What v-coprimality can do for you

Muhammad Zafrullah

Idaho State University

Pocatello,Idaho

The purpose of this talk is to introduce the audience to the notion of v-coprimality, its applications, its morphs and its generalizations.

Terminology and notation.

- D__ an integral domain, $K = qf(D)$
- $F(D)$ __ nonzero fractional ideals of D, $A^{-1} = \{x \in K : xA \subseteq D\}$, $A_v = (A^{-1})^{-1} = \bigcap_{A \subseteq cD} cD,\ c \in K \backslash \{0\}$
- A function, $*$ on $F(D)$ is a star operation, if for all $a \in K\backslash\{0\}$, and $A, B \in F(D)$

(1*) $(a)^* = (a)$, $(aA)^* = aA^*$, (2*) $A \subseteq A^*$ and $A \subseteq B \Rightarrow A^* \subseteq B^*$ (3*) $(A^*)^* = A^*$.

Given $A, B \in F(D)$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$ ($*$-mult.)

To each $*$ associate $*_f$ defined by $A^{*_f} = \cup\{F^* : 0 \neq F$ is a D-submodule of $A\}$ for $A \in F(D)$. Call $*$ of finite type if $A^* = A^{*_f}$ for all $A \in F(D)$. The well known t-operation is given by $t = v_f$.

The identity function $A \mapsto A$ on $F(D)$ is the

d-operation. If $\{D_\alpha\}$ is a family of overrings of D such that $D = \cap D_\alpha$ then the function $A \mapsto A^* = \cap AD_\alpha$ is also a star operation.

$A \in F(D)$ is a $*$-ideal of finite type if $A = B^*$ for some f.g. $B \in F(D)$ and A is $*$-invertible if $(AB)^* = D$ for some $B \in F(D)$. A t-invertible t-ideal is of finite type. D is a Prufer v-multiplication domain(PVMD) if f.g. nonzero ideals of D are t-invertible.

Good sources: Gilmer's [Multiplicative Ideal Theory, Marcel Dekker, New York, 1972 and Anderson and Cook's [Comm. Algebra 28(2000), 2461-2475]

- Definition and basics

Two nonzero elements $x, y \in D$ are called *v-coprime* if $(x,y)_v = D$ (i.e. $xD \cap yD = xyD$ or equivalently $(x,y)^{-1} = D$.) Obviously a proper t-ideal cannot contain a pair of v-coprime elements.

- It is easy to establish that $(a,b)_v = D \Leftrightarrow ((a,b) \subseteq (c/d) \Rightarrow c|d$ (c divides d)). So $(x,y)_v \neq D \Leftrightarrow$ (there exist $c, d \in D$ such that $c \nmid d$ but$(a,b) \subseteq c/d$).
- Ordinarily $x, y \in D$ are said to be *coprime* if x and y have no nonunit common factor in D. Note that x, y being v-coprime implies x, y coprime but not conversely. (Try $(x,y)_v = D \wedge (x,y) \subseteq dD$.) However, negation of coprimality is much cleaner than the negation of v-coprimality. On the other hand in some integral domains, such as GCD domains the notions of coprime and v-coprime coincide. The v-coprimality (which can be traced back to Gilmer's work) is the ring theoretic equivalent of orthogonality in directed p.o. groups. [Given $G(D) = \{kD : k \in K^*\}$, for $rD, sD \in G(D)^+$ we have that $rD \vee sD \in G(D)$ translates to

$rD \cap sD$ being principal and if $rD \cap sD = rsD$ then in $G(D)$ we have $rD \vee sD = rDsD$ which forces rD and sD to be disjoint in $G(D)$.]

- Let $*$ be a general star operation and call $x,y \in D$ *$*-$coprime* if $(x,y)^* = D$. Since $A^* \subseteq A_v$ for every star operation $*$, we know that x,y being $*$-coprime implies x,y v-coprime but not conversely. (x,y d-coperime means $(x,y) = (x,y)_d = D$ means x,y are comaximal, but not all v-coprime pairs are comaximal.) Obviosly *no proper $*$-ideal contains a pair of $*$-coprime elements.*
- Proposition 1. For a general star operation $*$ on $F(D)$ $r, s \in D$ are $*$-coprime to $x \in D$ if and only if $(rs,x)^* = D$.

  Proof. Suppose r and s are $*$-coprime to x and consider $(x,rs)^* = (x,rx,rs)^*$
  $= (x,(rx,rs)^*)^* = (x,r(x,s)^*)^* = (x,r)^* = D.$
  Conversely suppose $(rs,x)^* = D$ and consider
  $(x,r)^* = (x,rs,r)^* = ((x,rs)^*,r)^* = (D,r)^* =$
- Corollary 2. For a general star operation $*$ on

$F(D)$
(i) $(r,x)^* = D \Leftrightarrow (r^n,x)^* = D \Leftrightarrow (r^n,x^m)$ for any natural m,n.
(ii) $(r,x)^* = D$ and $r|xy \Rightarrow r|y$
(iii) $(r,x)^* = D \Rightarrow D = D_r \cap D_x$
(iv) $(r,x)_v = D \Leftrightarrow D = D_r \cap D_x$
(v) Let $x = \frac{r}{s} \in K \backslash D$, if s has a nonunit factor that is $*$-coprime with r then x cannot be integral over D.
Proof. (i) is direct, for (ii) note:
$(r) = (r,xy)^* = (r,ry,xy)^*=$
$((r,y(r,x)^*)^* = (r,y)^*$. For (iii) let $(r,x)^* = D$ and consider $h \in D_r \cap D_x$. Then for some naturals m,n $hr^m, hx^n \in D$. So $D \supseteq (hr^m,hx^n)^* = h(r^m,x^n)^* = hD$ (by (i)).
For (iv) use (iii) to establish that $(r,x)_v = D \Rightarrow D = D_r \cap D_x$. For the converse assume $D = D_r \cap D_x$ and note that $(r,x)^* = (r,x)D_r \cap (r,x)D_x = D$. But $(r,x)^* = D \Rightarrow (r,x)_v = D$. For (v) use the fact that if $\frac{r}{s}$ is integral over D then $s|r^n$ for some n then use (i).
Using the similarity between *disjoint* and *v-coprime* we can associate with each

multiplicative set S the set $S^{\perp} = \{t \in D : (t,s)_v = D$ for all $s \in S\}$ and state:
Proposition 3. For each multiplicative set S of D we have (i) $S = (S^{\perp})^{\perp}$and (ii) $D = D_S \cap D_{S^{\perp}}$.
Proof. (i) direct (though lengthy) and for (ii) we can use the same argument as in the above Corollary.

- The notion of v-coprimality is useful when we need to sift through factors in the presence of properties weaker than the GCD property. Recall that D is an almost GCD(AGCD) domain (monoid) if for each pair of nonzero elements x,y there is a natural number n such that $(x^n,y^n)_v$ is principal. (If for each pair x,y, $n = 1$ we have the GCD domain.) I introduced this notion in [Z, Manmath,51(1985), 29-62], studied it further in [AZ, AB domains, J. Alg 142(1991), 285-309] and in [DLMZ, J. Alg 245(2001) no 1, 161-181]. Here is a brief demonstration. ((D,M) t-local≡ D is local with M a t-ideal.)
- Proposition 4. Let (D,M) be a t-local AGCD

domain. For each pair $x,y \in D\backslash\{0\}$ there is a natural number n such that $x^n|y^n$ or $y^n|x^n$.

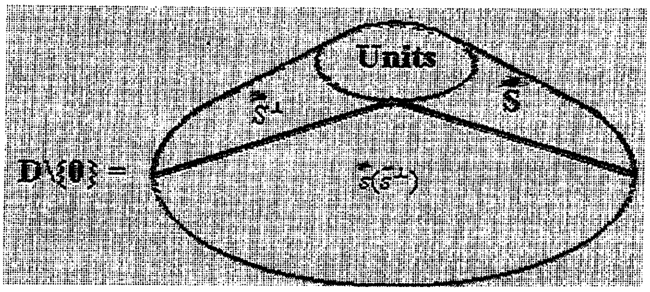
Proof. Let $x,y \in M\backslash\{0\}$. Since D is AGCD, $(x^n,y^n)_v = dD$ for some natural n and $d \in D$. Or $(\frac{x^n}{d},\frac{y^n}{d})_v = D$. Because M is a t-ideal, $\frac{x^n}{d},\frac{y^n}{d}$ cannot both be in M, forcing one of $\frac{x^n}{d},\frac{y^n}{d}$ to be a unit and making d an associate of x^n or of y^n.

- The notion of v-coprimality has recently played a decisive role in the study of almost GCD monoids and in the study of the, so called, inside factorial domains [AZ, AGCD monoids, Semigroup Forum], [CAZ, inside factorial weakly Krull domains], both can be downloaded from my web page: www. lohar.com. (The integral domain definitions can be directly translated to the monoid case.)

Splitting (Multiplicative) Sets (The main application)

- Definition. A saturated multiplicative set S of D is a *splitting multiplicative set* if each $x \in D\backslash\{0\}$ can be written as $x = ds$ where $s \in S$ and d is v-coprime to every member of

S. It follows that if S is splitting then $S^{\perp} = \{t : (t,s)_v = D, s \in S\}$ called the multiplicative complement of S is also a splitting set. The notion of a splitting set seems to make the following picture in my mind.



$\bar{S} = S \setminus$ Units
$\bar{S}^{\perp} = S^{\perp} \setminus$ Units

You can make your own, but it will help to make a picture.

- Here are a few characterizations of splitting sets whose proofs can be found in [AAZ, splitting the t-class group, JPAA, 74(1991)]

Proposition 5. TFAE for a sturated multiplicative set S.

1. S is a splitting set
2. $< SD >$ the p.o. subgroup of $G(D)$ generated by $\{sD : s \in S\}$ is a cardinal summand of $G(D)$ the group of divisibility of D, i.e., there is a p.o. subgroup H of $G(D)$ such that $< SD > \oplus_c H = G(D)$

3. If A is a principal integral ideal of D_S then $A \cap D$ is a principal ideal of D. (prin. integ. ideals of D_S contract to prin. ideals of D).
4. There is a multiplicative subset T of D such that (a) each element d of $D\backslash\{0\}$ can be written as $d = st$ where $s \in S$ and $t \in T$ and (b) any of the following equivalent conditions holds:
(i) If $s_1t_1 = s_2t_2$, where $s_i \in S$ and $t_i \in T$ then $s_2 = s_1u$ and $t_2 = t_1u^{-1}$, where $u, u^{-1} \in D$. ($d = st$ is unique upto assocs.)
(ii) If $d = st$ ($s \in S, t \in T$), then $dD_S \cap D = tD$
(iii) For each $s \in S$ and $t \in T$, $(s,t)_v = D$.
(iv) For each $t \in T$, $tD_S \cap D = tD$.

- Definition. A splitting set S is an lcm splitting set if in addition every element of S has an lcm with every element of D.

Proposition 6. TFAE for a saturated multiplicative set S :
i. S is lcm splitting
ii. $s_1D \cap s_2D$ is principal for $s_i \in S$
iii. $s_1D \cap s_2D = sD$ for $s,\ s_i \in S$

iv. $D_{S^\perp}$ is a GCD domain.

- Examples and applications of splitting sets

  1. In a Noetherian (Krull) domain D the saturation of every multiplicative set S generated by nonzro principal primes.

  2. In any domain the set $U(D)$ of units of D is a splitting set and so is $D\backslash\{0\}$.

  3. Divisibility/Nagata type Theorems: Let S be an lcm splitting set in D generated by prin. primes. Then D is a UFD, satisfies ACC on principal ideals, is atomic, or the elements of D satisfy a property satisfied by finite products of principal primes if and only if the same holds for D_S. :[AAZ,J.Alg.152(92)78-93]

  4.Structure: D is a weakly factorial domain (locally finite intersection of localizations at height one primes such that every t-invertible t-ideal of D is principal) if and only if every saturated multiplicative set in D is a splitting set. [AZ, PAMS,109(1990), 907-913]. (*It would be interesting to see a po group equivalent of this result.*)

  5. Structure: Let S be a splitting set of D

i. If P is a prime t-ideal, then P intersects S or P intersects $S^{\perp}$ but not both. (Any prime ideal that intersects both contains a v-coprime pair.)

ii. If A is a nonzero ideal of D then $A_t D_S = (AD_S)_t$. [AAZ, JPAA, 74(1991)]. So, P is a prime t-ideal of D if and only if PD_S or PD_T is a prime t-ideal of the respective quotient ring.

6. Nagata type theorem: If S is lcm splitting and D_S is a GCD domaim (PVMD) then so is D.

7. Recall that under t-multiplication the group of t-invertible t-ideals modulo the group of nonzero principal ideals is called the t-class group, $Cl_t(D)$. If S is a splitting then $Cl_t(D) \simeq Cl_t(D_S) \times Cl_t(D_{S^{\perp}})$. [AAZ, JPAA, 74(1991)]

8. If D is a GCD domain and S a multiplicative set of D then the $D + XD_S[X]$ construction is a GCD domain if and only if S is a splitting set of D [Z, JPAA, 50(1988) 93-107].

9. If S is a splitting set generated by primes in

a Noetherian domain D, and if the integral closure of D_S is a UFD then so is the integral closure of D [DZ, PAMS 130 (2002), no. 6, 1639–1644]

- Splitting sets originated in an effort to produce generalizations of the following theorem of Nagata: Let D be Noetherian and let S be a multiplicative set of D generated by prime elements of D, if D_S is a UFD then so is D. They first appeared in Gilmer and Parker[Michigan Math. J. 21 (1974), 65–86] as UF sets, which can now be described as splitting sets generated by height one primes and as lcm splitting sets. (Samuel in his Tata notes No 30, 1964 on unique factorization domains had restated Nagata's theorem for Krull domains.) Then Mott and Schexnayder[Krelle's Joutnal, 283/284(1976), 388-401] gave them the proper setting, which also means that a splitting set splits the group of divisibilty of the domain into a cardinal product of two subgroups.
- The success with the splitting sets led us to

the notion of t-splitting sets.

t-Splitting Sets. A multiplicative set S of D is a t-splitting set if for each nonzero nonunit $d \in D$ we have $(d) = (AB)_t$ where A and B are ideals with $A \cap S \neq \phi$ and B is such that $(B,s)_t = D$ for each $s \in S$. t-splitting sets S are characterized by: If A is a principal ideal of D_S then $AD_S \cap D$ is t-invertible. It was shown in [AAZ, Arab. J. Sci. Eng. Sect. C 26(2001), no. 1, 3–16] that for D a PVMD and S a multiplicative set of D the construction $D + XD_S[X]$ is a PVMD if and only if S is a t-splitting set of D. (GCD domains are a special case of PVMD's). In [CDZ, JPAA, (2003),71-86] the t-splitting sets are further explored and there we bring forth Nagata type Theorems that do not seem to have anything to do with the GCD property or the UFD property. Here is a quick example:

Proposition 7. [CDZ,JPAA, Cor. 3.8] Let X be an indeterminate over D, $G = \{f \in D[X] \, : \, (A_f)_v = D\}$ and let S be a nonempty subset of G. Then $D[X]$ is a Krull (resp. Mori, integrally closed, completely integrally closed, essential,UMT, Prufer v-multiplication) domain if $D[X]_S$ is.

- t-Splitting Sets of Ideals [CDZ (submitted)] Let D be an integral domain, $\boldsymbol{S}$ a multiplicative set of ideals of D and $D_{\boldsymbol{S}} = \{x \in K : xA \subseteq D$ for some $A \in \boldsymbol{S}\}$ the $\boldsymbol{S}$–transform of D as in Arnold and Brewer[J. Algebra, 18(1971), 254-263]. If I is an ideal of D, then $I_{\boldsymbol{S}} = \{x \in K : xA \subseteq I$ for some $A \in \boldsymbol{S}\}$ is an ideal of $D_{\boldsymbol{S}}$ containing I. Denote by $\boldsymbol{S}^{\perp}$ the set of ideals B of D with $(A+B)_t = D$ for all $A \in \boldsymbol{S}$. Call $\boldsymbol{S}^{\perp}$ the t-complement of $\boldsymbol{S}$. Denote by $sp(\boldsymbol{S})$ the "saturation" of $\boldsymbol{S}$ (set of all ideals C of D such that $C_t \supseteq A$ for some $A \in \boldsymbol{S}$). Call $\boldsymbol{S}$ a t-splitting set of ideals if every nonzero principal ideal dD can be written as $dD = (AB)_t$ where $A \in sp(\boldsymbol{S})$ and $B \in \boldsymbol{S}^{\perp}$.
- It turns out that $\boldsymbol{S}$ being t-splitting is equivalent to $sp(\boldsymbol{S})$ being t-splitting and that if $\boldsymbol{S}$ is generated by principal ideals and t-splitting then it is the usual t-splitting set defined above. Moreover if $\boldsymbol{S}$ is t-splitting

then (i) so is $\boldsymbol{S}^{\perp}$ (ii) for each $C \in \boldsymbol{S}$, C_t contains a t-invertible t-ideal of $sp(\boldsymbol{S})$. (So, a splitting set of ideals $\boldsymbol{S}$ is v-finite in Gabelli's terminology [Dekker LNPure APPL.M, 206, 117-142.]) In fact if $\boldsymbol{S}_i$ is the set of all t-invertible t-ideals in $sp(\boldsymbol{S})$ then $\boldsymbol{S}_i$ is a t-splitting set with t-complement $\boldsymbol{S}^{\perp}$. It turns out that a lot of results proved for (t-)splitting sets carry through to this more general setting albeit with some new interpretations. Here's a sampling of some of the results proved in [CDZ (submitted)]

Proposition 8. Let $\boldsymbol{S}$ be a t-splitting set of ideals of D. Then for every nonzero ideal I of D we have $I_t = (AB)_t$ with $A \in sp(\boldsymbol{S})$ and $B \in \boldsymbol{S}^{\perp}$ and this "splitting" of I is unique up to t-closures.

Proposition 9. Let $\boldsymbol{S}$ be a multiplicative set of ideals of D. Then $\boldsymbol{S}$ is t-splitting iff S is v-finite and every nonzero principal ideal of $D_{\boldsymbol{S}}$ contracts to a t-ivertible t-ideal.

Proposition 10. A splitting set of ideals induces a natural cardinal product decomposition of the ordered monoid of fractional t-ideals of D under the t-product and ordered by the usual reverse inclusion. Finally here's something to remind you of the earlier "Nagata type Theorems".

Proposition 11. Let ***F*** be a family of height one t-invertible prime t-ideals of D such that every nonzero nonunit of D belongs to at most a finite number of members of ***F***. Let ***S*** be a multiplicative set generated by members of ***F***. The the following hold:

i. D is a PVMD iff so is D_S ,

ii. D is a Krull domain iff so is D_S.

iii. D is of finite t-character iff so is D_S.

- Finally, a word about a gap that needs to be filled. In jumping from splitting sets to t-splitting sets we overlooked the possibility of studying say d-∗-splitting sets, d for divisibility. It appears to me that there is a whole world of results parallel to those we know about splitting sets. Let me give you a couple of examples:
- Call a saturated multiplicative set S a d-d-splitting set if every element $d \in D\backslash\{0\}$ can be written as $d = st$ where $s \in S$ and t is

d-coprime to every member of S. Recall that d-coprime $\equiv comaximal$. Example: A saturated multiplicative set S generated by height one principal maximal ideals such that no nonzero member of D is divisible by an infinite set of nonassociated primes from S.

- Proposition. If S is a d-d-splitting set generated by principal maximal ideals, Then D is a PID (Noetherian, Prufer) iff D_S is.