we conclude that a GCD domain is essential.

Lemma 1.3. *Let $\{D_\alpha\}_{\alpha \in J}$ be a family of flat overrings of an integral domain D such that $D = \cap_{\alpha \in \Lambda} D_\alpha$. If for each $\alpha \in \Lambda$, D_α is essential then so is D.*

Proof. Let $\{P_{\alpha\beta}\}_{\beta \in K_\alpha}$ be the defining family of valued primes of D_α. Since D_α is a flat overring of D, we get $(D_\alpha)_{P_{\alpha\beta}} = D_{P'_{\alpha\beta}}$ where $P'_{\alpha\beta} = P_{\alpha\beta} \cap D$. Thus, $D_\alpha = \cap_{\beta \in K_\alpha} D_{P'_{\alpha_\beta}}$. Now if we set $\Delta = \cup K_\alpha$, then $D = \cap_{\alpha \in \Lambda} D_\alpha = \cap_{\alpha\beta \in \Delta} D_{P'_{\alpha_\beta}}$ and $D_{P'_{\alpha_\beta}}$ is a valuation domain for all $\alpha_\beta \in \Delta$. □

Proposition 1.4. *If D is a locally GCD domain, then D is essential.*

Proof. Let M be a maximal ideal of D. Then D_M is a GCD domain, and hence D_M is essential. Thus, by Lemma 1.3, D is essential because $D = \cap D_M$ where M ranges over maximal ideals of D. □

As the definition of the t-class group hinges on the group of t-invertible t-ideals, the nature of the t-class group of a locally GCD domains will be determined by the nature of its t-invertible t-ideals. Let's go a bit general on this. Recall that an integral domain D is called a *$*$-domain* if $(\cap(a_i))(\cap(b_j)) = \cap(a_ib_j)$ for all $a_1, a_2, ..., a_m; b_1, ..., b_n \in D^*$. It may be noted that the $*$-property is equivalent to the property that for all $a_1, a_2, ..., a_m; b_1, ..., b_n \in K \setminus \{0\}$, where K is the quotient field of D, we have $(\cap(a_i))(\cap(b_j)) = \cap(a_ib_j)$. The reason is that (i) $D^* \subseteq K \setminus \{0\}$, and (ii) if $*$ holds, let $a_1, a_2, ..., a_m; b_1, ..., b_n \in K \setminus \{0\}$ and let $x, y \in D^*$ such that $xa_i, yb_j \in D$, then by $*$, $(\cap(xa_i))(\cap yb_j) = \cap(xa_iyb_j)$, and hence cancelling xy from both sides gives $(\cap(a_i))(\cap(b_j)) = \cap(a_ib_j)$. This notion was introduced in [35].

Lemma 1.5. *Let A be a nonzero fractional ideal of a $*$-domain D such that $A = B_v$ where B is finitely generated and suppose that $A^{-1} = C_v$ where C is finitely generated. Then A is invertible. Consequently, in a $*$-domain, every t-invertible t-ideal is invertible, i.e., $Cl_t(D) = Pic(D)$.*

Proof. According to [34, Proposition 1.6], D is a $*$-domain if and only if for every pair of nonzero finitely generated fractional ideals A, B of D, we have $(AB)^{-1} = A^{-1}B^{-1} = (A_vB_v)^{-1}$. Applying this to our situation we get $(AA^{-1})^{-1} = (B_vC_v)^{-1}$ $= B^{-1}C^{-1} = A^{-1}A$ since $B^{-1} = A^{-1}$ and $C^{-1} = (A^{-1})^{-1} = A$. But as $AA^{-1} \subseteq D$ and $(AA^{-1})^{-1} \supseteq D$, $AA^{-1} = (AA^{-1})^{-1}$ implies that $AA^{-1} = D$. Now if A is a t-invertible t-ideal, then A satisfies the requirements of the lemma. □

An integrally closed domain D is called a *Schreier domain* if for all nonzero $x, y, z \in D$, $x|yz$ implies that $x = rs$ for some $r, s \in D$ with $r|y$ and $s|z$.

Proposition 1.6. *If D is a locally GCD domain, then D is a $*$-domain, and hence $Cl_t(D) = Pic(D)$.*

Proof. We know that a GCD domain is Schreier [14, Theorem 2.4] and a Schreier domain is a $*$-domain [35, Corollary 1.7]; hence a GCD domain is a $*$-domain. Also, recall that a locally $*$-domain is a $*$-domain [35, Theorem 2.1]. Thus, a locally GCD domain is a $*$-domain, and hence $Cl_t(D) = Pic(D)$ by Lemma 1.5. (The fact that a locally GCD domain D satisfies $Cl_t(D) = Pic(D)$ follows also from [3, Theorem 2.1] where it is shown that this equality can be tested locally.) □

As in [2], we say that D is a *generalized GCD domain* (GGCD domain) if $aD \cap bD$ is invertible for all $a, b \in D^*$. It is known that D is a GGCD domain if and only if every v-ideal of finite type is invertible [2, Theorem 1]. Clearly, a GCD domain is a GGCD domain. Also, D is a PvMD if and only if $aD \cap bD$ is t-invertible for all $a, b \in D^*$ [28, Corollary 1.8]. So by Theorem 1.1 and Proposition 1.6,

$$\text{GCD domain} \Rightarrow \text{GGCD domain} \Leftrightarrow \text{locally GCD domain} + \text{P}v\text{MD}.$$

In [7, Corollary 1.3], it was shown that if D is of finite t-character, then D is a GGCD domain if and only if D is a locally GCD domain. (D is said to be of *finite t-character* if each nonzero nonunit of D is contained in only finitely many maximal t-ideals of D.)

Corollary 1.7. (cf. [35, Corollary 3.4]) *The following are equivalent for a PvMD D.*

(1) *D is a GGCD domain.*
(2) *D is a locally GCD domain.*
(3) $Cl_t(D) = Pic(D)$.
(4) *D is a $*$-domain.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Proposition 1.6.

(3) $\Rightarrow$ (1) Let $a, b \in D^*$. Then $aD \cap bD$ is t-invertible because D is a PvMD, and since $Cl_t(D) = Pic(D)$, $aD \cap bD$ is invertible. Thus, D is a GGCD domain.

(2) $\Leftrightarrow$ (4) [35, Corollary 3.4]. □

We next give some necessary and sufficient conditions for a locally GCD domain to be a GGCD domain.

Lemma 1.8. *Let $a, b \in D^*$. If the ideal (a, b) is not t-invertible, then there is a maximal t-ideal M of D such that $(a) \cap (b) = Ma \cap Mb$.*

Proof. (a, b) not being t-invertible means that there is a maximal t-ideal M such that $((a, b)(a, b)^{-1})_t \subseteq M$. Hence, $(a, b)^{-1} \subseteq M : (a, b) = Ma^{-1} \cap Mb^{-1} \subseteq (a, b)^{-1} = (a^{-1}) \cap (b^{-1})$. Multiplying by ab we get $(a) \cap (b) = Ma \cap Mb$. □

Corollary 1.9. *The following are equivalent for a locally GCD domain D.*

(1) *D is a GGCD domain.*
(2) *D is a PvMD.*
(3) *$aD \cap bD$ is t-invertible for all $a, b \in D^*$.*
(4) *$aD \cap bD$ is a v-ideal of finite type for all $a, b \in D^*$.*
(5) *$(M((a) \cap (b)))_w = Ma \cap Mb$ for all $a, b \in D^*$ and for all maximal t-ideals M of D.*

Proof. (1) $\Rightarrow$ (5) Let Q be a maximal t-ideal of D. Then D_Q is a valuation domain and hence a Prüfer domain. Moreover in a Prüfer domain every ideal is flat and so MD_Q is a flat ideal of D_Q. Hence, $(M((a) \cap (b))D_Q = MD_Q((a) \cap (b))D_Q = MD_Q((a)D_Q \cap (b)D_Q) = MD_QaD_Q \cap MD_QbD_Q = MaD_Q \cap MbD_Q = (Ma \cap Mb)D_Q$. Thus, $(M((a) \cap (b)))_w = Ma \cap Mb$.

(5) $\Rightarrow$ (2) Suppose that (a, b) is not t-invertible. Then by Lemma 1.8, $(a) \cap (b) = Ma \cap Mb$ for some maximal t-ideal M of D. Note that $(M((a) \cap (b)))_w = Ma \cap Mb$ by assumption; hence $(M((a) \cap (b)))D_M = (Ma \cap Mb)D_M = ((a) \cap (b))D_M$. This gives $MD_M((a) \cap (b))D_M = ((a) \cap (b))D_M$. Since D_M is a GCD domain we have that $((a) \cap (b))D_M$ is principal. Cancelling $((a) \cap (b))D_M$ from both sides we have $MD_M = D_M$ which is impossible. So every nonzero two generated ideal of D is t-invertible, and hence D is a PvMD.

(2) $\Leftrightarrow$ (3) [28, Corollary 1.8].

(3) ⇒ (4) This follows because a t-invertible t-ideal is a v-ideal of finite type.

(4) ⇒ (1) By Theorem 1.1, $aD \cap bD$ is locally principal, and hence $aD \cap bD$ is flat. Thus, the result follows directly from the fact that a flat v-ideal of finite type in an integral domain is invertible [38, Proposition 1]. □

Remark 1.10. (1) By Corollary 1.9, if D is a locally GCD domain, then D is a GGCD domain if and only if $aD \cap bD$ is t-invertible for all $a, b \in D^*$. So it's natural to ask if we can change "t-invertible" to "v-invertible". However, the requirement that $(a) \cap (b)$ is v-invertible for all $a, b \in D^*$ is already met by a locally GCD domain, in fact, by any essential domain [18, Proposition 2.1].

(2) By Corollary 1.9, a locally GCD domain that is not a PvMD is not a GGCD domain. It so happens that there are a lot of locally GCD domains that are not PvMDs, and hence not GGCDs. A classical example can be found in [29, Example 2.1]. This example is a redo of an example given by Heinzer and Ohm [24] of an essential domain that is not a PvMD. With some effort it was shown in [29] that every quotient ring of the ring of [29, Example 2.1] was essential. Towards the end of [29, Example 2.1], it was shown that the domain constructed in [24] is actually locally factorial and so a locally GCD domain. This establishes the existence of locally GCD domains that are not PvMDs. We shall encounter other examples of non-PvMD locally GCD domains in Section 3, in connection with the $D + XD_S[X]$ construction from locally GCD domains.

Corollary 1.9 tells us one way of deciding that a locally GCD domain is a GGCD domain. Here is another.

Proposition 1.11. *The following are equivalent for a locally GCD domain D.*

(1) *D is a GGCD domain.*
(2) *PD_P is a t-ideal of D_P for every maximal t-ideal P of D.*
(3) *For every prime t-ideal Q of D and for every multiplicative set S of D with $Q \cap S = \emptyset$, QD_S is a prime t-ideal of D_S.*
(4) *For all nonzero finitely generated ideals A of D and for every multiplicative set S of D, we have $(AD_S)_v = A_v D_S$.*
(5) *For every $a, b \in D^*$ and for every maximal ideal M of D, we have $((a, b)D_M)_v = (a, b)_v D_M$.*

Proof. (1) ⇒ (4) Since A is finitely generated, $(AD_S)^{-1} = A^{-1}D_S$ [33, Lemma 4]. Also, since D is a GGCD domain, A^{-1} is invertible, and hence finitely generated. Thus, $(AD_S)_v = (A^{-1}D_S)^{-1} = A_v D_S$.

(4) ⇒ (3) Let Q be a prime t-ideal of D. Then, by (4), $(QD_S)_t = \cup\{A_v D_S \mid (0) \neq A \subseteq Q$ is finitely generated $\} \subseteq QD_S \subseteq (QD_S)_t$. Thus, $(QD_S)_t = QD_S$.

(3) ⇒ (2) Clear.

(2) ⇒ (1) Let P be a maximal t-ideal of D. Then PD_P is a t-ideal of D_P, and since D_P is a GCD domain (hence a PvMD) by Theorem 1.1, $D_P = (D_P)_{PD_P}$ is a valuation domain. Thus, D is a PvMD, and so a GGCD domain by Corollary 1.9.

(4) ⇒ (5) Clear.

(5) ⇒ (1) Since D_M is a GCD domain, $(a, b)_v D_M = ((a, b)D_M)_v$ is principal. Hence, $(a, b)_v$ is locally principal, and thus $(a, b)_v$ is invertible [38, Corollary 2]. □

Recall that D is a GCD domain if and only if $(a,b)_v$ is principal for every $a,b \in D^*$. Also, note that if (a,b) is principal, then $(a,b)_v = (a,b)$. Hence, a GCD domain D is a Bezout domain if and only if $(a,b)_v = (a,b)$ for every $a,b \in D^*$. We next give a locally GCD domain analogue of this result.

Corollary 1.12. *The following are equivalent for an integral domain D.*

(1) *D is a Prüfer domain.*
(2) *D is a GGCD domain and $(a,b)_v = (a,b)$ for every $a,b \in D^*$.*
(3) *D is a locally GCD domain and $((a,b)D_M)_v = (a,b)D_M$ for every $a,b \in D^*$ and every maximal ideal M of D.*
(4) *D is a locally GCD domain and $M((a) \cap (b)) = Ma \cap Mb$ for each maximal ideal M and for all $a,b \in D^*$.*

Proof. (1) $\Rightarrow$ (3) The result follows from the fact that D_M is a valuation domain for all maximal ideals M of a Prüfer domain D.

(3) $\Rightarrow$ (2) Since (a,b) is finitely generated, $((a,b)D_M)_v = ((a,b)_v D_M)_v$ [33, Lemma 4], and hence $((a,b)D_M)_v = (a,b)_v D_M = (a,b)D_M$. Thus, D is a GGCD domain by Proposition 1.11 and $(a,b)_v = (a,b)$ [20, Theorem 4.10].

(2) $\Rightarrow$ (1) Let $a,b \in D^*$. Then $(a,b)_v$ is invertible because D is a GGCD domain, and thus (a,b) is invertible. Hence, D is a Prüfer domain.

(1) $\Leftrightarrow$ (4) This follows from [1, Proposition 1] because a Prüfer domain is a locally GCD domain. □

2. Locally factorial domains

Let D be an integral domain. As already mentioned, D is called *a locally factorial domain* if D_M is a UFD for each maximal ideal M of D. An essential part of being a UFD or of being factorial is being Krull. So we start with a somewhat novel characterization of Krull domains. Kang in [26, Theorem 3.6] proved that D is a Krull domain if and only if every minimal prime of a nonzero principal ideal of D is t-invertible. Using this result we prove the following lemma.

Lemma 2.1. *An integral domain D is factorial if and only if every minimal prime of a nonzero principal ideal of D is principal.*

Proof. Indeed, as a nonzero principal ideal is invertible and hence t-invertible, so by [26, Theorem 3.6], D is Krull. Hence, as in a Krull domain the minimal primes of nonzero principal ideals are of height one, we have a Krull domain whose height one primes are all principal which is a UFD. The converse is obvious. □

Using Lemma 2.1, we can state the following characterization of locally factorial domains.

Proposition 2.2. *An integral domain D is locally factorial if and only if every minimal prime of a nonzero principal ideal is locally principal.*

Proof. Let D be locally factorial and let P be a minimal prime of a nonzero principal ideal of D. Then for any maximal ideal M, PD_M is a minimal prime of a principal ideal of D_M and hence principal if $P \subseteq M$, otherwise $PD_M = D_M$. Thus every minimal prime of a nonzero principal ideal is locally principal. Conversely, suppose that every minimal prime of a nonzero principal ideal is locally principal and let M be a maximal ideal of D. Now let $\mathcal{P}$ be a minimal prime of a nonzero principal ideal of D_M. Then it is easy to see that $\mathcal{P} = PD_M$ where P is a minimal prime of a principal ideal. But then PD_M is principal and so is $\mathcal{P}$. As $\mathcal{P}$ is arbitrary, we conclude, by Lemma 2.1, that D_M is factorial. □

Proposition 2.2 shows that a number of results involving local factoriality can be stated using minimal primes of nonzero principal ideals. This brings to mind a celebrated characterization of UFDs from Kaplansky's book [27, Theorem 178].

Theorem 2.3. *Let R be an integral domain. The following three conditions are necessary and sufficient for R to be a UFD.*

(1) *R_M is a UFD for each maximal ideal M.*
(2) *Every minimal prime ideal in R is finitely generated.*
(3) *Every invertible ideal in R is principal.*

With a great deal of hindsight, of course, we can restate Theorem 2.3 as follows.

Theorem 2.4. *The following three conditions are necessary and sufficient for an integral domain D to be a UFD.*

(1) *Minimal primes of nonzero principal ideals are locally principal.*
(2) *Every minimal prime of a nonzero principal ideal is a v-ideal of finite type.*
(3) *Every invertible ideal in D is principal, i.e., $Pic(D) = 0$.*

Proof. By Proposition 2.2, (1) accounts for Theorem 2.3(1). In the presence of (1), (2) accounts for Theorem 2.3(2), because a nonzero locally principal ideal I is invertible if and only if I is of finite type [3, Theorem 2.1], and an invertible ideal is finitely generated. □

Throwing out (3) from Theorem 2.4, we have the following characterization of locally factorial Krull domains. Of course it is a well known result but we include it here to show the ease with which it can be proven with the approach of this paper and with Proposition 2.2.

Theorem 2.5. *The following are equivalent for an integral domain D:*

(1) *D is a locally factorial Krull domain.*
(2) *Every minimal prime of a nonzero principal ideal is locally principal and a v-ideal of finite type.*
(3) *Every minimal prime of a nonzero principal ideal is invertible.*

Proof. (1) $\Rightarrow$ (2) D being locally factorial implies that every minimal prime of a nonzero principal ideal is locally principal. Also, D being Krull implies that the minimal prime is a t-invertible t-ideal and hence a v-ideal of finite type.

(2) $\Rightarrow$ (3) This follows from the fact that a nonzero locally principal ideal is invertible if and only if it is of finite type [3, Theorem 2.1].

(3) $\Rightarrow$ (1) We note that every minimal prime of a principal ideal being invertible means it's locally principal which implies that D is locally factorial. Again, as an invertible ideal is t-invertible, that implies that D is a Krull domain. □

Let's note that (1) and (2) in Theorem 2.4 combine to give locally factorial Krull domains and that seems to indicate that locally factorial domains and locally factorial Krull domains are different. Indeed they are different and in the following we give two examples which can be useful in many contexts. As these examples are already amply discussed in literature we shall only give references to them.

Example 2.6. (a) Example 2.1 of [29] is given as an example of a locally factorial domain that is not a PvMD. Since a Krull domain is a PvMD, this example is not a Krull domain. Also, the ring D described in [29, Example 2.1], being an ascending union of UFDs has the property that $Pic(D) = (0)$.

(b) Almost Dedekind domains are integral domains D such that D_M is a rank one discrete valuation domain for every maximal ideal M of D. These domains, discussed in Section 36 of [20], obviously pass as locally factorial domains. Using [20, Theorem 37.2] one can conclude that an almost Dedekind domain is a Dedekind domain if and only if it is a Krull domain. The first ever example of a non-Krull almost Dedekind domain was due to Nakano [30]. Perhaps because this example was steeped in algebraic number theory and in German, Gilmer who formally introduced almost Dedekind domains in [19] chose to refer to this example as an example on page 426 of Nakano's paper. By [20, Theorem 36.7] one can always get a Bezout almost Dedekind domain that is not a PID. Thus in this case too we can find a non-Krull locally factorial domain D with $Pic(D) = (0)$.

One reason why we have chosen to give somewhat elaborate treatment to Example 2.6 is the following statement that one of the authors found in [40]. The statement goes as: Proposition 3.16. *R is factorial if and only if R is locally factorial and $Pic(R) = 0$.* Of course, as a locally GCD domain is integrally closed and as a Noetherian integrally closed domain is Krull, the above statement would be true if R is assumed to be Noetherian, but removed from that context it would be false. So, perhaps, it would be safer to restate the above proposition as: R is factorial if and only if R is locally factorial Krull and $Pic(R) = 0$.

Incidentally, for Noetheran domains, conditions apparently far less than locally factorial imply the locally factorial property. We start with a generalization of Noetherian domains that goes by the name of Mori domains to push the point home. An integral domain is called a *Mori domain* if it satisfies the ascending chain conditions on integral v-ideals. It is clear that Noetherian domains and Krull domains are Mori domains; and D is a Mori domain if and only if every nonzero ideal A of D is strictly v-finite, i.e., there is a nonzero finitely generated subideal J of A such that $A_v = J_v$ [37, Corollary 1.2].

Lemma 2.7. *A Mori domain D is a GCD domain if and only if D is a factorial domain.*

Proof. Let Q be a nonzero prime ideal of D. Then Q contains a prime t-ideal P of D, and since D is a Mori domain, there is a finitely generated ideal I of D such that $I_v = P$. Hence, if D is a GCD domain, then $P = I_v$ is principal, i.e., P is a principal prime ideal. Thus, D is a factorial domain [27, Theorem 5]. The converse is clear. □

An integral domain D is a *π-domain* if every nonzero principal ideal of D is a finite product of prime ideals. It is known that

$$\text{factorial domain} \Rightarrow \pi\text{-domain} \Rightarrow \text{Krull domain}.$$

Also, D is a π-domain if and only if D is a Krull domain and minimal prime ideals of D are invertible, if and only if D is a locally factorial domain of finite t-character (cf. [20, Theorem 46.7]). Also, if D is a π-domain, then $Cl_t(D) = Pic(D)$, and hence a π-domain D is a factorial domain if and only if every invertible ideal of D is principal (cf. [32, Theorem 1]).

Theorem 2.8. *The following are equivalent for a Mori domain D.*

(1) *D is a locally factorial domain.*
(2) *D is a locally GCD domain.*
(3) *D is a GGCD domain.*
(4) *D is a π-domain.*
(5) *Every minimal prime of a nonzero principal ideal is locally principal.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Leftrightarrow$ (3) This follows from [7, Corollary 1.3] because a Mori domain is of finite t-character [25, Proposition 1.4].

(2) $\Rightarrow$ (4) Let M be a maximal ideal of D. Then D_M is a GCD domain that is a Mori domain by (2) and [31, Théorème 2], and hence D_M is a factorial domain by Lemma 2.7. Thus, D is a π-domain because D is a locally factorial domain of finite t-character.

(4) $\Rightarrow$ (5) Let P be a prime ideal of D minimal over a principal ideal. Since D is a π-domain, P must be a height-one prime ideal, and hence P is invertible. Thus, P is locally principal.

(5) $\Rightarrow$ (1) Let M be a maximal ideal of D, and let Q be a nonzero prime ideal of D_M. Then $Q \cap D \neq (0)$, and so there is a prime ideal P of D such that $P \subseteq Q \cap D$ and P is minimal over a principal ideal. Hence, PD_M is a principal prime ideal by (5) and $PD_M \subseteq Q$. Thus, D_M is factorial [27, Theorem 5]. □

We have already characterized locally GCD domains saying that D is locally GCD if and only if for all $a, b \in D^*$, $aD \cap bD$ is locally principal. In some cases there are some conditions on ideals that end up giving something more than locally GCD property. The first such condition that comes to mind is GD: For all $A, B \in F(D)$, $(AB)^{-1} = A^{-1}B^{-1}$. This condition was first studied in [34] (see also [8]) and domains with GD were called *generalized Dedekind domains* (G-Dedekind domain) in [34] and pseudo-Dedekind domains in [8]. Now, as it was shown in [34, Theorem 1.1], D is a G-Dedekind domain if and only if A_v is invertible for each $A \in F(D)$. In particular, in a G-Dedekind domain D, A_v is invertible for each finitely generated ideal which is a characterizing property of GGCD domains which are locally GCD domains, as we already know.

The next result appears in [34, Theorem 1.10] that is a number of equivalent conditions for an integral domain to be a π-domain. Indeed these indicate that a Krull domain becomes locally factorial under conditions that are apparently far less than the locally factorial property.

Theorem 2.9. *The following are equivalent for an integral domain D.*

(1) *D is a G-Dedekind Krull domain.*
(2) *D is a G-Dedekind domain and Mori.*
(3) *D is Krull and locally factorial.*
(4) *D is Krull and a $*$-domain.*
(5) *D is Krull and, for all $a, b, c, d \in D \setminus \{0\}$,*

$$((a) \cap (b))((c) \cap (d)) = (ac) \cap (ad) \cap (bc) \cap (bd).$$

(6) *D is Mori and locally factorial.*
(7) *D is Mori and a $*$-domain.*
(8) *D is Mori and GGCD.*
(9) *Every t-ideal of D is invertible.*
(10) *Every associated prime ideal of D is invertible.*
(11) *D is Krull such that the product of any two v-ideals is again a v-ideal.*
(12) *D is G-Dedekind, every quotient ring of D is G-Dedekind and every rank one prime ideal of D is invertible.*

An integral domain D is said to be a *TV-domain* if every t-ideal of D is divisorial, i.e., $I_t = I_v$ for all $I \in F(D)$. TV-domains were studied in [25] where it was shown among other results that a Mori domain is a TV domain. Also, recall that a prime ideal minimal over a proper nonzero ideal of the form $(a) :_R (b)$ is called an associated prime of a principal ideal. Associated primes of principal ideals were studied in [13]. It can be shown that the minimal prime of a principal ideal is a t-ideal [23]. So in a TV-domain every associated prime of a principal ideal is divisorial. Now according to [25, Theorem 2.3], D is a Krull domain if and only if every associated prime of D is t-invertible.

Theorem 2.10. *The following are equivalent to each of the conditions in Theorem 2.9.*

(13) *D is a Krull domain and a locally GCD domain.*
(14) *D is a TV-domain that is also G-Dedekind.*

Proof. (3) $\Rightarrow$ (13) Clear.

(13) $\Rightarrow$ (4) This follows from Corollary 1.7 because a Krull domain is a PvMD.

(2) $\Rightarrow$ (14) This follows because a Mori domain is a TV-domain.

(14) $\Rightarrow$ (10) Note that an associated prime is a t-ideal which is divisorial in a TV-domain. Now take an associated prime P in D. Because D is G-Dedekind, P_v is invertible. But P being divisorial $P_v = P$ and so in D every associated prime of a principal ideal is invertible, which is (10). □

Next, while for a locally factorial domain D to be Krull, D has to be Mori but there are simpler looking conditions that will make a locally factorial domain into locally factorial Krull.

Theorem 2.11. *Let D be a locally factorial domain. Then D is locally factorial Krull if and only if there is a family $\{P_\alpha\}$ of prime ideals of D such that $D = \cap D_{P_\alpha}$ and the intersection is locally finite.*

Proof. If D is Krull then $\{P_\alpha\}$ can be taken to be the family of height one primes. Conversely, if there is a family $\{P_\alpha\}$ of prime ideals of D such that $D = \cap D_{P_\alpha}$ and the intersection is locally finite, then each of D_{P_α} is factorial and hence a Krull domain. Also, a locally finite intersection of Krull domains is a Krull domain. □

Corollary 2.12. *The following hold for a locally factorial domain D.*

1. *D is locally factorial Krull if D is of finite character.*
2. *D is locally factorial Krull if and only if D is of finite t-character.*

We can prove several routine results for locally factorial domains. For instance, if D is locally factorial then so is D_S and so is $D[X]$ as we did in the locally GCD case. Also one may wonder if a locally GCD domain is of t-dimension one. The answer is no. Example 2.1 of [29] is not a PvMD and so must have t-dimension greater than one, for if a locally factorial domain is of t-dimension one, it must be a PvMD as D is a PvMD if and only if D_M is a valuation domain for every maximal t-ideal M. This leaves us with the question: Is there a non-Krull locally factorial domain of t-dimension one that is not Prufer? The answer is yes, take a non-Dedekind almost Dedekind domain D and X an indeterminate then $D[X]$ is such an example.

3. The $D + XD_S[X]$ construction from locally GCD domains

Let D be an integral domain, S be a multiplicative set in D, X be an indeterminate over D, and $D^{(S)} = D + XD_S[X]$, i.e.,

$$D^{(S)} = \{f(X) \in D_S[X] \mid f(0) \in D\}.$$

Let's elaborate on the importance of the $D + XD_S[X]$ construction. This construction was studied in [15] as a way of constructing examples. It was shown in [15, Theorem 1.1] that $D + XD_S[X]$ is a GCD domain if and only if D is a GCD domain and $GCD(d, X)$ exists for all $d \in D^*$. This result later developed into "$D + XD_S[X]$ is a PvMD if and only if D is a PvMD and (d, X) is a t-invertible ideal for all $d \in D^*$" [6, Theorem 2.5]. So we can ask: when $D + XD_S[X]$ is a locally GCD domain, given that D is locally GCD ? In this section, we study some necessary and sufficient conditions for $D + XD_S[X]$ to be a locally GCD domain.

Clearly, the map $\varphi : D + XD_S[X] \to D$, given by $\varphi(a + Xf) = a$, is a retract, i.e., φ is a ring homomorphism such that $\varphi(a) = a$ for each $a \in D$. Our first result shows that if $D + XD_S[X]$ is a locally GCD domain, then so is D.

Lemma 3.1. *Let $A \subseteq B$ be an extension of integral domains having a retract $\varphi : B \to A$.*

1. *If B is a GCD domain, then so is A.*
2. *If B is a locally GCD domain, then so is A.*

Proof. (1) is well-known: If $aB \cap bB = cB$ with $a, b \in A$ and $c \in B$, then we derive easily that $aA \cap bA = \varphi(c)A$.

(2) Let P be a maximal ideal of A and $Q = \varphi^{-1}(P)$. It follows easily that $Q \cap A = P$. Then φ induces the ring homomorphism $q : B_Q \to A_P$ given by $q(b/s) = \varphi(b)/\varphi(s)$ for $b \in B$ and $s \in B \setminus Q$. It follows that q is a retract of $A_P \subseteq B_Q$, so part (1) applies. □

The next result is already known [36, Corollary 4.2], but we give its proof for the completeness of this section.

Lemma 3.2. *D is a locally GCD domain if and only if $D[X]$ is a locally GCD domain.*

Proof. ($\Rightarrow$) Let M be a maximal ideal of $D[X]$, and let $P = M \cap D$. Then D_P is a GCD domain, and hence $D_P[X]$ is a GCD domain. Thus, $D[X]_M$ is a GCD domain because $D[X]_M = (D_P[X])_{M_P}$. ($\Leftarrow$) This follows directly from Lemma 3.1(2). □

Remark 3.3. Let $A \subseteq B$ be an extension of integral domains. According to [16], B is called a *locally polynomial ring* over A if for every prime ideal $P \subseteq A$, $B_P = B \otimes_A A_P$ is a polynomial ring over A_P. It is easy to adapt the proof of Lemma 3.2 to show that if A is a locally GCD domain and B is a locally polynomial ring over A, then B is a locally GCD domain. For example, let $D = \mathbb{Z}[\{X/p_n, Y/p_n\}_n]$, where $\{p_n\}_n$ is the set of prime numbers. Then D is a locally GCD domain (cf. [16, Introduction]), while D is not a PvMD [11]. Thus, D is not a GCD domain.

It is known that $D + XD_S[X]$ is a GCD domain if and only if D is a GCD domain and S is a splitting set in D. We next give a locally GCD domain analogue which has many interesting applications. To do this, we first recall that a splitting set S of D is an *lcm splitting set* if $sD \cap dD$ is principal for all $s \in S$ and $d \in D$. It is known that if S is an lcm splitting set, then D is a GCD domain if and only if D_S is a GCD domain [21, Theorem 3.1]. The proof of (2) $\Rightarrow$ (1) of the next theorem is almost the same as in [36, Proposition 4.1], except that the proof is a bit more streamlined.

Theorem 3.4. *The following are equivalent for a multiplicative subset S of D.*

1. *$D + XD_S[X]$ is a locally GCD domain.*
2. *D is a locally GCD domain and the saturation of S in D_P is a splitting set in D_P for every maximal ideal P of D with $P \cap S \neq \emptyset$.*
3. *D is a locally GCD domain and $aD_S \cap D$ is a locally principal ideal for every $a \in D^*$.*

Proof. (1) $\Rightarrow$ (2) By Lemma 3.1, D is a locally GCD domain, and hence we need only prove that the saturation of S in D_P is splitting for every maximal ideal P of D with $P \cap S \neq \emptyset$. Let $M = P + XD_S[X]$. Then M is a maximal ideal of $D^{(S)}$ with $M \cap S \neq \emptyset$ [15, Theorem 2.1]. Now as $D \backslash P \subseteq D^{(S)} \backslash M$ we conclude that $(D^{(S)})_M$ is a ring of fractions of $(D^{(S)})_{D \backslash P} = D_P + XD_{S(D \backslash P)}[X]$. Next $D^{(S)} \backslash M = \{a + Xf \mid a \in D \backslash P \text{ and } f \in XD_S[X]\}$. So $(D^{(S)})_M = (D_P + XD_{S(D\backslash P)}[X])_{D^{(S)} \backslash M} = (D_P + XD_{S(D\backslash P)}[X])_W$, where $W = \{u + Xf \mid u \text{ is a unit in } D_P \text{ and } f \in XD_{S(D\backslash P)}[X]\}$.
Note that (i) $D_P + XD_{S(D\backslash P)}[X]$ is a Schreier domain [15, Remark before Theorem 1.1] because D_P is a GCD domain and (ii) each of $1 + Xf$ is a product of atoms which generate height one primes. Hence, W is an lcm splitting set of $D_P + XD_{S(D\backslash P)}[X]$, once we note that every element g of $D_P + XD_{S(D\backslash P)}[X]$ can be written as $g = h(1 + Xf)$ where h is not divisible by any of the members of W and $1 + Xf \in W$. Thus $(D_P + XD_{S(D\backslash P)}[X])_W$ is a GCD domain if and only if $D_P + XD_{S(D\backslash P)}[X]$ is a GCD domain. But $D_P + XD_{S(D\backslash P)}[X]$ is a GCD domain

if and only if the saturation of S in D_P is a splitting set (as S is a multiplicative set of D_P and $D_{S(D\setminus P)} = (D_P)_S$).

(2) $\Rightarrow$ (1) Let M be a maximal ideal of $D + XD_S[X]$. Then there are three possibilities: (i) $M \cap D = (0)$, (ii) $M \cap D \neq (0)$ but $M \cap S = \emptyset$, and (iii) $M \cap S \neq \emptyset$.

(i) If $M \cap D = (0)$, then $(D+XD_S[X])_M = ((D+XD_S[X])_{D^*})_{M_{D^*}} = K[X]_{MK[X]}$ is a valuation domain, where K is the quotient field of D.

(ii) If $M \cap D \neq (0)$ but $M \cap S = \emptyset$, then $(D+XD_S[X])_M = ((D+XD_S[X])_S)_{M_S} = (D_S[X])_{MD_S[X]}$ is a GCD domain because $D_S[X]$ is a locally GCD domain, by Corollary 1.2 and Lemma 3.2.

(iii) Finally, assume $M \cap S \neq \emptyset$, and let $M \cap D = P$. Then P is a maximal ideal of D and $M = P + XD_S[X]$ [7, Lemma 2.1]. Now $(D + XD_S[X])_M \supseteq (D + XD_S[X])_{D\setminus P} = D_P + XD_{S(D\setminus P)}[X]$ which is a GCD domain [36, Corollary 1.5] because D_P is a GCD domain and the saturation of S (hence of $S(D \setminus P)$) in D_P is a splitting set by assumption. Now as $(D + XD_S[X])_M$ is a ring of fractions of the GCD domain $D_P + XD_{S(D\setminus P)}[X]$, we have that $(D + XD_S[X])_M$ is a GCD domain.

(2) $\Rightarrow$ (3). Let $a \in D^*$ and P be a maximal ideal of D. If $P \cap S = \emptyset$, then $(aD_S \cap D)D_P = aD_P$ is principal. If $P \cap S \neq \emptyset$, then $(aD_S \cap D)D_P = aD_{S(D\setminus P)} \cap D_P$ is principal [4, Corollary 1.3].

(3) $\Rightarrow$ (2) This follows from [4, Corollary 1.3]. □

The next result is a locally GCD domain analogue of the fact that if D is a UFD and if S is any multiplicative set in D, then $D + XD_S[X]$ is a GCD domain [15, Corollary 1.2].

Corollary 3.5. *If D is a locally factorial domain, then $D + XD_S[X]$ is a locally GCD domain for each multiplicative set S of D.*

Proof. Let P be a maximal ideal of D such that $P \cap S \neq \emptyset$. Then D_P is a UFD, and hence every saturated multiplicative subset of D_P is a splitting set [9, Theorem]. Thus, by Theorem 3.4, $D + XD_S[X]$ is a locally GCD domain. □

Corollary 3.5 can be put to use immediately to get various examples of locally GCD domains from locally factorial domains. What is interesting is that if we start with a locally factorial domain D, we end up with a locally GCD domain $D + XD_S[X]$ for any multiplicative set S of D. Also, if D is a locally factorial domain that is not a UFD, then $D + XD_S[X]$ is not a GCD domain for some multiplicative set S of D. This is because $D + XD_S[X]$ being a GCD domain for every multiplicative set S of D forces D to be a GUFD and a locally factorial GUFD is a UFD.

The t-dimension of an integral domain D is defined by

$$t\text{-dim}D = \sup\{n \mid P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n \text{ is a chain of prime } t\text{-ideals of } D\}.$$

Hence, t-dim$D = 1$ if and only if D is not a field and every prime t-ideal of D is a maximal t-ideal. As already pointed out, D is called an *almost Dedekind domain* if D_M is a local PID for every maximal ideal M of D. Obviously, an almost Dedekind domain D is a Prüfer domain (see Gilmer [20] for basic material on this type of Prüfer domains and [17] for some factorization properties of almost

Dedekind domains), and it is a Bezout domain if and only if $Pic(D) = 0$. Also, Dedekind domains are almost Dedekind and the (Krull) dimension of an almost Dedekind domain is one.

Corollary 3.6. *Let D be an almost Dedekind domain but not a field.*

(1) *$D + XD_S[X]$ is a locally GCD domain for every multiplicative set S of D.*
(2) *D is a Dedekind domain if and only if $D + XD_S[X]$ is a GGCD domain for every multiplicative set S of D.*
(3) *D is a PID if and only if $D + XD_S[X]$ is a GCD domain for every multiplicative set S of D.*
(4) *t-$dim(D + XD_S[X]) = 2$ if and only if $D[X] \subsetneq D + XD_S[X]$.*

Proof. (1) This is an immediate consequence of Corollary 3.5, because an almost Dedekind domain is a locally factorial domain.

(2) Recall that an almost Dedekind domain D is a Dedekind domain if and only if D is of finite character (i.e., each nonzero nonunit of D is contained in only finitely many maximal ideals of D) [20, Theorem 37.2], if and only if every multiplicative subset of D is a t-splitting set [6, p. 8]; a locally GCD domain is a GGCD domain if and only if it is a PvMD (Corollary 1.9); and $D + XD_S[X]$ is a PvMD if and only if D is a PvMD and S is a t-splitting set [6, Theorem 2.5]. Thus, by (1), D is a Dedekind domain if and only if $D + XD_S[X]$ is a GGCD domain for every multiplicative set S of D.

(3) This result is an immediate consequence of the following two facts: (i) D is a weakly factorial GCD domain if and only if $D + XD_S[X]$ is a GCD domain for every multiplicative subset S of D [36, Theorem 3.1] and (ii) an almost Dedekind domain is a PID if and only if it is a weakly factorial domain [20, Theorem 37.2].

(4) This follows from the facts that $\dim(D + XD_S[X]) = 2$ [15, Theorem 2.6], t-$\dim(D[X]) = 1$, and if P is a prime ideal of D with $P \cap S \neq \emptyset$, then $P + XD_S[X]$ is a prime t-ideal if and only if P is a prime t-ideal [7, Lemma 2.1]. □

As in [36], we say that a prime ideal P *intersects a multiplicative set S in detail* if for all nonzero prime ideals $Q \subseteq P$, $Q \cap S \neq \emptyset$. It is easy to show that P intersects S in detail if and only if $D_{S(D \setminus P)} = K$, the quotient field of D.

Corollary 3.7. (cf. [36, Proposition 4.1]) *Let S be a multiplicative set in D such that M intersects S in detail for every maximal ideal M of D with $M \cap S \neq \emptyset$. Then D is a locally GCD domain if and only if $D + XD_S[X]$ is a locally GCD domain.*

Proof. Let M be a maximal ideal of D such that $M \cap S \neq \emptyset$. Then M intersects S in detail, that is, $D_{S(D \setminus M)} = K$; so, in this case, the saturation $\overline{S}$ of S in D_M is $D_M \setminus \{0\}$. Hence, $\overline{S}$ is a splitting set of D_M. Thus, the result follows directly from Theorem 3.4. □

As remarked in [36, p. 104], Corollary 3.7 is hard to apply. But in some cases, it delivers locally GCD domains that are not PvMDs. Of these, [36, Example 2.6] is a very easy example of a locally GCD domain that is a non-GCD Schreier domain and hence a non-PvMD.

Remark 3.8. (1) Let S be a multiplicative set of D such that $D \subsetneq D_S$; so there is a nonunit $s \in S$ of D. Clearly, $X \cdot (\frac{1}{s})^n \in D + XD_S[X]$ for all integers $n \geq 1$, but $\frac{1}{s} \notin D + XD_S[X]$. Hence, $D + XD_S[X]$ is not completely integrally closed, while a locally factorial domain is locally completely integrally closed, and so completely integrally closed. Therefore, $D + XD_S[X]$ is not a locally factorial domain even though D is factorial.

(2) Let D be a locally factorial domain with $\dim D \geq 2$ that is not factorial, and let M be a maximal ideal of D. Choose $0 \neq a \in M$, and put $S = \{a^n \mid n \geq 0\}$. Then $D + XD_S[X]$ is a locally GCD domain and $S \cap M \neq \emptyset$. But, since $\dim D \geq 2$, we may assume that D_M is a factorial domain of $\dim D_M \geq 2$. So there is at least one prime ideal P of D such that $P \subsetneq M$ and $P \cap S = \emptyset$. This shows that the assumption about S of Corollary 3.7 is not necessary for $D + XD_S[X]$ to be a locally GCD domain.

Corollary 3.9. *Let K be the quotient field of D. Then D is a locally GCD domain if and only if $D + XK[X]$ is a locally GCD domain.*

Proof. Let $S = D^*$. Then S is a multiplicative set of D that satisfies the condition of Corollary 3.7 and $D + XK[X] = D + XD_S[X]$. Thus, the result follows directly from Corollary 3.7. $\square$

Let S be a multiplicative set of D. Clearly, if P is a height-one prime ideal of D intersecting S, then P intersects S in detail. Hence, if $\dim D = 1$, then every prime ideal of D intersecting S intersects S in detail; so by Corollary 3.7, we have

Corollary 3.10. *Let D be a locally GCD domain of $\dim D = 1$. Then $D + XD_S[X]$ is a locally GCD domain for every multiplicative set S of D.*

Let E be the ring of entire functions, and let S be the multiplicative set of E generated by the principal primes of E. It is well known that E is a Bezout domain in which every principal prime ideal is a maximal ideal of height-one [20, Exercise 19 on p. 147]. Hence, by Corollary 3.7, $E + XE_S[X]$ is a locally GCD domain [36, page 95].

Example 3.11. *Let E be the ring of entire functions, and let S be a saturated multiplicative set of E generated by prime elements.*

1. *$E + XE_S[X]$ is a locally GCD domain.*
2. *If $E + XE_S[X]$ is a GGCD domain, then $E + XE_S[X]$ is a GCD domain.*
3. *$E + XE_S[X]$ is a GCD domain if and only if S is generated by finitely many prime elements.*

Proof. (1) If M is a maximal ideal of E with $M \cap S \neq \emptyset$, then M is of height-one, and hence M intersects S in detail. Thus, by Corollary 3.7, $E + XE_S[X]$ is a locally GCD domain because E is a Bezout domain (and so a locally GCD domain).

(2) Since a GGCD domain is a PvMD, $E + XE_S[X]$ is a PvMD, and hence S is a splitting set [6, Theorem 2.5] because $Cl_t(E) = 0$. Thus, $E + XE_S[X]$ is a GCD domain [36, Corollary 1.5].

(3) Let z be a prime element of E and $0 \neq d \in E$. It is well known that there is an integer $n = n(z) \geq 0$ such that $z^n | d$ but $z^{n+1} \nmid d$ in E; $z \nmid d$ if and only if

$(z, d) = E$; and for any set $\{z_i\}$ of prime elements, there exists a nonzero $x \in E$ such that $z_i | x$ for all z_i. Hence, S is finitely generated if and only if S is a splitting set. Thus, by [36, Corollary 1.5], $E + XE_S[X]$ is a GCD domain if and only if S is finitely generated. □

References

[1] M.M. Ali, *Generalized GCD rings IV*, Beitrage zur Algebra und Geom. in press.

[2] D.D. Anderson and D.F. Anderson, *Generalized GCD domains*, Comment. Math. Univ. St. Pauli 28 (1979), 215-221.

[3] D.D. Anderson, *Globalization of some local properties in Krull domains*, Proc. Amer. Math. Soc. 85 (1982), 141-145.

[4] D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Factorization in integral domains, II*, J. Algebra 152 (1992), 78-93.

[5] D. D. Anderson, D. F. Anderson, and M. Zafrullah, *A generalization of unique factorization,* Boll. Un. Mat. Ital. A(7) 9 (1995), 401-413.

[6] D.D. Anderson, D.F. Anderson and M. Zafrullah, *The ring $D + XD_S[X]$ and t-splitting sets*, Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001), 3-16.

[7] D.D. Anderson, G.W. Chang, and M. Zafrullah, *Integral domains of finite t-character*, J. Algebra 396 (2013), 169-183.

[8] D.D. Anderson and B.G. Kang, *Pseudo Dedekind domains and divisorial ideals in $R[X]_T$*, J. Algebra 122 (1989), 323-336.

[9] D.D. Anderson and M. Zafrullah, *Weakly factorial domains and groups of divisibility*, Proc. Amer. Math. Soc. 109 (1990), 907-913.

[10] D.D. Anderson and M. Zafrullah, *On a theorem of Kaplansky*, Boll. Un. Mat. Ital. (7) 8-A (1994), 397-402.

[11] J.T. Arnold and R. Matsuda, *An almost Krull domain with divisorial height one primes,* Canad. Math. Bull. 29 (1986), 50-53.

[12] A. Bouvier, *Le groupe des classes d'un anneau intégré*, 107ème Congrès National des Sociétés Savantes, Brest, 1982, fasc. IV, 85-92.

[13] J. Brewer and W. Heinzer, *Associated primes of principal ideals*, Duke Math. J. 41 (1974), 1-7.

[14] P.M. Cohn, *Bezout rings and their subrings*, Proc. Cambridge Philos. Soc. 64 (1968), 251-264.

[15] D. Costa, J. Mott and M. Zafrullah, *The construction $D + XD_S[X]$*, J. Algebra 53 (1978), 423-439.

[16] P. Eakin and J. Silver, *Rings which are almost polynomial rings*, Trans. Amer. Math. Soc. 174 (1972), 425-449.

[17] M. Fontana, E. Houston, and T. Lucas, *Factoring ideals in integral domains*, Lecture Notes of the Unione Matematica Italiana, 14. Springer, Heidelberg; UMI, Bologna, 2013.

[18] M. Fontana and M. Zafrullah, *On v-domains: a survey*, in "Commutative algebra – Noetherian and non-Noetherian Perspectives", pp. 145-179, Springer, New York, 2011.

[19] R. Gilmer, *Integral domains which are almost Dedekind*, Proc. Amer. Math. Soc. 15 (1964), 813-818.

[20] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[21] R. Gilmer and T. Parker, *Divisibility properties in semigroup rings*, Michigan Math. J. 21 (1974), 65-86.

[22] F. Halter-Koch, *Ideal Systems – An Introduction to Multiplicative Ideal Theory*, Dekker, New York, 1998.

[23] Hedstrom and E. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra 18 (1980), 37-44.

[24] W. Heinzer and J. Ohm, *An essential ring which is not a v-multiplication ring,* Canad. J. Math. 25 (1973), 856-861.

[25] E. Houston and M. Zafrullah, *Integral domains in which each t-ideal is divisorial*, Michigan Math. J. 35 (1988), 291-300.

[26] B.G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra 124 (1989), 284-299.
[27] I. Kaplansky, *Commutative Rings*, Polygonal Publishing House, Washington, New Jersey, 1994.
[28] S. Malik, J. Mott and M. Zafrullah, *On t-invertibility,* Comm. Algebra 16 (1988), 149-170.
[29] J. Mott and M. Zafrullah, *On Prüfer v-multiplication domains,* Manuscripta Math. 35 (1981), 1-26.
[30] N. Nakano, *Idealtheorie in einem speziellen unendlichen algebraischen Zahlkörper*, J. Sci. Hiroshima Univ. Ser. A. 16 (1953), 425-439.
[31] J. Querré, *Intersections d'anneaux integres,* J. Algebra 43 (1976), 55-60.
[32] M. Zafrullah, *On a result of Gilmer*, J. London Math. Soc. 16 (1977), 19-20.
[33] M. Zafrullah, *On finite conductor domains*, Manuscripta Math. 24 (1978), 191-203.
[34] M. Zafrullah, *On generalized Dedekind domains,* Mathematika 33 (1986), 285-296.
[35] M. Zafrullah, *On a property of pre-Schreier domains,* Comm. Algebra 15 (1987), 1895-1920.
[36] M. Zafrullah, *The $D + XD_S[X]$ construction from GCD domains,* J. Pure Appl. Algebra 50 (1988), 93-107.
[37] M. Zafrullah, *Ascending chain conditions and star operations,* Comm. Algebra 17 (1989), 1523-1533.
[38] M. Zafrullah, *Flatness and invertibility of an ideal,* Comm. Algebra 18 (1990), 2151-2158.
[39] M. Zafrullah, *Putting t-invertibility to use,* in Non-Noetherian Commutative Ring Theory, pp. 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.
[40] http://people.fas.harvard.edu/~amathew/chfactorization.pdf.

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