CORRIGENDUM TO "INTEGRAL DOMAINS OF FINITE t-CHARACTER" [J. ALGEBRA 396 (2013), 169-183]

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The main purpose of this note is to report that Example 1.4 of [1] has a serious flaw and to point to correct examples that already exist in the paper. We shall also indicate the positive aspects of the example in question. Our terminology is the same as used in [1] and so we shall proceed without any new introductions.

Example 1.4 of [1] is as stated follows:

Example 1. Let $R = \mathbb{R}[\![X,Y,Z]\!]$ be the power series ring over the field $\mathbb{R}$ of real numbers, $M = (X,Y,Z)\mathbb{R}[\![X,Y,Z]\!]$, and $D = \mathbb{Q} + M$, where $\mathbb{Q}$ is the field of rational numbers. Then R is a 3-dimensional local Noetherian Krull domain with maximal ideal M, and D is a quasi-local domain with maximal ideal M such that $\text{Spec}(R) = \text{Spec}(D)$ and M is a v-ideal of D. Hence D is of finite t-character. But, if P is a prime ideal of D with $\text{ht}P = 2$, then P is a prime ideal of R such that $D_P = R_P$. Clearly, R_P is a 2-dimensional Krull domain and $\text{ht}PR_P = 2$, and thus $PD_P = PR_P$ is not a t-ideal. Thus D is not well-behaved.

To explain the error we shall need to explain what a well-behaved domain is, and use a result from the literature to establish that the ring D in Example 1 is a well-behaved domain, contrary to the claim in a prior to the example. We shall also discuss the source of this error and the positive aspects of the example.

The notion of a well-behaved domain was introduced in [6] where a prime t-ideal P of an integral domain D was said to be *well-behaved* if PD_P is also a t-ideal, D was called *conditionally well-behaved* if every maximal t-ideal is well-behaved, and D was called *well-behaved* if every prime t-ideal is well-behaved. So the essential error in Example 1 is that while it shows that for every height 2 prime ideal P, PD_P is not a prime t-ideal, it does not produce a height 2 prime t-ideal. Now we proceed to show that there isn't any height 2 prime t-ideal in D.

It was shown in [6] that if D is such that for every nonzero finitely generated ideal I of D and for every multiplicative set S of D the ideal I_vD_S is v-ideal then D is well-behaved [6, Proposition 1.4]. From this, on pages 201-202 of [6] it was concluded that D is well-behaved if it has the property that for each finitely generated nonzero ideal I the ideal I^{-1} is a v-ideal of finite type, i.e., there is a finitely generated fractional ideal J such that $I^{-1} = J_v$. So Noetherian domains and Mori domains (domains satisfying the ascending chain condition on integral v-ideals) are all well-behaved. Now if we can show that the domain D in Example 1 is actually a Mori domain, then Example 1 is false. That the domain D in Example 1 is a Mori domain has been established in [3, Theorem 4.18]. Theorem 4.18 of [3]

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says: Consider a pullback diagram, where R is an integral domain, M is a maximal ideal of R, and A is a subring of k:

$$\begin{array}{ccc} D = \varphi^{-1}(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ R & \xrightarrow{\varphi} & k = R/M \end{array}$$

Then D is Mori if and only if R is Mori and A is a field. Here if we set $R = \mathbb{R}[\![X, Y, Z]\!]$ which is Noetherian and hence Mori and $M = (X, Y, Z)\mathbb{R}[\![X, Y, Z]\!]$, we have that $\mathbb{Q} \subset \mathbb{R} = R/M$, $D = \mathbb{Q} + M$, and D is Mori. So with the argument of Example 1, there is no prime t-ideal of height 2 in the domain $D = \mathbb{Q} + M$ of Example 1 to ensure that the domain is not well-behaved.

Having seen that the conclusion of Example 1 is false, we proceed to put the rest of the example to, hopefully, good use. We note that the main arguments include the facts that $\mathrm{Spec}(R) = \mathrm{Spec}(D)$, verifiable from [2, Corollary 3.11], and $PR_P = PD_P$ for each height 2 prime ideal P which follows because $r = \frac{ar}{a} \in D_P$ for any $r \in \mathbb{R}$ and $a \in M \setminus P$. So, if we keep in mind the fact that the maximal ideal of the quasi-local ring D in Example 1 is a t-ideal, the argument involved in the example gives us the following result.

Proposition 2. (1) The ring $D = \mathbb{Q}+ (X, Y, Z)\mathbb{R}[\![X, Y, Z]\!]$ described in Example 1 contains no prime t-ideal of height 2. (2) Each height 2 prime ideal P of D is such that $P_t = M = (X, Y, Z)\mathbb{R}[\![X, Y, Z]\!]$.

That a prime ideal contained in a prime t-ideal may not be a t-ideal was suspected until Mimouni [4, Example 2.8] came up with an example. Indeed in the terminology of [4] each prime ideal of height 2 of D is a w-ideal that is not a t-ideal. We note that our example is more efficient and less restricted.

What is ironic is that this example was not necessary to make the point it is purporting to make! That is, there are examples actually in the paper that would serve the same purpose of providing integral domains of finite t-character that are not well-behaved. Of these one example comes from Section 2 of [6]. This example of [6] is an example of a conditionally well-behaved domain which is shown to be of finite t-character in [1, Example 3.9]. The other examples may be found in Remark 3.2(2) of [1]. (The example alluded to can be repeated as follows: *Let* $\mathbb{Z}$ (*resp.,* $\mathbb{Q}$) *be the ring* (*resp., field*) *of integers* (*resp., rational numbers*), p *be a nonzero prime number,* X *and* Y *be indeterminates over* $\mathbb{Q}$, $R = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!]$, K *be the quotient field of* R, Z *be an indeterminate over* K *and let* $D = R + ZK[\![Z]\!]$. *Then* D *is of finite* t*-character, and* $N := (X, Y)\mathbb{Q}[\![X, Y]\!] + ZK[\![Z]\!] = \cap p^n D$ *is a prime* t*-ideal. But if* $S = \{p^n \mid n = 0, 1, 2, \dots\}$, *then* $D_S = \mathbb{Q} + (X, Y)\mathbb{Q}[\![X, Y]\!] + ZK[\![Z]\!]$ *and* $ND_S = (X, Y)\mathbb{Q}[\![X, Y]\!] + ZK[\![Z]\!]$ *which is not a* t*-ideal because* X *and* Y *are* v*-coprime in* D_S.)

One may wonder why one should be so concerned about a prime t-ideal being well behaved, or about a domain being well-behaved. A good description of the reasons is presented in the last paragraph on page 94 of [5] as echoed in the introduction in [6].

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References

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