Factoriality in Riesz groups

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Abstract

Throughout let $G = (G, +, \leq, 0)$ denote a Riesz group, where $+$ is not necessarily a commutative operation. Call $x \in G$ homogeneous if $x > 0$ and for all $h, k \in (0, x]$ there is $t \in (0, x]$ such that $t \leq h, k$. In this paper we develop a theory of factoriality in Riesz groups based on the fact that if $x \in G$ and x is a finite sum of homogeneous elements then x is expressible, uniquely, as a sum of finitely many mutually disjoint homogeneous elements. We then compare our work with existing results in lattice ordered groups and in (commutative) integral domains.

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Let $G = (G, +, \leq, 0)$ be a partially ordered group (po-group) with $+$ not necessarily commutative. A po-group G is said to be a *group with interpolation property* if it satisfies the *Riesz interpolation property*: for all $a_1, a_2, b_1, b_2 \in G$, with $a_i \leq b_j$ $(i, j = 1, 2)$ there exists $c \in G$ such that $a_i \leq c \leq b_j$. A directed p.o. group with interpolation property is called a *Riesz group*. It is easy to see that a lattice ordered group is a Riesz group. There is of course an abundant supply of "non lattice ordered" Riesz groups. (See [10], [8] and [17].) Two elements $x, y \in G$ are *disjoint* if $x \wedge y = 0$, where $a \wedge b$ denotes $\inf_G(x, y)$.Call an element $h \in G$ *homogeneous* if h is strictly positive, i.e., $h > 0$, and for all $u, v \in (0, h]$, u, v are non-disjoint. Two homogeneous elements h, k are said to be related if $h \wedge k \neq 0$. We show that every finite sum of homogeneous elements in a Riesz group is uniquely expressible as a finite sum of mutually disjoint homogeneous elements. This leads us to the notion of a *factorial core* $F(G)$ of every Riesz group. Here $F(G)$ is such that $(F(G))^+$ consists of finite sums of homogeneous elements of G, taking 0 as an empty sum. We show that for a Riesz group G, $F(G)$ is an *o-ideal*, i.e., a convex normal directed subgroup of G. We call a Riesz group G *factorial* if $G = F(G)$ and indicate the connections of our results with the existing literature. A Riesz group G may be said to satisfy *Conrad's F-condition* if *every strictly positive element of G exceeds at most a finite number of mutually disjoint elements.* We show that in a Riesz group that satisfies Conrad's F-condition every strictly positive element exceeds at most a finite number of disjoint homogeneous elements. Our work improves upon the work by Rachunek [13] who studied Riesz groups that contain at most a finite number of disjoint strictly positive elements. Apparently the introduction of factoriality has helped us treat the general case successfully. This notion of factoriality comes from generalizations of the notion of unique factorization in integral domains, an interested reader may consult Anderson's recent survey article [1] to see a description of various efforts at generalizing the notion of factoriality.

In what follows we shall use $a \wedge b \neq 0$, for $a, b > 0$, to indicate that a and b are not disjoint, i.e., there exists $d \leq a, b$ such that $d \not\leq 0$. This usage is notational and does not signify that $a \wedge b$ exists. Following [8] we call a Riesz group G an antilattice if for $a, b \in G$, $a \wedge b \in G$ implies that a and b are comparable, i.e., $a \leq b$ or $b \leq a$. A cardinal sum of a family $\{G_i\}_{i \in I}$ of partially ordered groups is their direct sum $G = \sum_{i \in I} G_i$ where an element $g = \sum_{i \in I} g_i$ is positive if each g_i is positive. For $g \in G^+$, $[0, g] = \{h : 0 \leq h \leq g\}$ and if $g > 0$, $(0, g]$ represents the set $\{h : 0 < h \leq g\}$. We denote by G^* the set of strictly positive elements of G. We note following Rachunek [13] that a Riesz group is regular, i.e. if $\inf_{G^+}(a, b)$ exists then $\inf_G(a, b)$ exists and both are equal.

In section 1 we study the basic properties of homogeneous elements, such as the relatedness of homogeneous elements is an equivalence relation and that the sum of two related homogeneous elements is a homogeneous element related to each. In short we set up the scene for the notion of factoriality. In section 2 we define the notion of factoriality and show among various structural results that if a strictly positive element of a Riesz group is a sum of homogeneous elements then it is uniquely expressible as a sum of mutually disjoint homogeneous elements. In section 3 we establish that there do exist homogeneous elements and there do exist subgroups $F(G)$ of some Riesz groups G such that every strictly positive element of $F(G)$ is expressible as a sum of finitely many homogeneous elements. Finally in section 4 we give examples of factorial Riesz groups and explore links between the notions of factoriality in po-groups and in the rings.

1 Basic properties of homogeneous elements

The purpose of this section is to provide the basic tools that we shall need throughout the paper. This includes a study of homogeneous elements. We show that the sum of two related homogeneous elements is again homogeneous related to them. Thus the set $A^{(b)}$ of all homogeneous elements, related to b, of a Riesz group G is a subsemigroup of G. We also show that $AL^{(b)} = \langle A^{(b)} \rangle$ is an antilattice and that if b and c are two disjoint homogeneous elements of G then $AL^{(b)} \cap AL^{(c)} = \{0\}$.

For a start we note here that by the definition if h is homogeneous in a Riesz group then each k with $0 < k \leq h$, is homogeneous.

Proposition 1.1 *Let G be a Riesz group and let $x, y \in G^*$. Then the following hold.*

(1) $x \wedge y \neq 0$ if and only if there exists a $t \in G$ with $0 < t \leq x, y$.

(2) x is a homogeneous element if and only if $(0, x]$ is lower directed, i.e., for all $a, b \in (0, x]$ there is $t \in (0, x]$ such that $t \leq a, b$. So if x is homogeneous and there is h with $0 < h \leq x$ then h is homogeneous.

(3) If $x \wedge y = 0$ and there is a $u \in G$ with $0 < u \leq y$ then $x \wedge u = 0$. Consequently if $x, y > 0$ and $x \wedge y = 0$ then for each pair $(i, j) \in (0, x] \times (0, y]$ we have $i \wedge j = 0$.

(4) Suppose that h and k are two homogeneous elements of G. Then the following are equivalent:

(a) $h \wedge k = 0$,

(b) for each pair $(a, b) \in (0, h] \times (0, k]$, $a \wedge b = 0$,

(c) there is at least one pair $(a, b) \in (0, h] \times (0, k]$ for which $a \wedge b = 0$.

(5) Suppose that h and k are two homogeneous elements of G. Then the following are equivalent:

(r) $h \wedge k \neq 0$, i.e., h and k are related

(s) for each pair $(a, b) \in (0, h] \times (0, k]$ we have $a \wedge b \neq 0$,

(t) there is at least one pair $(a, b) \in (0, h] \times (0, k]$ such that $a \wedge b \neq 0$.

(6) Relatedness is an equivalence relation on the set of all homogeneous elements

(7) If $x, y \in G^*$ and $x \wedge y = 0$ and if h is a homogeneous element then h must be disjoint with at least one of x, y. More generally if there are mutually disjoint positive elements $b_1, b_2, ..., b_n$ then h must be disjoint with at least $n - 1$ of the b_i.

(8) If $x \in G^+$ and if there are mutually disjoint positive elements $b_1, b_2, ..., b_n$ with $x \leq b_1 + b_2 + ... + b_n$ then $x = a_1 + a_2 + ... + a_n$ where $0 \leq a_i \leq b_i$ and a_i are mutually disjoint. Consequently if x is homogeneous then $x \leq b_i$ for some i.

Proof. (1) Recall that for $x, y \in G$, $(x \wedge y = 0) \Leftrightarrow \forall_{u \leq x,y}(u \leq 0)$. Equivalently, we get $(x \wedge y \neq 0) \Leftrightarrow \exists_{u \leq x,y}(u \nleq 0)$. So, if $x \wedge y \neq 0$ then we have a u with $u \nleq 0$ such that $u, 0 \leq x, y$. By the Riesz interpolation property there must be a t with $u, 0 \leq t \leq x, y$. Now $t = 0$ will contradict the fact that $u \nleq 0$, so $t > 0$. The converse is obvious. Clearly (2) follows directly from (1). For (3) note that if for some $(i, j) \in (0, x] \times (0, y]$ we have $i \wedge j \neq 0$ then by (1) there is a t such that $0 < t \leq i, j$, but then $0 < t \leq x, y$. To establish (4) note that (a) $\Rightarrow$ (b) follows from (3) and (b) $\Rightarrow$ (c) is logically natural. For (c) $\Rightarrow$ (a) suppose that there is $(a, b) \in (0, h] \times (0, k]$ for which $a \wedge b = 0$, yet $h \wedge k \neq 0$. Then by (1) there is a pair $(t, t) \in (0, h] \times (0, k]$ and by (2) t is homogeneous. Now as $t, a \in (0, h]$ we have a $t_1 \in (0, h]$ with $t_1 \leq t, a$. Similarly we can find a $t_2 \in (0, k]$ where $t_2 \leq t, b$. But then $t_1, t_2 \in (0, t]$ and so there must be a $z \in (0, t]$ such that $z \leq t_1, t_2$ and by transitivity of $\leq$ we have $0 < z \leq a, b$, contradicting the assumption that $a \wedge b = 0$. We get (5) directly from (4). For (6) only transitivity needs explaining and for that let r, s, t be three homogeneous elements with $r \wedge s \neq 0 \neq s \wedge t$. By (1) there exist u, v with $0 < u \leq r, s$ and $0 < v \leq s, t$. Now as $u, v \in (0, s]$ and as s is homogeneous there is a $z \in (0, s]$ such that $z \leq u, v$ and this leads to the conclusion that r and t are nondisjoint. For (7), note that if h were nondisjoint with both x, y then by (1) there would be $h_1, h_2 \in (0, h]$ such that $h_1 \leq x$ and $h_2 \leq y$. Now since h is homogeneous we have a $z \in (0, h]$ such that $z \leq h_1, h_2$ and hence $0 < z \leq x, y$ contradicting the disjointness of x, y. This observation can be easily extended to establish the more general statement. Finally for (8) recall from [8, Theorem 2.2(5)] that G is a Riesz group if and only if for $a, b_1, b_2, ..., b_n \in G^+$ with $a \leq b_1 + b_2 + ... + b_n$ there exist $a_1, a_2, ..., a_n \in G^+$

with $a_i \leq b_i$ and $a = a_1 + a_2 + ... + a_n$. So $x \leq b_1 + b_2 + ... + b_n$ implies that $x = x_1 + x_2 + ... + x_n$ where $0 \leq x_i \leq b_i$ and using (3) we can conclude that x_i are mutually disjoint. Next if x is a homogeneous element x cannot, by definition, have disjoint nonzero summands, and this forces all but one of x_i to be zero. □

Proposition 1.2 *Let h and k be two related homogeneous elements in a Riesz group G. Then $h + k$ is a homogeneous element related to both of h and k.*

Proof. Let $u, v \in (0, h+k]$. Since G is Riesz $u = a+b$, $v = c+d$ where $a, c \in (0, h]$ and $b, d \in (0, k]$. Now suppose by way of contradiction that $u \wedge v = 0$. Then by (3) $a \wedge (c + d) = 0$ and again by (3) $a \wedge c = 0$ but this contradicts the fact that h is homogeneous. □

Corollary 1.3 *Let b be a homogeneous element in a Riesz group G. Then the set $A^{(b)} = \{x \in G : x$ is a homogeneous element related to $b\}$ is a convex directed subsemigroup of G.*

Proof. That $A^{(b)}$ is a subsemigroup follows from Proposition 1.2 and the fact that $(A^{(b)}, +) \subseteq (G, +)$. For convexity let $x, y \in A^{(b)}$ with $x \leq y$ and let z be such that $x \leq z \leq y$. Then $z > 0$, off the bat. As y is homogeneous and as $0 < z \leq y$ we conclude that z is homogeneous and related to y. That z is related to b now follows from (6) of Proposition 1.1. That $A^{(b)}$ is directed follows once we note that for $x, y \in A^{(b)}$ $x \wedge y \neq 0$ so, by (1) of Proposition 1.1, there is a t such that $0 < t \leq x, y$ and of course $x, y \leq x + y$. □

We shall call the set $A^{(b)}$, defined in Corollary 1.3, the semigroup associated with the homogeneous element b. Our next step is to study the subgroup generated by $A^{(b)}$. To facilitate our work in that direction we prove the following Lemma. Yet, before we do that, it seems fair to recall the easily verified fact that if in a p.o. group G, $x \wedge y$ exists for some x, y in G then for any $p \in G$ we have $p + (x \wedge y) = (p + x) \wedge (p + y)$.

Lemma 1.4 *Let y be a homogeneous element of a Riesz group G and let $x \in G$. Then $y^x = -x+y+x$ is homogeneous. If, moreover, x is a homogeneous element related to y then y^x is related to y.*

Proof. Because $y > 0$ we have $-x + y + x > 0$. Now let $h, k \in (0, y^x]$. Then $h^{-x}, k^{-x} \in (0, y]$ and as y is homogeneous we have a t such that $0 < t \leq h^{-x}, k^{-x}$ or $0 < t^x \leq h, k$. Next, suppose that x is homogeneous related to y. Then by Proposition 1.1, if y^x is not related to x then we must have $x \wedge y^x = 0$. But then $0 = x + (x \wedge y^x) - x = x \wedge y$ which contradicts the fact that x and y are related. □

Proposition 1.5 *Let b be a homogeneous element of a Riesz group G. Then the set $AL^{(b)} = \{x - y : x, y \in A^{(b)} \cup \{0\}\}$ is a convex directed subgroup of G. Moreover $(AL^{(b)})^* = A^{(b)}$ and $(AL^{(b)})^+ = A^{(b)} \cup \{0\}$.*

Proof. Our proof consists in showing that the group $< A^{(b)} >$ generated by $A^{(b)}$ can be put into the form of $AL^{(b)}$. For this we note that every nonzero element of $< A^{(b)} >$ is a finite sum of elements of $A^{(b)}$ and/or of inverses of elements of $A^{(b)}$. Using Lemma 1.4 we can push the negatives to the right as in the following example: Consider $z = x_1 - x_2 + x_3$, where $x_i \in A^{(b)}$. Then $z = x_1 - x_2 + x_3 + x_2 - x_2$ which gives $z = x_1 + (x_3)^{x_2} - x_2$ where $x_1 + (x_3)^{x_2} \in A^{(b)}$, by Lemma 1.4. Next as $A^{(b)}$ is convex it is easy to see that $AL^{(b)}$ is convex. Finally let $0 < x \in AL^{(b)}$. Then since $x = u - v$ for $u, v \in A^{(b)}$, we have $0 < u - v \leq u$ and so $u - v$ is a homogeneous element and this establishes the claim that $(AL^{(b)})^+ = A^{(b)} \cup \{0\}$. □

Proposition 1.6 *If b is a homogeneous element of a Riesz group G, then $AL^{(b)}$ is an antilattice.*

Proof. That $AL^{(b)}$ is a Riesz group follows from the fact that $AL^{(b)}$ is a convex directed subgroup of G. For the antilattice part let $x, y \in AL^{(b)}$ and suppose that $x \wedge y \in AL^{(b)}$. Then $x \wedge y = u - v$ where $u, v \in A^{(b)}$. Adding $v - u$ to the equation $x \wedge y = u - v$ we get $0 = (x \wedge y) + (v - u)$ or $(x + v - u) \wedge (y + v - u) = 0$, i.e., $(x + v - u)$, $(y + v - u)$ are disjoint. We claim that at least one of $(x + v - u), (y + v - u)$ is zero. For if not then $(x + v - u), (y + v - u) > 0$ and so $(x + v - u), (y + v - u) \in A^{(b)}$ and thus related to b. But according to (7) of Proposition 1.1, b cannot be nondisjoint with two disjoint elements, a contradiction. Now if $(y + v - u) = 0$ then $y = u - v = x \wedge y$ which gives $y \leq x$ and if $(x + v - u) = 0$ we get $x \leq y$. □

For any two related homogeneous elements $b, c \in G$ we have $A^{(b)} = A^{(c)}$ and hence $AL^{(b)} = AL^{(c)}$ ((6) of Proposition 1.1)). This leads to the question of relationship between $AL^{(b)}$ and $AL^{(c)}$ when b and c are disjoint.

Proposition 1.7 *Let b and c be two disjoint homogeneous elements of a Riesz group G. Then $AL^{(b)} \cap AL^{(c)} = \{0\}$.*

Proof. By (4) of Proposition 1.1, $A^{(b)} \cap A^{(c)} = \phi$. Now suppose that $0 \neq x \in AL^{(b)} \cap AL^{(c)}$. Since $x \in AL^{(b)}$ we have, by Proposition 1.5, $x = u - v$ where $u, v \in A^{(b)} \cup \{0\}$. Similarly since $x \in AL^{(c)}$ we have $x = p - q$ where $p, q \in A^{(c)} \cup \{0\}$. This gives $u - v = p - q$ or $u = p - q + v$. Now, being disjoint v and q commute. So, $u = p + v - q$ or $u + q = p + v$ where $u, q, p, v \in G^+$ and we have $u \leq p + v$ and $v \leq u + q$. If $u = 0$ then $v \leq q$ which is possible only if $v = 0$, because $v > 0$ would contradict the disjointness of b and c. Thus u, q, p, v are strictly positive and hence homogeneous. But then by (8) of Proposition 1.1 $u \leq p + v$ implies that $u \leq v$ and $v \leq u + q$ implies $v \leq u$. This gives $0 = u - v = x \neq 0$ a contradiction. □

2 Unique factorization and finite sums of homogeneous elements

In this section we use the properties of homogeneous elements in Riesz groups, established in section 1, to introduce the notion of factoriality. We show that if $b_1, b_2, ..., b_n$ are mutually disjoint homogeneous elements then $\sum AL^{(b_i)}$ is a cardinal sum. Then we extend this notion to: If $h(G)$ is a set of homogeneous elements such that for $u, v \in h(G)$, $u = v$ or $u \wedge v = 0$. We show that if, in a Riesz group G, an element x is a sum of homogeneous elements then x is uniquely expressible as a sum of mutually disjoint homogeneous elements.

In any partially ordered group $x \leq g$ if and only if there is $h \geq 0$ such that $x + h = g$. So if $0 \leq x \leq g$ we can call x a positive summand or a factor of g. Using ideas established in Proposition 1.1(8) we prove the following result.

Proposition 2.1 *In a Riesz group G the following hold.*

(1) If $x \in G$ is expressible as a finite sum of finitely many mutually disjoint homogeneous elements then this expression is unique up to order of the mutually disjoint summands. That is if $x \in G$ is expressible as $x = b_1 + b_2 + ... + b_n = c_1 + c_2 + ... + c_m$ where b_i are mutually disjoint homogeneous elements and c_j are mutually disjoint homogeneous elements then $m = n$ and c_j are just a permutation of b_i.

(2) If $x \in G$ is expressible as a finite sum of homogeneous elements then x is expressible uniquely as a single homogeneous element or as a finite sum of mutually disjoint homogeneous elements.

Proof. (1) Let $x = b_1 + b_2 + ... + b_n = c_1 + c_2 + ... + c_m$ as described above. By Proposition 1.1 (8), for each i, $b_i \leq c_j$ for exactly one j and (for this j) $c_j \leq b_k$. Now if $k \neq i$ then c_j is a homogeneous element related to two disjoint homogeneous elements contradicting (7) of Proposition 1.1. So, $k = i$ and consequently each b_i is equal to some c_j. This can be used to complete the proof. For (2) let $x = b_1 + b_2 + ... + b_n$ where b_i are homogeneous elements. If all b_i are related to a single homogeneous element then $b_1 + b_2 + ... + b_n$ is a homogeneous element, by Proposition 1.2. If there are some summands that are disjoint then, using the fact that in a p.o. group disjoint elements commute, we can regroup $b_1 + b_2 + ... + b_n$ into blocks of mutually related homogeneous elements. This will give us $x = B_1 + B_2 + ... + B_r$ where $r \leq n$ and B_i are mutually disjoint homogeneous elements. Now we can apply (1) above. □

We can now say that a Riesz group G is a factorial group if every element of $G^+ \backslash \{0\}$ is expressible as a finite sum of homogeneous elements of G. Each antilattice is a good, but a very specialized, example of a factorial Riesz group. In this section we show that cardinal sums of antilattices are indeed factorial Riesz groups. We also show that if a Riesz group G has at least one homogeneous element then G contains a convex normal subgroup $F(G)$ that is a factorial Riesz group. We also show that $F(G) = \sum AL^{(b)}$ where b ranges over the maximal set of mutually disjoint homogeneous elements of G and $\sum$ represents the cardinal sum.

Lemma 2.2 *Let $b_1, b_2, ..., b_n$ be mutually disjoint homogeneous elements of a Riesz group G and let $x_i \in AL^{(b_i)}$ for $i = 1, ..., n$. Then $x_1 + x_2 + ... + x_n \geq 0$ if and only if, for each i, $x_i \geq 0$. Moreover $x_1 + x_2 + ... + x_n = 0$ if and only if, for each i, $x_i = 0$.*

Proof. Clearly if each $x_i \geq 0$ then $x_1 + x_2 + ... + x_n \geq 0$. Conversely let $x_i \in AL^{(b_i)}$. Then $x_i = u_i - v_i$ where $u_i, v_i \in A^{(b_i)} \cup \{0\}$. Then $(u_1 - v_1) + (u_2 - v_2) + ... + (u_n - v_n) \geq 0$. Now, in a p.o. group, $-x$ and $-y$ commute if and only if x and y commute. Moreover disjoint elements commute. Using these facts we can push negative terms to the right of the inequality to get $u_1 + u_2 + ... + u_n \geq v_1 + v_2 + ... + v_n \geq 0$. This gives $0 \leq v_i \leq u_1 + u_2 + ... + u_n$ for $1 \leq i \leq n$. Using (8) of Proposition 1.1 we have $v_i = a_1 + a_2 + ... + a_n$ where $0 \leq a_i \leq u_i$. Since $v_i \wedge u_j = 0$ for $j \neq i$ we have $a_j = 0$ for $j \neq i$ and this forces $v_i \leq u_i$, which in turn gives $x_i = u_i - v_i \geq 0$. For the moreover part note that $x_i \geq 0$ puts x_i in G^+ and in G^+ $x_1 + x_2 + ... + x_n = 0$ if and only if, for each i, $x_i = 0$. □

As mentioned above, our aim is to show that $F(G)$ is a factorial Riesz group whenever $H(G)$ the set of all homogeneous elements of G is nonempty. To do this we shall show that $F(G)$ is a cardinal sum of antilattices associated with mutually disjoint homogeneous elements. To this end the following lemma will be useful.

Lemma 2.3 *Let b and c be two disjoint homogeneous elements in a Riesz group G and let $\underline{A}^{(x)} = A^{(x)} \cup \{0\}$ where x is a homogeneous element. Then the following hold.*

(1) $\underline{A}^{(b)} + \underline{A}^{(c)} = \underline{A}^{(c)} + \underline{A}^{(b)}$
(2) $AL^{(b)} + AL^{(c)} = AL^{(c)} + AL^{(b)}$
(3) $AL^{(b)} + AL^{(c)} = < \underline{A}^{(b)} \cup \underline{A}^{(c)} > = < A^{(b)} \cup A^{(c)} >$
(4) $(AL^{(b)} + AL^{(c)})^+ = \underline{A}^{(b)} + \underline{A}^{(c)}$
(5) $AL^{(b)} + AL^{(c)}$ is convex and directed.

Proof. (1) and (2) can be easily proved using the fact that disjoint elements (and their negatives) commute. For (3) note that $AL^{(b)} + AL^{(c)} \subseteq < A^{(b)} \cup A^{(c)} >$ as $0 = b - b = c - c$. Now let $0 \neq x \in < A^{(b)} \cup A^{(c)} >$. Then $x = \sum \epsilon_i u_i$ where $\epsilon_i = \pm 1$ and u_i are homogeneous elements related to b or c. Using the fact that disjoint elements commute we can write $x = \sum \zeta_i y_i + \sum \eta_j z_j$ where $\eta_j, \zeta_i = \pm 1$, $y_i \in A^{(b)}$ and $z_j \in A^{(c)}$ but then $\sum \zeta_i y_i \in < A^{(b)} > = AL^{(b)}$ and $\sum \eta_j z_j \in AL^{(c)}$. Thus $AL^{(b)} + AL^{(c)} \subseteq < A^{(b)} \cup A^{(c)} > \subseteq AL^{(b)} + AL^{(c)}$. For (4) note that $x \in (AL^{(b)} + AL^{(c)})^+$ implies, by (3), that $x = a_1 + a_2 \geq 0$ where $a_1 \in AL^{(b)}$ and $a_2 \in AL^{(c)}$ and by Lemma 2.2 this means that $a_i \geq 0$ forcing $a_1 \in \underline{A}^{(b)}$ and $a_2 \in \underline{A}^{(c)}$. In (5), for the convexity we need to show that for each $c \in (AL^{(b)} + AL^{(c)})^+$ and for each $x \in G^+$ with $x \leq c$ we have $x \in (AL^{(b)} + AL^{(c)})^+$. Now, by (4) $0 \leq x \leq c = u_1 + u_2$ where $u_1 \in \underline{A}^{(b)}, u_2 \in \underline{A}^{(c)}$, but then by (8) of Proposition 1.1 we have $x = a_1 + a_2$ where $0 \leq a_i \leq u_i$ $(i = 1, 2)$. This forces $a_1 \in \underline{A}^{(b)} = (AL^{(b)})^+$ because $AL^{(b)}$ is convex by Proposition 1.5. The same conclusion for a_2 relative to $\underline{A}^{(c)}$. For directedness note that every element of $AL^{(b)} + AL^{(c)}$ can be expressed as a difference of elements from $(AL^{(b)} + AL^{(c)})^+ = \underline{A}^{(b)} + \underline{A}^{(c)}$. □

Now if $H(G) \neq \phi$ then $H(G)$ can be partitioned into equivalence classes, each consisting of related homogeneous elements. By Proposition 1.6, each equivalence class can then be turned into a single antilattice. Proposition 1.7, implies that antilattices representing distinct classes intersect trivially and Lemma 2.3 implies that the elements of two such classes commute. So we can talk about the direct sum of these antilattices and this direct sum is a cardinal sum by Lemma 2.2. This leads us to the following theorem.

Theorem 2.4 *Let G be a Riesz group such that $H(G) \neq \phi$. Then there exists a maximal set S of mutually disjoint elements of $H(G)$ and the sum $F(G) = \sum_{s\in S} AL^{(s)}$ is the cardinal sum of all the (distinct) antilattices associated with homogeneous elements of S and hence is a Riesz subgroup of G. Moreover,*

(1) $F(G) =< \cup A^{(s)} >=< H(G) >$
(2) $(F(G))^{+}\backslash\{0\} = \sum_{s\in S} A^{(s)}$ and so $(F(G)) = \sum_{s\in S} \underline{A}^{(s)}$
(3) $F(G)$ is independent of the choice of S,
(4) Every homogeneous element of G belongs to $F(G)$.
(5). $F(G)$ is a factorial Riesz group.

Proof. We form S by choosing a homogeneous element from each equivalence class. This ensures that S is maximal. Lemma 2.3 in conjunction with Proposition 1.7 implies that $F(G)$ is a direct sum. Lemma 2.2 implies that $\sum_{s\in S} AL^{(s)}$ is a cardinal sum. That $F(G)$ is a Riesz subgroup follows from the fact that each of $AL^{(s)}$ is a Riesz subgroup. In case of (1) obviously $F(G) \subseteq < H(G) >$. Now let $x \in < H(G) >$. Then $x = \sum_1^n \epsilon_i x_i$ where, for each i, $\epsilon_i = \pm 1$ and x_i is a homogeneous element. Now each of the x_i is related to one of $s_1, s_2, ..., s_r \in S$, where $r \leq n$. Using the fact that disjoint elements commute and we can regroup the x_i so that $x = U_1 + U_2 + ... + U_r$ where each $U_i = \sum_{j=1}^k \gamma_{ij} x_{ij}$ where $\gamma_{ij} = \pm 1$ and each x_{ij} is related to s_i. Now using Lemma 1.4, as in Proposition 1.5 we can write $U_i = u_i - v_i$ where $u_i, v_i \in A^{(s_i)} \cup \{0\}$. Thus $x \in \sum_1^r AL^{(s_i)} \subseteq F(G)$. For (2) let $x \in (F(G))^{+}\backslash\{0\}$. Then $x \in \sum_{s\in S} AL^{(s)}$ such that $x \geq 0$. So, $x = u_1 + u_2 + ... + u_r$ where $u_i \in AL^{(s_i)}$ and $x = u_1 + u_2 + ... + u_r \geq 0$. This forces $u_i \geq 0$, by Lemma 2.2 Now as $x \neq 0$ some of the u_i are strictly positive. So, $x = u_{11} + u_{12} + ... + u_{1k}$ where u_{1i} ranges over the strictly positive members of $\{u_1, u_2, ..., u_r\}$. But then each of u_{1i} is a homogeneous element related to s_i for some $s_i \in S$. So $x = u_{11} + u_{12} + ... + u_{1k} \in \sum_{i=1}^k A^{(s_i)} \subseteq \sum_{s\in S} A^{(s)}$. So $(F(G))^{+}\backslash\{0\} \subseteq \sum_{s\in S} A^{(s)}$. The reverse containment follows from the fact that a finite sum of homogeneous elements is strictly positive. The rest is immediate. In case of (3) the independence of $F(G)$ of the choice of S follows from the fact that each equivalence class is represented by a unique antilattice as remarked prior to Proposition 1.7. (4) is obvious, in view of (2). For (5) note that by (2) every strictly positive element of $F(G)$ is a sum of mutually disjoint homogeneous elements. So, by definition, $F(G)$ is a factorial Riesz group. □

Let G be a directed p.o. group, a convex normal directed subgroup H of G is called an o-ideal.

Theorem 2.5 *Let G be a Riesz group with $H(G) \neq \phi$. Then $F(G)$ is an o-ideal of G.*

Proof. We first note that if b and c are two disjoint homogeneous elements of G, then for any $x \in G$, b^x and c^x are disjoint. Of course b^x and c^x are homogeneous by Lemma 1.4. Suppose that b^x and c^x are nondisjoint then there exists a t such that $0 < t \leq b^x, c^x$ by Proposition 1.1(1). But then $0 < t^{-x} \leq b, c$ contradicting the disjointness of b and c. But this means that $H(G)$ and hence S is invariant under inner automorphisms of G. This establishes the normality, in G, of $F(G)$. Convexity and directedness are proved on the same lines as in (5) of Lemma 2.3. □

Now comes the question: Given a Riesz group G, under what conditions is G a cardinal sum of its antilattice subgroups associated with mutually disjoint homogeneous elements? One answer is when $G = F(G)$, but a more intriguing answer can be supplied. For this let us call an antilattice subgroup A of G a maximal antilattice subgroup if for any antilattice subgroup H of G with $A \subseteq H$ we have $A = H$. Let us also recall that $F(G) =< H(G) >$ and make the following elementary observation.

If B is any group under an operation $*$, and A is a subgroup of B then for all $a * b \in B \backslash A$ either $a \in B \backslash A$ or $b \in B \backslash A$. This, indirectly, leads to the conclusion that a set S in a group B generates B if and only if for each proper subgroup A of B we have $(B \backslash A) \cap S \neq \phi$.

Theorem 2.6 *A Riesz group G is a cardinal sum of its maximal antilattice subgroups if and only if for every proper subgroup H of G, $G \backslash H$ contains a homogeneous element of G.*

Proof. If G is an antilattice then every strictly positive element of G is homogeneous. Moreover if a proper subgroup H of a directed p.o. group G contains all positive elements of G then $H = G$. So, for every proper subgroup H of an antilattice G, $G \backslash H$ contains a homogeneous element. Now let $G = \sum_{i \in I} L_i$ be a cardinal sum of its subgroups L_i where L_i are maximal convex antilattice subgroups. Then as $L_i \cap L_j = \{0\}$ for $i \neq j$ the strictly positive elements of L_i for each i are unique to L_i and for each $b \in L_i^*$, $L_i = AL^{(b)}$. Also for $j \neq i$, if $L_j = AL^{(c)}$ then b and c are disjoint. (For if $b \wedge c \neq 0$ then there is t such that $0 < t \leq b, c$ which would contradict $L_i \cap L_j = \{0\}$.) So every element x of G can be written as $x = u_1 + u_2 + ... + u_r$ where $u_i = a_i - b_i$ where $a_i, b_i \in (L_i)^+$, $(i = 1, 2, ..., r)$. Now if H is a proper subgroup of G and $x \in G \backslash H$ then $x = \sum(a_i - b_i)$ can be reduced to a single summand by the observation prior to the statement of the theorem; which is a homogeneous element or the negative of a homogeneous element. If this single element is a homogeneous element we are done and if it is the negative of a homogeneous element say $-h$ where h is homogeneous, then since H is a subgroup, $-h \in G \backslash H$ implies that $h \in G \backslash H$. Conversely, if for every proper subgroup K of G, $G \backslash K$ contains a homogeneous element the $G =< H(G) > (= F(G))$. □

3 Existence questions

In the above two sections we have developed a general theory of factoriality in Riesz groups, but it raises several questions:

(1) Are there (non-antilattice) Riesz groups with $H(G) \neq \phi$? Is there a class of Riesz groups G such that $H(G) \neq \phi$?

(2) Let G be Riesz with $H(G) \neq \phi$. Given $x \in G$, how do we decide that $x \in F(G)$?

We start with the second question first. Let us first note that if $x = 0$ then $x \in F(G)$, definitely. So let us concentrate on $x \neq 0$. Now, as a Riesz group is directed, we note that $x = a - b$ where $a, b \in G^+$. Now, $x \in F(G)$ if and only if $x = a - b$ can also be written as $x = a_1 - b_1$ where $a_1, b_1 \in (F(G))^+$. So, question (2) can be reformulated as:

(2') Let G be Riesz with $H(G) \neq \phi$. Given $x \in G^+ \backslash \{0\}$, how do we decide that x is a finite sum of homogeneous elements?

To these questions we address our attention now. However, on our way, we tackle question 1 in the following lemma.

Lemma 3.1 *Let G be a Riesz group and let $0 < x \in G$ such that x exceeds at most a finite number of mutually disjoint strictly positive elements of G. Then x exceeds at most a finite number of mutually disjoint homogeneous elements of G.*

Proof. We first note the following two points: (1) In a Riesz group G, if for $x, y > 0$, $x \wedge y = 0$ and $0 < u, v \leq x$ then $u \wedge y = 0$ and $v \wedge y = 0$. So, if $u \wedge v = 0$ then u, v, y are mutually disjoint. (2) If $0 < y \leq x$ and if x exceeds at most a finite number of mutually disjoint elements of G then so does y. □

Next we observe that there are at most two possibilities for x: (i) For all u, v with $0 < u, v \leq x$ there is t with $0 < t \leq u, v$. In this case x is itself a homogeneous element and we can say that x exceeds no two disjoint elements. (ii) There exist at least two disjoint strictly positive elements preceding x. In this case, let n be the largest number of mutually disjoint strictly positive elements preceding x. Then let $\{x_1, x_2, ..., x_n\}$ be such that $0 < x_i \leq x$ with $x_i \wedge x_j = 0$ for $i \neq j$. We claim that each x_i is a homogeneous element. For if not and say, by a permutation of x_i, x_1 is not homogeneous then there is at least one pair x_{11}, x_{12} with $0 < x_{11}, x_{12} \leq x_1$ such that $x_{11} \wedge x_{12} = 0$. But then by note (1) at the beginning of the proof $\{x_{11}, x_{12}, x_2, ..., x_n\}$ is a set of mutually disjoint strictly positive elements preceding x. This contradicts the assumption that n is the largest such number for x.

The above proof establishes the existence of a maximal set of mutually disjoint homogeneous elements preceding $x > 0$, but does not show how to get to a homogeneous element preceding x. To isolate a homogeneous element we adopt the following procedure from [5]. (Conrad used this procedure for lattice ordered groups.):

Let x be a strictly positive element in a Riesz group G. If x exceeds no pair of disjoint strictly positive elements of G then x is a homogeneous element and we have nothing more to establish. Now suppose that x is not homogeneous then there is at least one pair y_1, y_2 with $0 < y_1, y_2 \leq x$ such that $y_1 \wedge y_2 = 0$. If any of y_i is homogenous we are done. If none of the y_1, y_2 is homogeneous then there exist y_{11}, y_{12} with $0 < y_{11}, y_{12} \leq y_1$ such that $y_{11} \wedge y_{12} = 0$. But then y_{11}, y_{12}, y_2 is a set of mutually disjoint strictly positive elements preceding x. Again, if y_{11} or y_{12} is a homogeneous element we are done. If not, then let $0 < y_{21}, y_{22} \leq y_{11}$ where $y_{21} \wedge y_{22} = 0$. This gives, as before, $y_{21}, y_{22}, y_{12}, y_2$ as a larger set of mutually disjoint strictly positive elements preceding x. Next if none of $\{y_{21}, y_{22}, y_{12}, y_2\}$ is homogeneous we can add $0 < y_{31}, y_{32} \leq y_{21}$ with $y_{31} \wedge y_{32} = 0$ to get $\{y_{31}, y_{32}, y_{22}, y_{12}, y_2\}$ where $y_{31}, y_{32}, y_{22}, y_{12}, y_2$ precede x. This process can be repeated indefinitely, increasing by one the sets of nonhomogeneous elements preceding x. But since x can exceed at most a finite number of mutually disjoint strictly positive elements, the procedure must stop after a finite number of steps. Thus we arrive at a stage $0 < y_{n1}, y_{n2} \leq y_{(n-1)1}$ where one of y_{n1}, y_{n2}, say $y_{n1}$is such that for all $0 < u, v \leq y_{n1}$, $u \wedge v \neq 0$. But then y_{n1} is a homogeneous element.

If a strictly positive element x of a Riesz group G exceeds at most a finite number of mutually disjoint strictly positive elements we say that x satisfies Conrad's F-condition. Moreover the group G satisfies Conrad's F-condition if every strictly positive element of G does. We so designate this condition because it generalizes to Riesz groups the condition Conrad [5] defined for all strictly positive elements of a lattice ordered (l.o.) group. A quick reward that we get from imposing Conrad's F-condition is that a Riesz group G that satisfies Conrad's F-condition has a nontrivial $F(G)$. One wonders if something like Conrad's F-condition can be imposed on partially ordered groups that are more general than Riesz groups. To see how Conrad's F-condition can appear in a more general situation see [6].

Now our task is to see if a strictly positive element x of G is expressible as a sum of homogeneous elements. Having isolated a homogeneous element h preceding $x > 0$ we have to see if we can write $x = x_1 + h_1$ where h_1 is a homogeneous element related to h such that $x_1 \wedge h_1 = 0$. For this consider $S(h, x) = \{k \in A^{(h)} : k \leq x\}$. We see that if $S(h, x)$ contains h_1 such that $h_1 \geq s$ for all $s \in S(h, x)$, then $(x - h_1) \wedge h_1 = 0$. For if $(x - h_1) \wedge h_1 \neq 0$ there exists t such that $0 < t \leq (x - h_1), h_1$, then t is homogeneous by (2) of Proposition 1.1 and $t + h_1$ is homogeneous, related to h by Propositions 1.1 and 1.2 and $0 < t \leq x - h_1$ gives $h_1 < t + h_1 \leq x$. So $t + h_1 \in S(h, x)$, contradicting the assumption that $h_1 \geq s$ for all $s \in S(h, x)$. Conversely if $x = x_1 + h_1$ where h_1 is homogeneous related to h such that $x_1 \wedge h_1 = 0$, then obviously $h_1 \in S(h, x)$ such that $h_1 \geq s$ for all $s \in S(h, x)$. This discussion proves the following lemma.

Lemma 3.2 *Let G be a Riesz group and let $0 < x \in G$. Suppose that h is a homogeneous element of G with $h \leq x$. Then there exists a homogeneous element h_1 related to h such that $x = x_1 + h_1$ with $x_1 \wedge h_1 = 0$ if and only if $S(h, x)$ contains an upper-bound.*

The following theorem is now easy to prove, once we note that if there exists another homogeneous element say $k \leq x$ then $k \leq x_1$ or $k \leq h_1$. If k is not related to h_1 then using the Riesz property and the properties of homogeneous elements we can show that $k \leq x_1$. Once this done the whole procedure can be repeated all over again to get $x = x_2 + k_1 + h_1$.

Theorem 3.3 *Let G be a Riesz group and let $0 < x \in G$. Then $x \in F(G)$ if and only if (i) x satisfies Conrad's F-condition and (ii) for every homogeneous element $h \leq x$ the set $S(h, x)$ contains an upper-bound. Consequently, G is a factorial group if and only if every strictly positive element of G satisfies (i) and (ii).*

Remark 3.4 *Since a l.o. group is a special case of a Riesz group the above result can be sharpened. Note that in a l.o. group an element $h > 0$ is homogeneous if and only if $(0, h]$ is linearly ordered. (If h is homogeneous then for all $x, y \in (0, h]$ $x \wedge y \in (0, h]$. If $x \wedge y$ does not equal either of x, y then $x - (x \wedge y)$, $y - (x \wedge y) \in (0, h]$, but $(x - (x \wedge y)) \wedge (y - (x \wedge y)) = 0$ a contradiction. The converse is obvious.) Indeed using the same reasoning we can show that in a l.o. group G, $A^{(h)}$ (and hence $AL^{(h)}$) is linearly ordered, under induced order from G, for homogeneous h. A homogeneous element, in a l.o. group, is called a basic element in [5]. To show how the above result can be sharpened we prove the following statement.*

Proposition 3.5 *Let G be a lattice ordered group, let $0 < x \in G$, and let b be a basic element preceding x. Then the following are equivalent:*

(1) $S(b, x)$ contains an upper bound,
(2) $x = x_1 + h_1$ where $x_1 \wedge h_1 = 0$ and $h_1 \in A^{(b)}$,
(3) there exists a basic element $h \in A^{(b)}$ such that $h \not\leq x$.

Proof. Lemma 3.2 supplies the proof of (1) $\Leftrightarrow$ (2). Moreover it is clear that (2) $\Rightarrow$ (3); just put $h = 2h_1$. Now let us prove (3) $\Rightarrow$ (1). Suppose that there exists $h \in A^{(b)}$ such that $h \not\leq x$ and let $h_1 = h \wedge x$. Then for any $k \in S(b, x)$ as $h \not\leq x$ and as $k \leq x$ we conclude that $h \not\leq k$, forcing $h > k$ because h is a basic element. So, $k \in S(b, x)$ implies that $k \leq h \wedge k = h_1$. Finally $h_1 \in S(b, x)$ because $h_1 \leq x$ and h_1 is homogeneous element related to b. □

Corollary 3.6 *Let G be a lattice ordered group. Then $0 < x \in F(G)$ if and only if (i) x satisfies Conrad's F-condition and (ii) for every basic element $b \leq x$ there exists a basic element h related to b such that $h \not\leq x$. Moreover, G is factorial if and only if each $0 < x \in G$ satisfies (i) and (ii).*

Recall that a p.o. group $(G, +, \leq)$ is Archimedean if in G, $na < b$ for all integers n implies that $a = 0$. Thus if $(G, +, \leq)$ is Archimedean and $0 < x \leq y$, we can show that for some natural number n we have $nx \not\leq y$.

Corollary 3.7 *An Archimedean l.o. group G is factorial if and only if G satisfies Conrad's F-condition.*

The proof can be extracted from the proof of the following.

Corollary 3.8 *Let G be a lattice ordered group that satisfies Conrad's F-condition. Then the following are equivalent:*

(1) G is Archimedean,

(2) For every basic element b of G, $b \leq x$ implies that for some n, $nb \nleq x$.

Proof. (1) $\Rightarrow$ (2) follows from the fact that G Archimedean supplies more than (2). For (2) $\Rightarrow$ (1), note that (2), in conjunction with the main hypothesis, implies that G is factorial. Now, by Remarks prior to Proposition 3.5, for every basic element b of G we have $AL^{(b)}$ totally ordered under the induced order. Now let $nx < y$ in $AL^{(b)}$ for all integers n. We claim that $x \ngtr 0$. For if $x > 0$ then x is a basic element. If $x > y$ we have the contradiction and if $x < y$ then by (2) there is a natural number m such that $mx \nleq y$ a contradiction. Hence $x \ngtr 0$. Next if $x < 0$ then $-x$ is a basic element with $-x < y$, and again for some natural number n we have $n(-x) \nleq y$. Or, as $n(-x) = -nx$, we conclude that there is an integer $-n$ such that $-nx \nleq y$. Now in a linearly ordered group a member x that can neither be strictly positive, nor strictly negative must be 0. So for each basic element $b \in G$, $AL^{(b)}$ is a fully ordered Archimedean group, hence o-isomorphic to a subgroup of the set of real numbers and hence Abelian see [7, p. 45]. Now G, being factorial as already mentioned, G is a cardinal sum of the $AL^{(b)}$ as proved in Theorem 2.4 and so is an Abelian group. Now let x, y be such that for all integers n we have $nx < y$. Let $y = y_1 + y_2 + ... + y_m$ where $y_i \in AL^{(b_i)}$. We show that each of y_i is 0. Using the Riesz property we can assume that $x = x_1 + x_2 + ... + x_m$ where $nx_i \leq y_i$ for each $1 \leq i \leq m$ and for each integer n. Now each $AL^{(b_i)}$ being Archimedean we have $x_i = 0$ as shown above. □

Part (3) of Proposition 3.5 gives the following result as a by-product.

Corollary 3.9 *Let $x > 0$ be an element in a l.o. group and suppose that x exceeds a basic element b. If there is no basic element $h \in AL^{(b)}$ such that $h \nleq x$ then x exceeds every element of $AL^{(b)}$.*

4 Examples and links

We believe that questions pertaining to divisibility in commutative integral domains and partially ordered groups are joined at the hip. The aim of this section is to explore the connections, pointing out the links of the work in this paper with results on various types of unique factorization in commutative integral domains. It seems pertinent to indicate the reasons for the "hip connection" remark. To see the hip connection, recall that if D is an integral domain with quotient field K then we can define a group $G(D) = \{kU : k \in K\backslash\{0\} = K^*\} = K^*/U$, where U is the group of units of D, under multiplication $(k_1U)(k_2U) = k_1k_2U$. Now $G(D)$ can be partially ordered by $k_1U \leq k_2U \Leftrightarrow \frac{k_2}{k_1} \in D$ (i.e. $k_1 \mid_D k_2$). Now if $k_1 \mid_D k_2$ then for any $k \in K^*$ we have $k_1k \mid_D k_2k$. So the partial order is compatible with the group multiplication and we have a partially ordered group in $G(D)$ which is called the group of divisibility for obvious reasons. Of course

$G(D)^+ = \{dU : d \in D^*\}$ and $G(D)$ is directed because for every kU in $G(D)$, k can be written as $k = \frac{a}{b}$ where $a, b \in D$. So $kU = \frac{a}{b}U = (aU)(bU)^{-1}$.

To see the role of the group of divisibility of a (commutative) domain in joining the study of partially ordered groups and the study of divisibility it seems necessary to bring in some more definitions. A fractionary ideal A of D is a D submodule of K such that for some $d \in D^*$ we have $dA \subseteq D$. Let $F(D)$ denote the set of nonzero fractionary ideals of D. For $A \in F(D)$ define $A^{-1} = \{x \in K : xA \subseteq D\}$. Clearly for each $A \in F(D)$ we have $A^{-1} \in F(D)$ and $(A^{-1})^{-1} = A_v \in F(D)$. We can also define $A_v = \cap\{kD : A \subseteq kD\}$ [9]. In the p.o. group terminology, if A is a nonempty subset of a p.o. group G, $U(A) = \{x \in G : x \geq a$ for all $a \in A\}$ and $L(A) = \{x \in G : x \leq a$ for all $a \in A\}$. For our purposes it is sufficient to see how, for $\phi \neq A \subseteq K^*$ an equivalent of $U(\{aU : a \in A\})$ can be used.

Note that $U(\{aU \in G(D) : a \in A\}) = \{xU \in G(D) : xU \geq aU$ for all $a \in A\}$ $= \{xU \in G(D) : xD \subseteq aD$ for all $a \in A\}$. Now as $xD \subseteq aD$ means $a \mid_D x$, and since division is associate blind, we can associate with $U(\{aU \in G(D) : a \in A\})$ the set $\{x \in K^*$: $a \mid_D x$ for all $a \in A\} = \{x \in K^*$: $a \mid_D x$ for all $a \in A\} = \{x \in K^* : x \in aD$ for all $a \in A\} = \cap\{aD : a \in A\backslash\{0\}\} = u(A)$. Call $A \subseteq K^*$ bounded from above if $\cap\{aD : a \in A\} \neq 0$. So if $A \subseteq K^*$ is bounded from above then $\cap\{aD : a \in A\backslash\{0\}\} \in F(D)$. While we are at it, it seems useful to give a ring theoretic representation of $L(A)$. Note that $L(\mathcal{A}) = \{xU \in G(D) : xU \leq aU$ for all $aU \in \mathcal{A}\}$. So $L(\mathcal{A}) = \{xU \in G(D) : x \mid_D a$ for all $aU \in \mathcal{A}\} = \{xU : xD \supseteq A\}$. Since $xD \supseteq A$ would hold for x and all associates of x we can associate with $L(\mathcal{A})$ the subset $l(A) = \{x : xD \supseteq A\}$. Of course $\mathcal{A}$ is (A is) bounded from below, if and only if $L(\mathcal{A}) \neq \phi$ ($l(A) \neq \phi$).This discussion, complements a similar discussion in [2] and supports the following statement.

Proposition 4.1 *Let A be a nonempty subset of K^*. If A is bounded from above then the ideal analogue of $U(\{aU : a \in A\})$ is $\cap\{aD : a \in A\}$. If A is bounded from below then* $u(l(A)) = \bigcap_{A \subseteq xD} xD = A_v$.

A fractionary ideal A is an integral ideal if $A \subseteq D$. For $a, b \in D^*$, the GCD of a, b, if it exists, is an element $d \in D$ such that $d \mid a, b$ and for all $r \mid a, b$ we have $r \mid d$. Now recall that an integral domain D is a GCD domain if for every pair $a, b \in D^*$, $GCD(a, b) = g$ exists. This GCD is unique up to associates. Now it is well known that D is a GCD domain if and only if for every pair $a, b \in D^*$, $LCM(a, b) = l$ exists. It is also known that $lD = aD \cap bD$. Now as we have seen above $aD \cap bD = u(\{a, b\})$ which can be translated to $U(\{aU, bU\})$. Now $lD = aD \cap bD$ means that $a, b \mid_D l$ and if $a, b \mid_D x$ then $l \mid_D x$. But this leads to $aU, bU \leq lU$ and if $aU, bU \leq xU$ then $lU \leq xU$ which lead to the conclusion that $lU = \sup(aU, bU) = aU \vee bU$ and $U(\{aU, bU\}) = lU(G^+)$. Now, a directed p.o. group G is a lattice ordered group if and only if for each pair $x, y \in G^+$ we have $x \vee y \in G^+$. Thus if D is a GCD domain then $G(D)$ is a lattice ordered group. Conversely, since the above translation can be reversed, if $G(D)$ is lattice ordered then D is a GCD domain. Then there is an interesting result known as

Krull-Kaplansky-Jaffard-Ohm theorem which states that we can associate with each Abelian l.o. group G a Bezout domain D such that $G = G(D)$. The reader may look up [11] for the history of this theorem. Commutative ring theorists exploited this result in constructing examples of integral domains with specific divisibility properties, as indicated in [12].

The following proposition includes all the results on factorial l.o. groups that we have proved directly or indirectly. It is interesting to note that while the following theorem is stated for general (not necessarily commutative) lattice ordered groups the results are reminiscent of the characterizations of semirigid GCD domains [14].

Proposition 4.2 *Let G be a lattice ordered group. Then the following are equivalent.*

(1) G is factorial according to our definition i.e. $G = F(G)$.

(2) G is a cardinal sum of its maximal, convex, fully ordered subgroups.

(3) For every proper subgroup B of G, $G \backslash B$ contains a basic element.

(4) G satisfies Conrad's F-condition and for all $0 < x \in G$, if b is a basic element preceding x then there exists a basic element h that is related to b and $h \nleq x$.

(5) G satisfies Conrad's F-condition and for every basic element $b \in G$, $AL^{(b)}$ is not bounded from above.

The above proposition is also related to [18], where a general theory of factoriality was devised so that a factorial group according to that theory is a lattice ordered group that is a cardinal sum of maximal convex, fully ordered subgroups. Indeed, as we have witnessed in this section, the overall assumption of the group being lattice ordered makes it much easier to define factoriality in terms of basic elements. The message of this paper is that where there are homogeneous (basic) elements in a Riesz (lattice ordered) group G, there is an o-ideal $F(G)$ that is factorial.

In [15] it was established that if in a GCD domain D a principal ideal xD has only finitely many minimal primes then x can be uniquely expressed as $x = x_1 x_2 ... x_n$ where, for each i with $1 \leq i \leq n$, $x_i D$ has a unique minimal prime. It would be of interest to have a l.o. group analog of this result.

Let us now look into some rather interesting cases where the Riesz property combined with some nice factorization property delivers a lattice ordered group.

Theorem 4.3 *Let G be a factorial Riesz group. If every homogeneous element of G is a basic element, then G is a lattice ordered factorial group.*

Proof. All we have to do is to note that in such a Riesz group any two related homogeneous elements are comparable. This follows from the fact that if b_1 and b_2 are related homogeneous elements then $b_1 + b_2$ is a homogeneous element related to b_1 and b_2. But then by the condition $b_1 + b_2$ is basic and so $b_1 \leq b_2$ or $b_2 \leq b_1$. Now to establish that G is lattice ordered we need to show that for $x, y \in G^+$ we have $x \vee y, x \wedge y \in G^+$. For this we note that $x = x_1 + x_2 + ... x_n$, $y = y_1 + y_2 + ... y_m$ where x_i are mutually disjoint basic elements and so are y_j. Since x_i

commute being disjoint (resp. y_j commute among themselves) we can rearrange and assume that for $1 \le i \le r$, x_i is related to y_i and that for $i, j > r$, x_i, y_j are disjoint. Construct $z = (x_1 \vee y_1) + ... + (x_r \vee y_r) + x_{r+1} + ... + x_n + y_{r+1} + ... + y_m$ and $w = (x_1 \wedge y_1) + ... + (x_r \wedge y_r)$. It is now routine to verify that $z = x \vee y$ and $w = x \wedge y$. □

An element $h > 0$ in a p.o. group G may be called irreducible if for each x with $0 \le x \le h$ implies that $x = 0$ or $x = h$. In a Riesz group an irreducible element acts like a prime. That is, if $h > 0$ is irreducible in a Riesz group and $h \le x + y$ then $h \le x$ or $h \le y$. For, in a Riesz group $h \le x + y$ implies that $h = h_1 + h_2$ where $0 \le h_1 \le x$ and $0 \le h_2 \le y$. But then $0 \le h_1, h_2 \le h$ and because h is irreducible precisely one of h_i is equal to h and the other is 0.

A p.o. group G is said to satisfy the descending chain condition (DCC) if every nonempty set of positive elements of G includes a minimal member. Clearly if G satisfies DCC, then (a) every strictly positive element of G exceeds an irreducible element of G and (b) every strictly positive element of G can be expressed as a sum of irreducible elements of G. These observations lead us to the following result.

Theorem 4.4 *Let G be a Riesz group that satisfies the DCC. Then G is an Abelian lattice ordered factorial group that is a cardinal sum of its cyclic subgroups generated by irreducible elements.*

Proof. We note that every irreducible element is a homogeneous element. Now let h be a homogeneous element of G. Then by DCC, h exceeds an irreducible element and because of homogeneity h cannot exceed two or more disjoint irreducible elements. Using DCC and homogeneity of h we can show, as indicated below, that $h = np$ for some natural number n. From this we conclude that every homogeneous element is basic. By Theorem 4.3, to show that G is lattice ordered all we need to show is that G is factorial. For this we proceed as follows. Let $0 < x \in G$ be typical strictly positive element of G. By the chain condition x is expressible as a finite sum of irreducible elements and so exceeds at most a finite number of mutually disjoint elements. That is x satisfies Conrad's F-condition. This takes care of (i) of Theorem 3.3. For (ii) we proceed as follows. Let $0 < x \in G$, then we have seen that x exceeds an irreducible element p. Consider the set $S = \{x - rp \ge 0 : r \in N\}$. Then by the chain condition S has a minimal positive element $x - rp$. Because of the minimality $x - rp$ is either 0 (in which case $x = rp$, true in case x is homogeneous.) or disjoint from p. In either case we have $x = x - rp + rp$ where $(x - rp) \wedge rp = 0$ and by Lemma 3.2 this is equivalent to (ii) of Theorem 3.3. So G is factorial and hence lattice ordered. Now if p is an irreducible element of G then every strictly positive element of $AL^{(p)}$ is an integral multiple of p as shown above. So $AL^{(p)}$ is cyclic and hence Abelian. Finally as G is factorial, by Theorem 2.4 there is a maximal set S of mutually disjoint homogeneous elements such that G is a cardinal sum of $AL^{(s)}$, where s ranges over S. But for each s we have $s = np$ for some irreducible p and so $AL^{(s)} = AL^{(p)}$. This forces G to be a cardinal sum of Abelian groups and hence makes it Abelian. □

Remark 4.5 *The above theorem is a Riesz group version of Theorem 21 of [3]. To see another link let us recall that an integral domain D is called a pre-Schreier domain if the group of divisibility of D is a Riesz group. In other words for $x, y, z \in D \backslash \{0\}$, $x|yz$ implies that $x = rs$ such that $r|y$ and $s|z$, see [16] for various characterizations of a pre-Schreier domain. Call an integral domain D atomic if every nonzero nonunit of D is expressible as a finite product of irreducible elements (atoms). Now an atomic pre-Schreier domain is a UFD see Corollary 1.8 of [16], see also Cohn [4]. However, Theorem 4.4, is slightly more than a mere translation, as Corollary 1.8 of [16] is in the framework of commutative rings.*

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