

Factorization in Integral Domains, II

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In this paper, we study several factorization properties in an integral domain which are weaker than unique factorization. We study how these properties behave under localization and directed unions. © 1992 Academic Press, Inc.

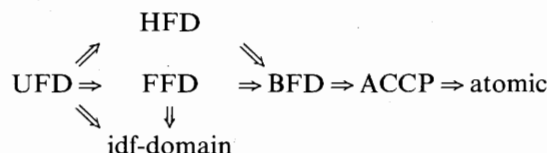
INTRODUCTION

In this paper, we continue our investigation begun in [1] of various factorization properties in an integral domain which are weaker than unique factorization. Section 1 introduces material on inert extensions and splitting multiplicative sets. In Section 2, we study how these factorization properties behave under localization. Special attention is paid to multiplicative sets generated by primes. Section 3 consists of "Nagata-type" theorems about these properties. That is, if the localization of a domain at a multiplicative set generated by primes satisfies a certain ring-theoretic property, does the domain satisfy that property? In the fourth section, we consider Nagata-type theorems for root closure, seminormality, integral

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closure, and complete integral closure. In the final section, we investigate how these factorization properties behave under directed unions.

We first recall the various factorization properties in addition to unique factorization which we will study here. Throughout, let R be an integral domain with quotient field K . Following Cohn [5], we say that R is *atomic* if each nonzero nonunit of R is a product of a finite number of irreducible elements (atoms) of R . We say that R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal integral ideals of R . The domain R is a *bounded factorization domain* (BFD) if R is atomic and for each nonzero nonunit $x \in R$ there is a bound on the lengths of factorizations of x into products of irreducible elements (equivalently, there is a bound on the lengths of chains of principal integral ideals starting at xR). Following Zaks [17], we say that R is a *half-factorial domain* (HFD) if R is atomic and each factorization of a nonzero nonunit of R into a product of irreducible elements has the same length. Following Grams and Warner [12], we say that the domain R is an *idf-domain* (for irreducible-divisor-finite) if each nonzero element of R has at most a finite number of nonassociate irreducible divisors. An atomic idf-domain will be called a *finite factorization domain* (FFD); these are precisely the domains in which each nonzero nonunit has only a finite number of nonassociate divisors (and hence, only a finite number of factorizations up to order and associates). BFDs and FFDs were introduced in [1]. These factorization properties have also been studied in [2]. In general,



Examples given in [1] show that no other implications are possible.

General references for any undefined terminology or notation are [7, 9, 13]. For an integral domain R , R^* is its set of nonzero elements, $U(R)$ its group of units, R' its integral closure, and \bar{R} its complete integral closure. By an ideal, we always mean an integral ideal. For nonzero $a, b \in R$, $(a, b) = 1$ means that a and b have no nonunit common factors. We also make the two harmless assumptions that all our multiplicative sets do not contain 0 and are saturated. A multiplicative set S is *generated by* $T \subset R$ if $S = \{ut_1 \cdots t_n \mid u \in U(R) \text{ and } t_i \in T\}$. Throughout, \mathbb{Z} , \mathbb{Q} , and \mathbb{R} denote the integers, rational numbers, and real numbers, respectively.

1. INERT EXTENSIONS AND SPLITTING MULTIPLICATIVE SETS

Following Cohn [5], we say that an extension of rings $A \subset B$ is an *inert extension* if whenever $xy \in A$ for nonzero $x, y \in B$, then $xu, yu^{-1} \in A$ for some $u \in U(B)$. Our first lemma is

LEMMA 1.1. *Let $A \subset B$ be an inert extension of integral domains. Then an irreducible element of A is either irreducible or a unit in B .*

Proof. Let $a \in A$ be irreducible and suppose that a is not a unit in B . If $a = xy$ for $x, y \in B$, then $xu, yu^{-1} \in A$ for some $u \in U(B)$. Hence $a = (xu)(yu^{-1})$ in A implies that either xu or yu^{-1} is a unit in A . Thus either x or y is a unit in B . Hence a is irreducible in B . ■

Easy examples show that either case may occur in Lemma 1.1. Also, an $a \in A$ may be irreducible in B , but not irreducible in A . However, if $A \subset B$ is an inert extension and $U(B) \cap A = U(A)$, then an $a \in A$ is irreducible in A if and only if it is irreducible in B . It is easily seen that none of our factorization properties need ascend or descend for an inert extension $A \subset B$ of integral domains.

We next define a special type of multiplicative set. A (saturated) multiplicative subset S of R is a *splitting multiplicative set* if for each $x \in R$, $x = as$ for some $a \in R$ and $s \in S$ such that $aR \cap tR = atR$ for all $t \in S$. (Thus $a \in S$ if and only if $a \in U(R)$.) Similar types of multiplicative sets have been studied in [10, Sect. 3; 14, Sect. 4]. (This should not be confused with splittable sets as defined by Zaks in [17, 18].) We first have (cf. [14, Proposition 4.1])

LEMMA 1.2. *Let R be an integral domain and S a multiplicative set of R . Then the following statements are equivalent for $x, y \in R$:*

- (a) $xR_S \cap R = yR$.
- (b) $x = ys$ for some $s \in S$ and $yR \cap tR = ytR$ for all $t \in S$.

Proof. (a) \Rightarrow (b) We may assume that x and y are nonzero. We have $x = ys$ for some $s \in R$ since $x \in xR_S \cap R = yR$. Also, $y = x(r/t')$ for some $r \in R$ and $t' \in S$. Thus $xr = yt'$, so $sr = t' \in S$. Hence $s \in S$ since S is saturated. We next show that $yR \cap tR = ytR$ for all $t \in S$. The " \supset " inclusion is clear. Conversely, let $z \in yR \cap tR$. Then $z = ya = tb$ for some $a, b \in R$. Thus $yas = tbs$, and hence $ax = tbs$. Thus $b = x(a/(st)) \in xR_S \cap R = yR$, so $b = yc$ for some $c \in R$. Hence $z = ytc \in ytR$. Thus the " \subset " inclusion holds and we have equality.

(b) \Rightarrow (a) Let $x = ys$ with $s \in S$ and $yR \cap tR = ytR$ for all $t \in S$. Then $xR_S = yR_S$, so we need only show that $yR_S \cap R = yR$. The " \supset " inclusion

is clear. Conversely, let $z \in yR_S \cap R$. Thus $z = y(r/t)$ for some $r \in R$ and $t \in S$. Hence $tz = yr \in yR \cap tR = ytR$, so $z \in yR$. Hence the “ \subset ” inclusion holds and we have equality. ■

COROLLARY 1.3. *A multiplicative set S of an integral domain R is a splitting multiplicative set if and only if principal ideals of R_S intersect to principal ideals of R .*

COROLLARY 1.4. *Let R be an integral domain and S a splitting multiplicative set of R . Let $x \in R$ be nonzero and $x = as$ with $a \in R$, $s \in S$, and $aR \cap tR = atR$ for all $t \in S$.*

(a) *If $x = a's'$ with $a' \in R$, $s' \in S$, and $a'R \cap tR = a'tR$ for all $t \in S$, then a and a' are associates and s and s' are associates.*

(b) *If $y = bt$ with $b \in R$, $t \in S$, and $bR \cap t'R = bt'R$ for all $t' \in S$, then $abR \cap t'R = abt'R$ for all $t' \in S$. Hence the decomposition for xy is $(ab)(st)$.*

(c) *x is prime (resp., irreducible) in R_S if and only if a is prime (resp., irreducible) in R .*

(d) *Each prime (resp., irreducible) element in R_S is an associate in R_S of a prime (resp., irreducible) element in R .*

Proof. (a) This follows directly from Lemma 1.2. (b) This follows because $abt'R = b(aR \cap t'R) = abR \cap bt'R = abR \cap bR \cap t'R = abR \cap t'R$. (c) If x is prime in R_S , then a is prime in R since $aR = xR_S \cap R$. The converse is clear. If a is irreducible in R , then x is irreducible in R_S by Lemma 1.1 (and Proposition 1.5) since $s \in U(R_S)$. Conversely, if $a = a_1a_2$ with neither factor a unit in R , then a , and hence x , is not irreducible in R_S . (d) This follows easily from (c). ■

In general, localization need not yield an inert extension $R \subset R_S$. However, we next show that this extension is inert when S is a splitting multiplicative set. In Proposition 1.9, we show that $R \subset R_S$ is also an inert extension when S is generated by primes.

PROPOSITION 1.5. *Let R be an integral domain and S a splitting multiplicative set of R . Then $R \subset R_S$ is an inert extension.*

Proof. Let $xy \in R$ for nonzero $x, y \in R_S$. Then $x = (as)/t$ and $y = (bu)/v$, where $a, b \in R$; $s, t, u, v \in S$; and $aR \cap t'R = at'R$ and $bR \cap t'R = bt'R$ for all $t' \in S$. Thus $absu = rtv$ for some $r \in R$. Since $bR \cap vtR = bvtR$, we have $asu = cvt$ for some $c \in R$. Hence $w = u/v \in U(R_S)$ and $xw, yw^{-1} \in R$. Thus $R \subset R_S$ is an inert extension. ■

In this paper, we are mainly interested in multiplicative sets generated by prime elements. Such multiplicative sets are always saturated, but need

not be splitting multiplicative sets (cf. Examples 1.8 and Example 2.3); precisely, we have

PROPOSITION 1.6. *Let R be an integral domain and S a multiplicative set of R generated by primes. Then the following statements are equivalent.*

- (a) S is a splitting multiplicative set.
- (b) $\bigcap p^n R = 0$ for each prime $p \in S$ and $\bigcap p_n R = 0$ for all sequences $\{p_n\}$ of nonassociative primes in S .
- (c) For each nonzero nonunit $x \in R$, there is a positive integer $n(x)$ such that whenever $p_1 \cdots p_n \mid x$ for primes $p_i \in S$, then $n \leq n(x)$.
- (d) Principal ideals of R_S intersect to principal ideals of R .

Proof. Clearly (b) and (c) are equivalent, and (a) and (d) are equivalent by Corollary 1.3. Suppose that (b) holds. For a nonzero nonunit $x \in R$, $x = as$, where $a \in R$, $s \in S$, and no prime $p \in S$ divides a . Thus $aR \cap tR = atR$ for all $t \in S$. Hence S is a splitting multiplicative set. Conversely, suppose that (a) holds. For a nonzero nonunit $x \in R$, $x = ap_1 \cdots p_n$ for some $a \in R$ and primes $p_1, \dots, p_n \in S$ such that no prime $p \in S$ divides a . Hence (c) holds with $n(x) = n$. ■

COROLLARY 1.7. *Let R be an atomic integral domain. Then any multiplicative set of R generated by primes is a splitting multiplicative set. In particular, this holds if R satisfies ACCP.*

Proof. Let x be a nonzero nonunit of R . Since R is atomic, $x = x_1 \cdots x_n$ for irreducible $x_1, \dots, x_n \in R$. Thus any prime p of R which divides x must be an associate of some x_i . Hence (c) holds with $n(x) = n$, so S is a splitting multiplicative set. ■

We remark that $\bigcap p^n R = 0$ for a prime p if and only if pR has height one (cf. [13, p. 7, Exercise 5]). Also, if R is an atomic integral domain and S is the multiplicative set of R generated by all primes of R , then by Corollary 1.4(d), R_S has no primes (Example 1.8(a) shows that it is necessary to assume that R is atomic). We next give four examples of multiplicative sets generated by primes which are not splitting multiplicative sets and a nontrivial example of a splitting multiplicative set which is not generated by primes. (Note that $S = R^*$ is always a splitting multiplicative set of R .)

EXAMPLES 1.8. (a) Let V be a two-dimensional valuation domain with principal maximal ideal $M = pV$ and height-one prime P . Then the multiplicative set S generated by p is not a splitting multiplicative set since $\bigcap p^n V = P$ is nonzero. Moreover, V may be chosen so that $V_p = V[1/p]$ is a DVR.

(b) (cf. [16, p. 264, (First) Example].) Let F be a field, X and Y indeterminates over F , $R = F[X, \{Y/X^n \mid n \geq 0\}]$, and S the multiplicative set generated by the prime X . Then $R_S = F[X, X^{-1}, Y]$ is a UFD. However, S is not a splitting multiplicative set since $Y \in \bigcap X^n R$.

(c) Let $R = \mathbb{Z} + X\mathbb{Q}[[X]]$ and $S = \mathbb{Z}^*$. Then R is a Bézout domain [9, p. 286, Exercise 13], S is a multiplicative set of R generated by primes, and $R_S = \mathbb{Q}[[X]]$ is a UFD, but R is neither atomic [1, Proposition 1.2] nor an idf-domain [1, Proposition 4.3]. Note that S is not a splitting multiplicative set since $\bigcap p^n R = X\mathbb{Q}[[X]]$ for each prime $p \in S$ and $\bigcap p_n R = X\mathbb{Q}[[X]]$ for each sequence $\{p_n\}$ of nonassociate primes in S .

(d) Let E be the ring (Bézout domain) of entire functions and S the multiplicative set generated by all primes of E . Then $\bigcap p^n E = 0$ for each prime $p \in E$. However, while the intersection of all nonzero principal prime ideals of E is zero, $\bigcap p_n E$ may be nonzero for a sequence $\{p_n\}$ of nonassociate primes of E . Thus S is not a splitting multiplicative set. (cf. [9, page 147, Exercises 16–21].)

(e) Let V be a nondiscrete one-dimensional valuation domain with quotient field F , X an indeterminate, and $R = V[X]$. Then $S = V^*$ is a splitting multiplicative set of R which is not generated by primes and $R_S = F[X]$ is a UFD. In fact, in this example V may be replaced by any GCD-domain which is not a UFD.

Even though a multiplicative set S of R generated by primes need not be a splitting multiplicative set, we next show that $R \subset R_S$ is always an inert extension. Note that this need not be true if S is merely assumed to be generated by irreducible elements. Indeed, if R is atomic, then each (saturated) multiplicative set of R is generated by irreducibles. Moreover, while a multiplicative set generated by primes is always saturated, a multiplicative set generated by irreducibles need not be saturated.

PROPOSITION 1.9. *Let R be an integral domain and S a multiplicative set of R generated by primes. Then $R \subset R_S$ is an inert extension.*

Proof. Suppose that $xy \in R$ for nonzero $x, y \in R_S$. Then $x = a/s$ and $y = b/t$, where $a, b \in R$; $s, t \in S$; and $(a, s) = (b, t) = 1$. Thus $st \mid ab$ in R , so $s \mid b$ and $t \mid a$ in R since s and t are each products of primes. Let $u = s/t \in U(R_S)$. Then $xu, yu^{-1} \in R$, so $R \subset R_S$ is an inert extension. ■

2. LOCALIZATIONS

It is well known that the localization of a UFD is a UFD. However, in [1], we gave examples to show that the localization of an atomic domain

(resp., domain which satisfies ACCP, BFD, idf-domain, or FFD) need not be an atomic domain (resp., satisfy ACCP, BFD, idf-domain, or FFD). We next show that if $R \subset R_S$ is an inert extension of integral domains, then each of these factorization properties, except the idf-property (see Example 2.3), does ascend from R to R_S . In particular, by Proposition 1.9 these properties are all preserved by localizing at multiplicative sets generated by primes. Thus, in some sense, the UFD case should not be viewed as exceptional since any (saturated) multiplicative set of a UFD is generated by primes. It is interesting to note that although these factorization properties need not be preserved by either localizations or inert extensions, they are preserved by the combination of the two.

THEOREM 2.1. *Let R be an integral domain and S a multiplicative set of R such that $R \subset R_S$ is an inert extension. Then R_S is atomic (resp., satisfies ACCP, a BFD, a FFD, or a UFD) if R is atomic (resp., satisfies ACCP, a BFD, a FFD, or a UFD).*

Proof. Suppose that R is atomic. Let $x = r/s \in R_S$ be a nonzero nonunit with $r \in R$ and $s \in S$. Then $r = r_1 \cdots r_n$ with each $r_i \in R$ irreducible. By Lemma 1.1, each r_i is either irreducible or a unit in R_S . Hence x is a product of irreducible elements in R_S and so R_S is atomic. Next, suppose that aR_S is properly contained in bR_S . We may assume that $a, b \in R$. Hence $a = b(r/s)$ for some $r \in R$ and $s \in S$. Thus $bu, (r/s)u^{-1} \in R$ for some $u \in U(R_S)$. Let $b' = bu$. Then $bR_S = b'R_S$ and aR is properly contained in $b'R$. Thus if R satisfies ACCP (resp., is a BFD), then R_S satisfies ACCP (resp., is a BFD). Finally, suppose that R is a FFD. Let y be a divisor of a nonzero nonunit x in R_S . Thus $x = yz$ for some $z \in R_S$. We may assume that $x \in R$. Hence $yu, zu^{-1} \in R$ for some $u \in U(R_S)$. Let x_1, \dots, x_n be the nonassociate divisors of x in R . Then $y = vx_i$ for some $v \in U(R_S)$ and $1 \leq i \leq n$. Hence R_S is a FFD. The UFD case is well known. ■

COROLLARY 2.2. *Let R be an integral domain and S a multiplicative set of R which is either generated by primes or a splitting multiplicative set. Then R_S is atomic (resp., satisfies ACCP, a BFD, a FFD, or a UFD) if R is atomic (resp., satisfies ACCP, a BFD, a FFD, or a UFD).*

We next give an example to show that the idf-property need not be preserved by localization at a multiplicative set generated by primes, and hence need not be preserved by localizations which are inert extensions. This failure is essentially because an idf-domain need not be atomic, and hence a multiplicative set generated by primes need not be a splitting multiplicative set. (Note that in Corollary 2.2 any multiplicative set generated by primes is a splitting multiplicative set since R is atomic.)

EXAMPLE 2.3. Let $R = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$. Then R is an idf-domain [1, Proposition 4.3] which is not atomic [1, Proposition 1.2] and 2 is prime in R . However, $R[1/2] = \mathbb{Q} + X\mathbb{R}[[X]]$ is atomic [1, Proposition 1.2], but is not an idf-domain since $\mathbb{R}^*/\mathbb{Q}^*$ is infinite [1, Proposition 4.2(a)]. Note that $\bigcap 2^n R = X\mathbb{R}[[X]]$, so 2 does not generate a splitting multiplicative set of R .

We do not know if the localization of a HFD is again a HFD or if HFDs are preserved by inert localizations. However, for splitting multiplicative sets we do get a positive result for both idf-domains and HFDs.

THEOREM 2.4. *Let R be an integral domain and S a splitting multiplicative set of R .*

- (a) *If R is an idf-domain, then R_S is an idf-domain.*
- (b) *If R is a HFD, then R_S is a HFD.*

Proof. (a) Let $x \in R_S$ be a nonzero nonunit and $y \in R_S$ be an irreducible divisor of x in R_S . We may assume that $x \in R$, $y \in R$, and $yR \cap sR = y_sR$ for all $s \in S$. By Corollary 1.4(c), y is irreducible in R . Now $x = y(r/t)$ for some $r \in R$ and $t \in S$. Thus $xt = yr = yta$ for some $a \in R$. Hence y is an irreducible divisor of x in R . Since R is an idf-domain, x has only a finite number of nonassociate irreducible divisors in R . Thus x has only a finite number of nonassociate divisors in R_S . Hence R_S is an idf-domain.

(b) By Corollary 2.2, R_S is atomic. Let $x_1 \cdots x_n = y_1 \cdots y_m$ be two products of irreducible elements in R_S . By Corollary 1.4(c), each $x_i = (a_i s_i)/t_i$ and $y_j = (b_j u_j)/v_j$, where each $a_i, b_j \in R$ is irreducible; $s_i, t_i, u_j, v_j \in S$; and $a_i R \cap sR = a_i sR$ and $b_j R \cap sR = b_j sR$ for all $s \in S$. Let $a = a_1 \cdots a_n$ and $b = b_1 \cdots b_m$. Then $as = bt$ for some $s, t \in S$. By Corollary 1.4(a) and (b), a and b are associates in R . Hence $m = n$ since R is a HFD and thus R_S is a HFD. ■

COROLLARY 2.5. *Let R be an integral domain and S a multiplicative set of R generated by primes. If R is a HFD, then R_S is a HFD.*

Proof. Since a HFD is atomic, by Corollary 1.7 any multiplicative set of R generated by primes is a splitting multiplicative set. ■

3. NAGATA-TYPE THEOREMS

In [15], Nagata showed that if an integral domain R is Noetherian and S is a multiplicative set of R generated by primes, then R is a UFD if R_S is a UFD. With the Noetherian hypothesis replaced by ACCP (as in [13,

Theorem 177]), this result is usually called Nagata's Theorem. Gilmer and Parker [10, Theorem 3.2] generalized this to the case in which S is a splitting multiplicative set generated by primes (also, cf. [14, Corollary 3.3]). Other versions relating divisor class groups of Krull domains are given in [7, pp. 35–36]. The examples given in Examples 1.8 show that some restrictions are necessary on R and S . In particular, Nagata's Theorem does not extend to an arbitrary splitting multiplicative set since for any integral domain R , $S = R^*$ is a splitting multiplicative set and $R_S = K$. Our next theorem is the Nagata-type theorem converse of Theorem 2.1. The ACCP case is also due to Gilmer and Parker [10, Theorem 3.2].

THEOREM 3.1. *Let R be an integral domain and S a splitting multiplicative set of R generated by primes. Then R is atomic (resp., satisfies ACCP, a BFD, an idf-domain, a FFD, or a UFD) if R_S is atomic (resp., satisfies ACCP, a BFD, an idf-domain, a FFD, or a UFD).*

Proof. First, suppose that R_S is atomic. Let $x \in R$. Then $x = as$, where $a \in R$, $s \in S$ is a finite product of primes, and no prime $p \in S$ divides a . Now $a = a_1 \cdots a_n$ with each $a_i \in R_S$ irreducible. Since no prime in S divides a , we may assume that each $a_i \in R$ and hence each a_i is irreducible in R . Thus x is a product of irreducibles in R , and hence R is atomic. We have already observed that the ACCP case has been proved in [10, Theorem 3.2]. Next, let R_S be a BFD. Let $x \in R$ and suppose that $x = x_1 \cdots x_n$ is a product of irreducibles in R . Suppose that x_{i+1}, \dots, x_n are the irreducible factors in S (and hence each is prime). Let $a = x_1 \cdots x_i$ and $s = x_{i+1} \cdots x_n$. By Corollary 1.4, in any factorization of x as a product of irreducibles $x = y_1 \cdots y_m$, the product of the y_i 's not in S is an associate of a and the product of the y_i 's in S is an associate of s . Since R_S is a BFD, there is an integer k such that any factorization of a in R_S has at most k irreducible factors. The number of prime factors in s is always $n - i$. Hence any factorization of x in R has at most $k + n - i$ irreducible factors. Thus R is a BFD. Suppose that R_S is an idf-domain. Let $x \in R$ be a nonzero nonunit. Then $x = ap_1 \cdots p_n$ with each $p_i \in S$ prime and no prime $p \in S$ divides a . In R_S , a has only a finite number of nonassociate irreducible divisors, a_1, \dots, a_m . We may assume that each $a_i \in R$ and hence by Corollary 1.4(c), each a_i is irreducible in R . Let $y \in R$ be an irreducible divisor of x in R . If $y \in S$, then y is an associate of some p_1, \dots, p_n . If $y \notin S$, then y is irreducible in R_S by Lemma 1.1, and hence $y = a_i(s/t)$ for some $s, t \in S$ and $1 \leq i \leq n$. Since neither y nor a_i is in S , $s/t \in U(R)$. Thus $a_1, \dots, a_m, p_1, \dots, p_n$ are the nonassociate irreducible divisors of x in R . Hence R is an idf-domain. Thus if R_S is a FFD, then so is R since a FFD is an atomic idf-domain. As mentioned earlier, the UFD case was proved by Gilmer and Parker in [10, Theorem 3.2]. ■

Note that by Corollary 1.7, we could just as well assume in Theorem 3.1 that R is atomic and S is generated by primes. (For the UFD case, this has been observed by Heinzer, see [11; p. 325].) However, Examples 1.8 show that Theorem 3.1 does not hold for either an arbitrary splitting multiplicative set or a multiplicative set generated by primes. Moreover, note that if S is a multiplicative set of R generated by primes and R_S satisfies any of the factorization properties in Theorem 3.1 except the idf-property, then R satisfies that property if and only if S is a splitting multiplicative set. We next give another case where R_S being a UFD implies that R is a UFD. (This result also follows directly from Theorem 3.1, Proposition 1.6, and Corollary 1.7, or it may be proved using primary decomposition.) Recall that a nonzero fractional ideal I of R is a t -ideal if $J_v \subset I$ for each nonzero finitely generated fractional ideal $J \subset I$.

PROPOSITION 3.2. *Let R be an integral domain and S a multiplicative set of R generated by primes such that R_S is a UFD. Then R is a UFD if and only if principal ideals of R_S intersect to principal ideals of R (i.e., S is a splitting multiplicative set).*

Proof. Suppose that R is a UFD. Let I be a nonzero principal ideal of R_S . Thus I is a t -ideal of R_S , and so $J = I \cap R$ is also a t -ideal of R . Hence J is principal since R is a UFD and each t -ideal in a UFD is principal. Conversely, suppose that principal ideals intersect to principal ideals. We show that each nonzero prime ideal P of R contains a nonzero principal prime ideal. If P intersects S , this is clear. Otherwise, P_S is a nonzero prime ideal of the UFD R_S and so contains a nonzero principal prime ideal Q . Then $Q \cap R$ is a nonzero principal prime ideal contained in P . Hence R is a UFD by [13, Theorem 5]. ■

Our next theorem is the Nagata-type theorem analogue for HFDs.

THEOREM 3.3. *Let R be an integral domain and S a splitting multiplicative set of R such that R_S is a HFD. Then R is a HFD if and only if each element of S is a product of irreducibles and whenever $s_1 \cdots s_m = t_1 \cdots t_n$ for irreducible $s_i, t_j \in S$, then $m = n$. In particular, if S is a splitting multiplicative set of R generated by primes, then R is a HFD if and only if R_S is a HFD.*

Proof. If R is a HFD, then certainly each element of S is a product of irreducibles and any two such products of irreducibles in S have the same length. Conversely, suppose that each element of S is a product of irreducibles and any two such products have the same length. We first show that R is atomic. Let $x \in R$. Then $x = as$ with $a \in R$, $s \in S$, and $aR \cap tR = atR$ for all $t \in S$. Since R_S is atomic, $a = a_1 \cdots a_n$ with each

$a_i \in R_S$ irreducible. By Corollary 1.4, we may assume that each $a_i \in R$ and is irreducible in R . Since s is a product of irreducibles, x is thus a product of irreducibles. Hence R is atomic. Suppose that $x = a_1 \cdots a_n = b_1 \cdots b_m$ with each $a_i, b_j \in R$ irreducible. Thus each factor $c = a_i, b_j$ either is in S or $cR \cap t'R = ct'R$ for all $t' \in S$. Suppose that $a_k \in S$ for $i+1 \leq k \leq n$ and $b_k \in S$ for $j+1 \leq k \leq m$. Let $a = a_1 \cdots a_i, b = b_1 \cdots b_j, s = a_{i+1} \cdots a_n,$ and $t = b_{j+1} \cdots b_m$. Then each $a_1, \dots, a_i, b_1, \dots, b_j$ is irreducible, and s and t are units in R_S ; so $i = j$ since R_S is a HFD. By Corollary 1.4(a) and (b), s and t are associates in R , so $n - i = m - j$. Thus $n = m$ and R is a HFD. The "in particular" statement follows from Corollary 2.5 and the fact that any two prime factorizations of a given element have the same number of prime factors. ■

Several other Nagata-type theorems are given in [8, 10, 11, 14, 16]. For example, the GCD and Mori (ACC on integral divisorial ideals) properties are investigated. Also, Mott and Schexnayder [14] relate several of these concepts to groups of divisibility.

4. CLOSURE PROPERTIES

In this section, we consider a few other properties and their relationship to localization. We show that Nagata-type theorems also hold for integrally closed, n -root closed, root closed, and seminormal integral domains. We recall that an integral domain R with quotient field K is *n -root closed* for a positive integer n if $x \in R$ whenever $x^n \in R$ for some $x \in K$; R is *root closed* if it is n -root closed for all $n \geq 1$; and R is *seminormal* if $x \in R$ whenever $x^2, x^3 \in R$ for some $x \in K$. It is well known that each of these properties is preserved by localization. We show that the converses also hold when the multiplicative set S is generated by primes. Note that we do not need to assume that S is a splitting multiplicative set for these properties. The case for complete integral closure is more subtle. Recall that R' and \bar{R} denote respectively the integral closure and complete integral closure of R .

PROPOSITION 4.1. *Let R be an integral domain and S a multiplicative set of R generated by primes. Then*

- (a) *R is seminormal if and only if R_S is seminormal.*
- (b) *R is n -root closed if and only if R_S is n -root closed.*
- (c) *R is root closed if and only if R_S is root closed.*

Proof. We prove (b); the proof of (a) is similar, and (c) follows from (b). It is well known that any localization of an n -root closed domain is

again n -root closed. Conversely, suppose that R_S is n -root closed and let $x^n \in R$ for $x \in K$. Then $x \in R_S$ since R_S is n -root closed. Hence $x = r/s$, where $r \in R$ and $s \in S$. Then $s^n \mid r^n$ in R forces $s \mid r$ in R since s is a product of primes. Thus $x \in R$, so R is n -root closed. ■

In [9, p. 555, Exercise 11], it is stated that if R satisfies ACCP and S is generated by primes, then R is integrally closed whenever R_S is integrally closed. Our next result shows that the ACCP hypothesis is not needed.

PROPOSITION 4.2. *Let R be an integral domain and S a multiplicative set of R generated by primes. Then $R = R_S \cap R'$. In particular, R is integrally closed if and only if R_S is integrally closed.*

Proof. The “ \subset ” inclusion is clear. Conversely, let $x = r/s \in R'$, where $r \in R$, $s \in S$, and $(r, s) = 1$. The standard proof (cf. [13, Theorem 50]) shows that $s \mid r^n$ in R and hence $s \in U(R)$. Thus $x \in R$ and we also have the “ \supset ” inclusion. The “in particular” statement is now clear since the intersection of two integrally closed domains is integrally closed and any localization of an integrally closed domain is integrally closed. ■

It is well known that the localization of a completely integrally closed domain need not be completely integrally closed (cf. [9, Sect. 13; 2, Remarks after Example 7.7]). In fact, [9, p. 148, Exercise 21] and [16, page 264, (Second) Example] show that this need not hold even if S is generated by primes. However, Roitman [16, Proposition 5.2] does show that if R is a completely integrally closed domain which satisfies ACCP and S is a multiplicative set of R generated by primes, then R_S is completely integrally closed. Our next proposition is a slight generalization of his result (in particular, his ACCP hypothesis may be weakened to R being atomic by Corollary 1.7).

PROPOSITION 4.3. *Let R be an integral domain and S a multiplicative set of R generated by primes. If R is completely integrally closed and $\bigcap p_n R = 0$ for all sequences $\{p_n\}$ of nonassociate primes in S , then R_S is completely integrally closed. In particular, if R is completely integrally closed and S is a splitting multiplicative set generated by primes, then R_S is completely integrally closed.*

Proof. Since R is completely integrally closed, we have $\bigcap p^n R = 0$ for each prime $p \in R$. Thus S is a splitting multiplicative set by Proposition 1.6. The proof now proceeds as in [16, Proposition 5.2]. The “in particular” statement is clear. ■

Examples 1.8 show that in general, R_S completely integrally closed does not imply that R is completely integrally closed even when S is generated

by primes. In [9, p. 555, Exercise 11], it is stated that if R satisfies ACCP and S is generated by primes, then R is completely integrally closed whenever R_S is completely integrally closed (this also holds if R is only assumed to be atomic). We next give a slight refinement of this result which is the complete integral closure analogue of Proposition 4.2.

PROPOSITION 4.4. *Let R be an integral domain and S a multiplicative set of R generated by primes. Then $R = R_S \cap \bar{R}$ if and only if $\bigcap p^n R = 0$ for each prime $p \in S$.*

Proof. If $0 \neq d \in \bigcap p^n R$ for some prime $p \in S$, then $1/p \in R_S \cap \bar{R} = R$, a contradiction. Conversely, suppose that $\bigcap p^n R = 0$ for each prime $p \in S$. Let $x = r/s \in R_S \cap \bar{R}$, where $r \in R$, $s \in S$, and $(r, s) = 1$. Then for some nonzero $d \in R$, $dx^n \in R$ for all $n \geq 1$. Thus for each n , $dx^n = s^n r_n$ for some $r_n \in R$. Since d is nonzero and each $\bigcap p^n R = 0$, $s \in U(R)$. Hence $x \in R$, so $R = R_S \cap \bar{R}$. ■

COROLLARY 4.5. *Let R be an integral domain and S a multiplicative set of R generated by primes. If R_S is completely integrally closed and $\bigcap p^n R = 0$ for each prime $p \in S$, then R is completely integrally closed. In particular, this holds if S is a splitting multiplicative set generated by primes.*

We note that Proposition 4.3 and Corollary 4.5 show that for a splitting multiplicative set S generated by primes, R is completely integrally closed if and only if R_S is completely integrally closed. In particular, this holds when R is atomic.

5. DIRECTED UNIONS

Let k be a field and $R_n = k[X^{1/n!}]$ for each integer $n \geq 1$. Then each R_n is a UFD, but the monoid domain $R = \bigcup R_n = k[X; \mathbb{Q}^+]$ is not a UFD; in fact, R is not even atomic. Also, any integrally closed domain is a directed union of integrally closed Noetherian domains (cf. [3].) Hence none of our factorization properties is preserved by directed unions and thus not much can be said about general directed unions. However, if we assume that each $R_\alpha \subset R_\beta$ is an inert extension (and hence each $R_\alpha \subset R = \bigcup R_\gamma$ is an inert extension), then we get the desired results. We will need the following lemma (cf. Example 5.4).

LEMMA 5.1. *Let $\{R_\gamma\}$ be a directed family of atomic integral domains such that each $R_\alpha \subset R_\beta$ is an inert extension and let $R = \bigcup R_\gamma$. If $x \in R$ is irreducible, then x is irreducible in some R_γ .*

Proof. Let $x \in R_\alpha$. Since R_α is atomic, $x = x_1 \cdots x_n$, where each $x_i \in R_\alpha$ is irreducible. Since x is irreducible in R , all but one of the x_i 's, say x_n , is

a unit in R . Thus all the x_i 's, except for x_n , are units in some R_γ with $\alpha \leq \gamma$. By Lemma 1.1, x_n is irreducible in R_γ . Hence x is irreducible in R_γ . ■

Our main result in this section is then

THEOREM 5.2. *Let $\{R_\gamma\}$ be a directed family of integral domains such that each $R_\alpha \subset R_\beta$ is an inert extension. Then $R = \bigcup R_\gamma$ is atomic (resp., satisfies ACCP, a BFD, a HFD, a FFD, or a UFD) if each R_γ is atomic (resp., satisfies ACCP, a BFD, a HFD, a FFD, or a UFD).*

Proof. First, suppose that each R_γ is atomic. Let $x \in R$; then $x \in R_\alpha$ for some α . In R_α , $x = x_1 \cdots x_n$ as a product of irreducibles. By Lemma 1.1, each x_i is either irreducible or a unit in R . Thus x is a product of irreducibles in R , so R is atomic. Next, suppose that $aR \subset bR$. Let $a \in R_\alpha$. Then $a = br$ for some $r \in R$, so $bu, ru^{-1} \in R_\alpha$ for some $u \in U(R)$. Let $b' = bu$. Then $bR = b'R$ and $aR_\alpha \subset b'R_\alpha$. Hence for each strictly increasing chain of principal ideals of length n in R starting at aR , we can construct a chain of principal ideals of length n in R_α starting at aR_α . Thus R satisfies ACCP (resp., is a BFD) if each R_α satisfies ACCP (resp., is a BFD). For the case in which each R_γ is a HFD, suppose that $x_1 \cdots x_n = y_1 \cdots y_m$ with each x_i, y_j irreducible in R . Then R is atomic and by Lemma 5.1, each x_i, y_j is irreducible in some R_α . Hence $m = n$ since R_α is a HFD. Thus R is a HFD. Next, suppose that each R_γ is a FFD. Let $x \in R$; then $x \in R_\alpha$ for some α . In R_α , let x_1, \dots, x_n be the nonassociate divisors of x . Suppose that $y \mid x$ for some $y \in R$. Then $ry = x$ for some $r \in R$. Hence $r, y \in R_\beta$ with $R_\alpha \subset R_\beta$. Thus $x = (ru)(yu^{-1})$ for some $u \in U(R_\beta)$ with $ru, yu^{-1} \in R_\alpha$. Thus $yu^{-1} = vx_i$ for some $v \in U(R_\alpha)$ and $1 \leq i \leq n$, so $y = (uv)x_i$ with $uv \in U(R_\beta) \subset U(R)$. Hence any divisor of x in R is an associate of some x_1, \dots, x_n . Thus R is a FFD. The proof for the UFD case is similar to that for the HFD case and will thus be omitted. ■

The case for UFDs have been observed by Cohn [6, p. 7] and the case for atomic domains by Zaks [19]. Cohn also notes that $R[\{X_\alpha\}]$ is a UFD for any family of indeterminates $\{X_\alpha\}$ when R is a UFD since $R[\{X_\alpha\}]$ is the directed union of $\{R[Y] \mid Y \subset \{X_\alpha\} \text{ finite}\}$ and $R[Y] \subset R[Z]$ is an inert extension if $Y \subset Z$. This observation together with Theorem 5.2 shows that any of our factorization properties which is preserved by adjoining a single indeterminate is also preserved by adjoining any family of indeterminates.

We next give an example to show that the directed union of a family $\{R_\gamma\}$ of idf-domains with each $R_\alpha \subset R_\beta$ an inert extension need not be an idf-domain.

EXAMPLE 5.3. Let V be a valuation domain with quotient field F such that F is the countable union of an increasing family of valuation overrings

$\{V_n\}$ of V . Let K be a proper field extension of F (thus K^*/F^* is infinite by Brandis' theorem [4]) and X an indeterminate. Then each $R_n = V_n + XK[[X]]$ is an idf-domain [1, Proposition 4.3]. However, $R = \bigcup R_n = F + XK[[X]]$ is not an idf-domain since K^*/F^* is infinite [1, Proposition 4.2(a)]. It is easily verified that $R_m \subset R_n$ is an inert extension whenever $m \leq n$.

Our final example shows that we may have $R = \bigcup R_\gamma$ a UFD and each $R_\alpha \subset R_\beta$ an inert extension, but no R_γ satisfies any of our factorization properties. It also shows that in Lemma 5.1 it is necessary to assume that each R_γ is atomic.

EXAMPLE 5.4. Let q_n be the product of the first n positive primes. Then $R_n = \mathbb{Z}[1/q_n] + X\mathbb{Q}[[X]]$ is a Bézout domain [9, p. 286, Exercise 13] which is neither atomic [1, Proposition 1.2] nor an idf-domain [1, Proposition 4.3]. However, $R = \bigcup R_n = \mathbb{Q}[[X]]$ is a UFD and $R_m \subset R_n$ is an inert extension whenever $m \leq n$. Note that X is irreducible in R , but X is not irreducible in any R_n .

Note added in proof. In the paper "Overrings of half-factorial domains," S. Chapman, W. W. Smith, and the second author give an example of a Dedekind HFD with a localization which is not a HFD.

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