FACTORIZATION IN INTEGRAL DOMAINS

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In this paper, we study factorization in an integral domain R, that is, factoring elements of R into products of irreducible elements. We investigate several factorization properties in R which are weaker than unique factorization.
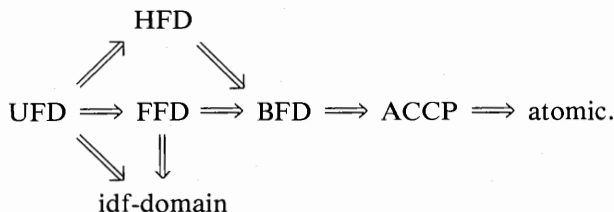
Introduction

Let R be an integral domain with quotient field K. In this paper, we study factorization in R, that is, factoring elements of R into products of irreducible elements. The classical situation is when this factorization exists and is unique up to order and associates, that is, when R is a unique factorization domain (UFD). This case has been studied extensively, and there are many excellent accounts of the theory ([14, 17, 28—30], for example). In this paper, we investigate various related factorization properties weaker than unique factorization. Our goal is to give a careful study of these properties and to give many examples, each as elementary as possible.

We first define the various factorization properties which we will study here. Following Cohn [13], we say that R is *atomic* if each nonzero nonunit of R is a product of a finite number of irreducible elements (atoms) of R. We say that R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal integral ideals of R. The domain R is a *bounded factorization domain* (BFD) if R is atomic and for each nonzero nonunit of R there is a bound on the length of factorizations into products of irreducible elements. We say that R is a *half-factorial domain* (HFD) is R is atomic and each

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factorization of a nonzero nonunit of R into a product of irreducible elements has the same length. This concept was introduced by Zaks in [33]. The domain R is an *idf-domain* (for irreducible-divisor-finite) if each nonzero element of R has at most a finite number of nonassociate irreducible divisors. They were introduced by Grams and Warner in [23]. We will be mainly interested in atomic idf-domains; they are precisely the domains in which each nonzero nonunit has only a finite number of nonassociate divisors (and hence, only a finite number of factorizations up to order and associates). We will call such a domain a *finite factorization domain* (FFD). In general,

$$
\begin{array}{ccc}
& \text{HFD} & \\
\nearrow & & \searrow \\
\text{UFD} \implies \text{FFD} \implies & \text{BFD} \implies \text{ACCP} \implies & \text{atomic.} \\
\searrow & \Downarrow & \\
& \text{idf-domain} &
\end{array}
$$

Examples will be given to show that no other implications are possible.

In the first section, we investigate atomic domains and the ACCP property. We also study several other factorization properties related to atomic domains and consider the question of when the polynomial ring $R[X]$ is atomic. Section 2 studies BFD's, while we consider HFD's in the third section. In the fourth section, we discuss idf-domains. The fifth section studies FFD's. In the final section, we investigate when these various factorization properties are preserved by ascent or descent for an extension $R \subset T$ of integral domains with $U(T) \cap R = U(R)$.

General references for any undefined terminology or notation are [6, 17, 18 or 26]. For an integral domain R, R^* is its set of nonzero elements, $U(R)$ its group of units, and R' its integral closure. The set of positive elements of a partially ordered abelian group G will be denoted by G^+. The set of nonzero principal integral ideals of R will be denoted by $\mathrm{Prin}(R)$; $\mathrm{Prin}(R)$ is a partially ordered monoid under inclusion. Throughout, ideal will always mean integral ideal. These factorization properties may also be interpreted as properties of $G(R)$, the group of divisibility of R. Here, $G(R)$ is the abelian group $K^*/U(R)$, written additively, and partially ordered by $aU(R) \le bU(R)$ if and only if $a \mid b$ (i.e., $ba^{-1} \in R$). (Thus $aR \to aU(R)$ is an order-reversing isomorphism from $\mathrm{Prin}(R)$ to $G(R)^+$.) For example, R is a UFD if and only if $G(R)$ is order isomorphic to a direct sum of copies of $\mathbb{Z}$ with the usual product order. Several examples involve monoid domain constructions. Given an integral domain R and torsionless grading monoid S, let $R[X; S] = \{\sum a_s X^s \mid a_s \in R$ and $s \in S\}$ with $X^s X^t = X^{s+t}$. An excellent reference for monoid domains is [19]. Throughout, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the integers, rational numbers, real numbers, and complex numbers, respectively.

1. Atomic domains and ACCP

An integral domain R is atomic if each nonzero nonunit of R is a product of irreducible elements (atoms) of R. It is well known that any UFD or Noetherian domain is atomic. At the other extreme, a domain need not have any irreducible elements at all. For example, a valuation domain whose maximal ideal is not principal has no irreducible elements. Another example of an integral domain with no irreducible elements is the monoid domain $\mathbb{C}[X; \mathbb{Q}^+]$. Other examples may be constructed from these via the $D+M$ construction (cf. the proof of Proposition 1.2). The easiest and usual way to show that a domain is atomic is to show that it satisfies some chain condition on ideals; the most common one is ACCP. Hence any Krull domain, and more generally, any Mori domain (ACC on integral divisorial ideals) is atomic. Somewhat surprisingly (cf. [13, Proposition 1.1]), the converse is not true; an atomic domain need not satisfy ACCP, but examples are hard to come by. The first such example is due to Grams [21]. For completeness and future reference, we include Grams' example. In [35], Zaks has also given several examples of atomic domains which do not satisfy ACCP.

Example 1.1 (Grams [21]). Let F be a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/3, 1/(2 \cdot 5), \dots, 1/(2^j p_j), \dots\}$, where $p_0 = 3$, $p_1 = 5, \dots$ is the sequence of odd primes. Let R be the monoid domain $F[X; T]$ and $N = \{f \in R \mid f \text{ has nonzero constant term}\}$. Then $A = F[X; T]_N$ is an atomic domain which does not satisfy ACCP. Note that A is one-dimensional [19, Theorems 21.4 and 17.1] and quasilocal since $N = R - M$, where $M = \{f \in R \mid f \text{ has zero constant term}\}$ is a maximal ideal of R.

We next determine when the $D+M$ construction (cf. [8]) yields atomic domains or domains which satisfy ACCP. Proposition 1.2 may be used to construct more examples of domains which are atomic but do not satisfy ACCP.

Proposition 1.2. *Let T be an integral domain of the form $K+M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D+M$. Then:*
   (a) *R is atomic if and only if T is atomic and D is a field,*
   (b) *R satisfies ACCP if and only if T satisfies ACCP and D is a field.*

Proof. First suppose that D is not a field. Then $m = d(m/d)$ for each $m \in M$ and $d \in D^*$. Thus no element of M is irreducible. Hence if R is either atomic or satisfies ACCP, D must be a field. So let D ($=k$) be a field.
   (a) Up to multiplication by a $\alpha \in K^*$ (resp., $\alpha \in k^*$), each element of T (resp., R) has the form m or $1+m$ for some $m \in M$. Each of these elements is irreducible in R if and only if it is irreducible in T (cf. [16, Lemma 1.5; 27]). If x is a product of irreducibles, we may assume that each irreducible factor has the form m or $1+m$ for some $m \in M$. Thus x is a product of irreducible elements in R if and only if it

is a product of irreducible elements in T. Hence R is atomic if and only if T is atomic.

(b) We first observe that a principal ideal of R or T may be generated by either m or $1+m$ for some $m \in M$. Let $m, n \in M$. It is easily verified that $(1+m)R \subset (1+n)R \Leftrightarrow (1+m)T \subset (1+n)T, mR \subset (1+n)R \Leftrightarrow mT \subset (1+n)T$, and $mR \subset nR \Rightarrow mT \subset nT$. Also, if $mT \subset nT$, then $mR \subset (\alpha n)R$ for some $\alpha \in K^*$. Hence, to each chain of principal ideals of length s in R starting at mR (resp., $(1+m)R$), there corresponds a chain of principal ideals of length s in T starting at mT (resp., $(1+m)T$), and conversely. Thus R satisfies ACCP if and only T satisfies ACCP. $\square$

The $D+M$ construction has been studied extensively since it has proven to be an excellent technique for constructing counterexamples. The classical situations are when either T is a valuation domain, for example, $T = K[\![X]\!]$, or $T = K[X]$, where $M = XT$ [15]. The general construction has been studied systematically in [8, 4, 16]. Another important case is when $T = K[X; S] = K + M$ is a monoid domain with $U(S) = \{0\}$ and maximal ideal $M = \{f \in T \mid f$ has zero constant term$\}$. We note that for any field K and abelian group G, there is a quasilocal Krull domain of the form $T = K + M$ with divisor class group $Cl(T) = G$ [4, Remark 3.8(3)]. Often, $T = K + M$ will be picked to be a UFD of Krull domain and $D = k$ a subfield of K. In this case, $R = k + M$ always satisfies ACCP, but R is not a Krull domain unless $k = K$. Suitable choices for K and D give R various properties. For example, R is Noetherian if and only if T is Noetherian, D is a field, and $[K : D]$ is finite [8, Theorem 4]. Specifically, let $T = \mathbb{R}[X] = \mathbb{R} + X\mathbb{R}[X]$. Then $R = \mathbb{Q} + X\mathbb{R}[X]$ is non-Noetherian, but satisfies ACCP, and hence is atomic. Also, if T is integrally closed, then $R = D + M$ is integrally closed if and only if D is integrally closed in K. Note that Grams' domain A (Example 1.1) has the form $F + M_M$; hence, for any subfield k of F, $k + M_M$ is an atomic domain which does not satisfy ACCP.

The difference between an integral domain R being atomic or satisfying ACCP is best seen in terms of $Prin(R)$ or $G(R)$. The domain R satisfies ACCP if and only if *each* chain in $Prin(R)$ is finite; while R is atomic precisely when for each $x \in R^*$, *some* maximal chain starting at (x) is finite (this follows since there are no principal ideals between $(y) \subset (z)$ if and only if y/z is irreducible). Irreducible elements of R correspond to minimal positive elements of $G(R)$. Thus R is atomic if and only if each positive element of $G(R)$ is a finite sum of minimal positive elements. Also, R satisfies ACCP if and only if there does not exist an infinite strictly decreasing sequence of positive elements in $G(R)$.

It is well known that an integral domain R satisfies ACCP if and only if $\bigcap (a_n) = \{0\}$ for each strictly descending chain $(a_1) \supset (a_2) \supset \cdots$ of principal ideals of R. (To see this, observe that a strictly ascending chain $(b_1) \subset (b_2) \subset \cdots$ of nonzero principal ideals of R yields a strictly descending chain $(a_1) \supset (a_2) \supset \cdots$ of principal ideals of R with $\bigcap (a_n) \neq \{0\}$, where each $a_n = b_1 b_n^{-1}$. The converse is similar.) Hence, if R satisfies ACCP, then $\bigcap (a^n) = \{0\}$ for each nonunit $a \in R$. However, the converse is false since any completely integrally closed (c.i.c.) or one-dimensional

domain satisfies $\bigcap (a^n) = \{0\}$ for each nonunit a, but need not satisfy ACCP (for example, a one-dimensional nondiscrete valuation domain or the ring of entire functions [18, Exercise 7, p. 285]).

The localization of an atomic domain or a domain satisfying ACCP need not satisfy that property. In [21, Example 1], Grams constructs an almost Dedekind domain $\bar{D}$ which satisfies ACCP. She observes that $\bar{D}[X]$ also satisfies ACCP, but its localization $\bar{D}(X)$, which is a Bézout domain, does not satisfy ACCP (in fact, is not even atomic) since an atomic Bézout domain is a PID. Probably the simplest example of an integral domain satisfying ACCP but with a localization which does not satisfy ACCP is the monoid domain $R = k[X; T]$, where k is a field and $T = \{q \in \mathbb{Q} \mid q \geq 1\} \cup \{0\}$ an additive submonoid of $\mathbb{Q}^+$ (cf. [19, p. 189]). For $S = \{X^t \mid t \in T\}$, $R_S = k[X; \mathbb{Q}]$ does not satisfy ACCP (see Example 2.7(a) for more details). Another example of a domain satisfying ACCP but with a localization that does not satisfy ACCP is given in [24, Example, p. 275]. Several more examples are given in Example 2.7 and in later sections.

It is well known that R satisfies ACCP if and only if $R[\{X_\alpha\}]$ satisfies ACCP for any family of indeterminates $\{X_\alpha\}$ (cf. [19, Theorem 14.6]). A similar result holds for $R[\![X]\!]$. More generally, the group ring $R[X; G]$ satisfies ACCP if and only if R satisfies ACCP and each nonzero element of the torsionfree abelian group G is of type $(0, 0, \ldots)$ [19, Theorem 14.17]. The problem of determining when the monoid domain $R[X; S]$ is an atomic domain or satisfies ACCP is still open. If $R[X]$ is atomic, then certainly R is atomic. We have been unable to determine if the converse is true. One problem is that in trying to look for a counterexample there are very few known examples of atomic domains which do not satisfy ACCP. (It may be verified that $A[X]$ is atomic for Grams' domain A given in Example 1.1.) So we ask the following:

Question 1. Is $R[X]$ atomic when R is atomic?

We next give some additional factorization conditions that R satisfies when $R[X]$ is atomic. If $R[X]$ is atomic, then for each $a, b \in R^*$, we can write $aX + b = a_1 \cdots a_s(cX + d)$, where each factor is irreducible. Thus $a = a_1 \cdots a_s c$ and $b = a_1 \cdots a_s d$, where c and d have no (nonunit) common factors, i.e., $\gcd(c, d) = 1$. This observation motivates our next definition. We say that an integral domain R is *strongly atomic* if for each $a, b \in R^*$, we can write $a = a_1 \cdots a_s c$ and $b = a_1 \cdots a_s d$, where $a_1, \ldots, a_s \in R$ ($s \geq 0$) are irreducible and $c, d \in R$ satisfy $\gcd(c, d) = 1$. The domain R is a *weak GCD-domain* if for each $a, b \in R^*$, there are $c, a', b' \in R$ so that $a = ca'$ and $b = cb'$, where $\gcd(a', b') = 1$. Finally, R is a *LT-domain* (lowest terms domain) if for each $a, b \in R^*$, there are $c, d \in R^*$ with $a/b = c/d$ and $\gcd(c, d) = 1$. We then have the following theorem:

Theorem 1.3. *Let R be an integral domain. Then*:
(a) *if R satisfies ACCP or $R[X]$ is atomic, then R is strongly atomic*,

(b) *if R is weak GCD-domain, then R is a LT-domain,*

(c) *R is strongly atomic if and only if R is an atomic weak GCD-domain.*

Proof. (a) Clearly R satisfies ACCP $\Rightarrow R[X]$ satisfies ACCP $\Rightarrow R[X]$ is atomic $\Rightarrow R$ is strongly atomic. The proof of part (b) is straightforward and hence will be omitted.

(c) Suppose that R is strongly atomic. Let $a \in R^*$ be a nonunit. Write $a = a_1 \cdots a_s c$ and $b = a_1 \cdots a_s d$, where each a_i is irreducible and $\gcd(c, d) = 1$. Then $a_1^2 \cdots a_s^2 c^2 = a^2 = a_1 \cdots a_s d$, so $d = a_1 \cdots a_s c^2$. But $\gcd(c, d) = 1$, so c is a unit. Hence a is a product of irreducible elements. Thus R is atomic. The remainder of the proof is straightforward. $\square$

Along these lines, we remark that if given $a_1, \ldots, a_n \in R^*$, there are $c_1, \ldots, c_n \in R$ with no common factors and irreducible $b_1, \ldots, b_m \in R$ such that $a_i = b_1 \cdots b_m c_i$ for each $1 \leq i \leq n$, then $T = R[\{X_\alpha\}]$ is atomic for any family $\{X_\alpha\}$ of indeterminates. Given $f \in T$, we can first factor f into a product of polynomials f_j whose only factors of less degree are constants. By hypothesis, we can then factor each f_j into a product of irreducibles. We record this observation as the following theorem:

Theorem 1.4. *The following statements are equivalent for an integral domain R.*
   (1) *For each $n \geq 2$ and $a_1, \ldots, a_n \in R^*$, there are $c_1, \ldots, c_n \in R$ with no common factors and irreducible $b_1, \ldots, b_m \in R$ such that $a_i = b_1 \cdots b_m c_i$ for each $1 \leq i \leq n$.*
   (2) *$R[\{X_\alpha\}]$ is atomic for any family $\{X_\alpha\}$ of indeterminates.*
   (3) *$R[X, Y]$ is atomic for indeterminates X and Y.*

Proof. We have already observed that (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (1), we first observe that for any field F and any $b, a_0, \ldots, a_n \in F^*$, $a_n X^n + \cdots + a_1 X + a_0 + bY \in F[X, Y]$ is irreducible. Hence, given $a_1, \ldots, a_n \in R^*$, $a_n X^{n-2} + \cdots + a_2 + a_1 Y = b_1 \cdots b_m (c_n X^{n-2} + \cdots + c_2 + c_1 Y)$, where $b_1, \ldots, b_m \in R$ are irreducible and $c_1, \ldots, c_n \in R$ have no common factors, since $R[X, Y]$ is atomic. Thus (1) holds. $\square$

Also of interest are those atomic domains with only a finite number of non-associate irreducible elements. They were first studied by Cohen and Kaplansky in [12] and have been studied by the first author in [1]. The following characterization is a special case of [1, Theorem 2].

Theorem 1.5. *The following statements are equivalent for an integral domain R.*
   (1) *R is an atomic domain with a finite number of nonassociate irreducible elements.*
   (2) *The semigroup of integral ideals of R is finitely generated.*
   (3) *R is a one-dimensional semilocal Noetherian domain such that for each non-principal maximal ideal M of R, R_M is analytically irreducible and R/M is finite.* $\square$

If an atomic domain R has only finitely many nonassociate irreducibles, then of course $G(R)$ is finitely generated. Somewhat surprisingly, the converse is false. From [20], it follows that $G(R)$ is finitely generated for R a Noetherian domain if and only if R is one-dimensional and semilocal with R_M analytically unramified and R/M finite for each nonprincipal maximal ideal M of R. In addition to semilocal PID's, typical examples of atomic domains with only a finite number of nonassociate irreducible elements include $F_1 + XF_2[\![X]\!]$, where $F_1 \subset F_2$ are finite fields, and $F[\![X^n, X^{n+1}, \dots]\!]$ for F a finite field. For a further investigation of these domains by Mott and the first author, see [3].

2. BFD's

In this section, we study bounded factorization domains. We recall that an atomic domain R is a bounded factorization domain (BFD) if for each nonzero nonunit x of R, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$ as a product of irreducible elements of R, then $n \le N(x)$. Clearly a UFD is a BFD, and a BFD satisfies ACCP. We first give an example of an integral domain which satisfies ACCP, but is not a BFD.

Example 2.1. Let k be a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/2, 1/3, 1/5, \dots, 1/p_j, \dots\}$, where p_j is the jth prime. Then the monoid domain $R = k[X; T]$ is a one-dimensional domain which satisfies ACCP, but is not a BFD. To verify this, we first note that each nonzero $a \in T$ may be written uniquely as $a = n_0 + n_1/p_1 + \cdots + n_j/p_j$, where $n_i \in \mathbb{Z}^+$, each $0 \le n_i \le p_i - 1$ $(i > 0)$, and $n_j \ne 0$ (cf. [21, Lemma 1.1]). Thus each X^{1/p_j} is irreducible, and hence R is not a BFD since $X = (X^{1/p_j})^{p_j}$ for each prime p_j. However, R does satisfy ACCP. For a $0 \ne f = b_1 X^{a_1} + \cdots + b_n X^{a_n} \in R$ with $a_1 < \cdots < a_n$ and $b_n \ne 0$, write $\beta(f) = a_n$. If ACCP fails, then there is a strictly increasing chain $(f_1) \subset (f_2) \subset \cdots$ of principal ideals in R. Then each $f_n = f_{n+1} g_{n+1}$ for some nonunit $g_{n+1} \in R$. Hence each $\beta(f_n) = \beta(f_{n+1}) + \beta(g_{n+1})$, and each term is positive. Then in T, we have $\beta(f_1) > \beta(f_2) > \cdots$ with each $\beta(f_n) - \beta(f_{n+1}) \in T$, but this is impossible by the above-mentioned unique representation of each nonzero $a \in T$.

On the positive side, Noetherian domains and Krull domains are BFD's. Several other proofs of these two facts will be given in later sections.

Proposition 2.2. *A Noetherian domain or a Krull domain is a BFD.*

Proof. Suppose that R is either a Noetherian domain or a Krull domain. Let x be a nonzero nonunit of R. Let $P_1, \dots, P_n$ be the height-one prime ideals of R that contain x. If $y \mid x$ for some nonunit $y \in R$, then y is an element of some P_i since then $(x) \subset (y)$ and any height-one prime ideal of y is also a height-one prime ideal

of x. Hence, if $x = x_1 \cdots x_m$ with $m \geq kn$ and each x_j is a nonunit of R, then $x \in P_i^k$ for some $1 \leq i \leq n$. If there is no bound on the length of factorizations of x, then $x \in \bigcap P_i^k = \{0\}$ for some $1 \leq i \leq n$, a contradiction. $\square$

Another class of BFD's may be obtained as follows. Let R be a quasilocal domain with maximal ideal M. If $\bigcap M^n = \{0\}$, then R is a BFD. For if $x = x_1 \cdots x_n$, with each x_i a nonunit of R, then $x \in M^n$. However, the converse is false; a quasilocal BFD may have $\bigcap M^n \neq \{0\}$. One may obtain such an example, which is actually a Krull domain, by localizing an example (due to Eakin) in [25, Example 5.7]. More generally, we have the following proposition:

Proposition 2.3. *Let R be an integral domain such that $\bigcap M^n = \{0\}$ for each maximal ideal M of R and $\bigcap M_n = \{0\}$ for any countably infinite set $\{M_n\}$ of maximal ideals of R. Then R is a BFD.* $\square$

We next give another characterization of BFD's. For any atomic integral domain R, we define the 'length function' $l_R : R^* \to \mathbb{Z}^+ \cup \{\infty\}$ by $l_R(x) = 0$ if $x \in U(R)$ and $l_R(x) = \sup\{n \mid x = x_1 \cdots x_n$ with each $x_i \in R$ irreducible$\}$ for x a nonunit of R. Then $l_R(xy) \geq l_R(x) + l_R(y)$ for all $x, y \in R^*$ and R is a BFD if and only if $l_R(x) < \infty$ for all $x \in R^*$. (Note that in general we may have $l_R(xy) > l_R(x) + l_R(y)$ for particular $x, y \in R^*$. Consider $R = k[X^2, X^3]$, where k is a field; then $l_R(X^2) = l_R(X^3) = 1$, so $3 = l_R(X^6) > l_R(X^3) + l_R(X^3) = 2$.) We next collect some other characterizations of BFD's.

Theorem 2.4. *The following statements are equivalent for an integral domain R.*
   (1) R is a BFD.
   (2) For each nonzero nonunit $x \in R$, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$, with each x_i a nonunit of R, then $n \leq N(x)$.
   (3) There is a function $l : R^* \to \mathbb{Z}^+$ such that $l(x) = 0$ if and only if $x \in U(R)$, and $l(xy) \geq l(x) + l(y)$ for all $x, y \in R^*$.

Proof. $(1) \Rightarrow (2)$ is clear. $(2) \Rightarrow (1)$. We need only show that R is atomic. If R is not atomic, then there are arbitrarily long factorizations in R, a contradiction. $(1) \Rightarrow (3)$. Let $l = l_R$ as defined above. $(3) \Rightarrow (2)$. Let $N(x) = l(x)$. Then $x = x_1 \cdots x_n$, with each x_i a nonunit of R, implies that $n \leq l(x_1) + \cdots + l(x_n) \leq l(x_1 \cdots x_n) = l(x)$. $\square$

This concept is easily interpreted in $\mathrm{Prin}(R)$ and $G(R)$. An integral domain R is a BFD if and only if for each $x \in R^*$ there is a bound on the lengths of chains in $\mathrm{Prin}(R)$ starting at (x). In $G(R)$, each positive element x must be the sum of at most a fixed number (depending on x) of (minimal) positive elements.

We next use the length function characterization for BFD's to show that the BFD property is preserved by adjoining an indeterminate.

Proposition 2.5. *The following statements are equivalent for an integral domain R.*

(1) *R is a BFD.*

(2) *$R[X]$ is a BFD.*

(3) *$R[\![X]\!]$ is a BFD.*

Proof. Clearly (1) is implied by either (2) or (3). Conversely, suppose that R is a BFD. Then $l_1 : R[X]^* \to \mathbb{Z}^+$ defined by $l_1(a_0 + \cdots + a_n X^n) = l_R(a_n) + n$ and $l_2 : R[\![X]\!]^* \to \mathbb{Z}^+$ defined by $l_2(a_n X^n + \cdots) = l_R(a_n) + n$ each satisfy the conditions of Theorem 2.4(3). Hence $R[X]$ and $R[\![X]\!]$ are each BFD's. $\quad\square$

Let T be a BFD with subring R. Then $l = l_T \mid_{R^*}$ defines a function $R^* \to \mathbb{Z}^+$. Now l satisfies the conditions in Theorem 2.4(3) if and only if $U(T) \cap R = U(R)$. Thus R is a BFD if T is a BFD and $U(T) \cap R = U(R)$. Note, though, that even in this case we need not have $l = l_R$. For example, let k be a field, $T = k[X]$, and $R = k[X^2, X^3]$. Then $l_R(X^2) = l_R(X^3) = 1$, while $l(X^2) = 2$ and $l(X^3) = 3$. Thus a domain R may have many 'length functions' which satisfy the conditions of Theorem 2.4(3). These observations together with Proposition 2.5 yield the following proposition:

Proposition 2.6. *Let R be a BFD and $\{X_\alpha\}$ any family of indeterminates. Then any subring T of $R[\{X_\alpha\}]$ which contains R is a BFD.*

Proof. By Proposition 2.5, any polynomial ring in finitely many indeterminates over a BFD is again a BFD. Since each polynomial involves only a finite number of indeterminates, $A = R[\{X_\alpha\}]$ is also a BFD. Thus T is a BFD since $U(T) = U(A)$ $(= U(R))$. $\quad\square$

We next construct three very different BFD's, each with a localization or integral closure which is not a BFD.

Example 2.7. (a) Let k be a field, $T = \{q \in \mathbb{Q} \mid q \geq 1\} \cup \{0\}$ an additive submonoid of $\mathbb{Q}^+$, and $R = k[X; T]$ the monoid domain. Then R is a one-dimensional BFD since each nonunit factor has degree at least one. However, its integral closure $R' = k[X; \mathbb{Q}^+]$ [19, Corollary 12.11] is not a BFD; in fact, R' is not even atomic since X has no irreducible factors. Also, $R_S = k[X; \mathbb{Q}]$, where $S = \{X^t \mid t \in T\}$, is not atomic since R_S is a GCD-domain [19, Theorem 14.2], but R_S does not satisfy ACCP [19, Theorem 14.17]. (We could also use $T = \{r \in \mathbb{R} \mid r \geq 1\} \cup \{0\}$.)

(b) Let R be an integral domain with quotient field K. In [2, Corollary 7.6], we showed that the domain $I(K, R) = \{f \in K[X] \mid f(R) \subset R\}$ of R-valued K-polynomials satisfies ACCP (resp., is a BFD) if and only if R satisfies ACCP (resp., is a BFD). Thus $R = I(\mathbb{Q}, \mathbb{Z})$, the ring of integer-valued polynomials, is a two-dimensional Prüfer BFD, but some localization of R is not a BFD (since the localization R_M at a height-two maximal ideal M is a two-dimensional valuation domain, which is not even atomic). It is also interesting to note that while R is c.i.c. (cf. [2, Theorem 7.2]), some localization R_M is not c.i.c.

(3) Let $\bar{\bar{\mathbb{Z}}}$ be the ring of all algebraic integers. Then $R = \mathbb{Z} + X\bar{\bar{\mathbb{Z}}}[X]$ is a two-dimensional BFD by [2, Example 5.1] and the remarks before [2, Theorem 7.5]. However, $R' = \bar{\bar{\mathbb{Z}}}[X]$ is not a BFD since the Bézout domain $\bar{\bar{\mathbb{Z}}}$ is not a BFD. (We could also let $R = \mathbb{Z} + X\bar{\bar{\mathbb{Z}}}[\![X]\!]$.)

Our next result about the $D + M$ construction is the analogue of Proposition 1.2. As an application, we see that for any pair of fields $K_1 \subset K_2$, $K_1 + XK_2[X]$ and $K_1 + XK_2[\![X]\!]$ are each BFD's. Note that they are not Noetherian if $[K_2 : K_1]$ is infinite.

Proposition 2.8. *Let T be an integral domain of the form $K + M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. Then R is a BFD if and only if T is a BFD and D is a field.*

Proof. First suppose that R is a BFD. Then D must be a field as in Proposition 1.2. The proof of Proposition 1.2 shows that R is a BFD if and only if T is a BFD. $\quad\square$

3. HFD's

We recall that R is a half-factorial domain (HFD) if R is atomic and for each nonunit $x \in R^*$, if $x = x_1 \cdots x_n = y_1 \cdots y_m$, with each x_i, y_j irreducible in R, then $m = n$. Clearly a UFD is a HFD and a HFD is also a BFD, and hence satisfies ACCP. Also, any Krull domain R with $\text{Cl}(R) = \mathbb{Z}/2\mathbb{Z}$ is a HFD [34, Theorem 1.4]. However, the Krull domain $R = k[X^3, XY, Y^3]$, where k is a field, is not a HFD since XY, X^3, and Y^3 are each irreducible in R and $(XY)^3 = X^3Y^3$ (note that $\text{Cl}(R) = \mathbb{Z}/3\mathbb{Z}$). Elementary examples of non-Krull HFD's are $\mathbb{Z}[\sqrt{-3}]$ [34, p.285] and $\mathbb{R} + X\mathbb{C}[X]$ (see below). Two of the simplest examples of integral domains which are not HFD's are $k[X^2, X^3]$, where k is a field, since X^2 and X^3 are each irreducible and $(X^2)^3 = (X^3)^2$, and $\mathbb{Z}[2\sqrt{2}]$ since 2 and $2\sqrt{2}$ are each irreducible and $(2\sqrt{2})^2 = 2^3$. In terms of $\text{Prin}(R)$, R is a HFD if and only if for each $x \in R^*$ there is a maximal chain in $\text{Prin}(R)$ starting at (x) and any two such maximal chains have the same length. In $G(R)$, this means that any positive element is a sum of minimal positive elements and any two such sums have the same number of summands.

HFD's were introduced by Zaks in [33], who gave a detailed study of Krull HFD's in terms of their divisor class groups in [34]. Most of the work on HFD's has been for Dedekind or Krull domains, with major emphasis on factorization in algebraic number rings. In fact, much of this work is motivated by the result of Carlitz [9] that an algebraic number ring is a HFD if and only if it has class number less than or equal to two. A related concept is that of a *congruence half-factorial domain* (CHFD): an atomic integral domain in which any two factorizations of an element into a product of irreducibles have the same length modulo r (for some fixed $r > 1$). These domains have recently been introduced and studied by Chapman and Smith

in [10] and [11]. Clearly a HFD is a CHFD, but there are CHFD's which are not HFD's. We will not consider CHFD's, but we note that [11] has an excellent bibliography on HFD's.

HFD's may also be characterized in terms of a length function analogous to that for a BFD as in Section 2. By [34, Lemma 1.3], an integral domain R is a HFD if and only if there is a 'length function' $l : R^* \to \mathbb{Z}^+$ with $l(xy) = l(x) + l(y)$ for all $x, y \in R^*$, im $l = \mathbb{Z}^+$, and $l(x) = 1$ if and only if x is irreducible. We note that R is a HFD if and only if $l_R(xy) = l_R(x) + l_R(y)$ for all $x, y \in R^*$.

In [2, Theorem 5.3], we showed that for fields $K_1 \subset K_2$, $R = K_1 + XK_2[X]$ is always a HFD. Similarly, $R = K_1 + XK_2[\![X]\!]$ is always a HFD. More generally, the subring $R = K_1 + M$ is a HFD whenever $T = K_2 + M$ is a UFD. These are special cases of our next result on when the $D + M$ construction yields a HFD.

Proposition 3.1. *Let T be an integral domain of the form $K + M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. Then R is a HFD if and only if D is a field and T is a HFD.*

Proof. As in Proposition 1.2, D is necessarily a field. The proof of Proposition 1.2 shows that a factorization into irreducibles in R has the same length as such a factorization in T. Hence R is a HFD if and only if T is a HFD. $\square$

As a generalization, let $\{K_n\}$ be an increasing sequence of subfields of a field K, and let $R = \sum K_n X^n \subset K[X]$. Then R is always a BFD, and hence satisfies ACCP; but R is not a HFD unless $K_n = K_{n+1}$ for all $n \geq 1$, i.e., $R = K_0 + XK_1[X]$. For suppose that $a \in K_n - K_{n-1}$ $(n \geq 2)$, then aX^n and $a^{-1}X^n$ are each irreducible in R, but $(aX^n)(a^{-1}X^n) = X^{2n}$ are factorizations of length 2 and $2n$ (≥ 4), respectively. (Similarly, we could let $R = \prod K_n X^n \subset K[\![X]\!]$.)

In general, HFD's do not behave very well under extensions. For example, while R is a HFD whenever $R[X]$ is a HFD; the converse need not hold. In [34, Theorem 2.4], it is shown that for a Krull domain R, $R[X]$ is a HFD if and only if either R is a UFD or $Cl(R) = \mathbb{Z}/2\mathbb{Z}$. In [2, Example 5.4], we showed that $R = \mathbb{R} + X\mathbb{C}[X]$ is a HFD, but $R[Y]$ is not a HFD since $(X(1 + iY))(X(1 - iY)) = X^2(1 + Y^2)$ are factorizations into irreducibles of different lengths.

4. idf-domains

In this section, we discuss idf-domains. We recall that R is an idf-domain (or R has the idf-property) if each nonzero element of R has at most a finite number of nonassociate irreducible divisors. For example, any UFD is an idf-domain. At the other extreme, a domain with no irreducible elements is vacuously an idf-domain. Some more examples of idf-domains will be given in the next section. Since an idf-domain need not be atomic, the idf-property does not imply any other of our fac-

torization properties. Note that the domain R is an idf-domain if and only if each nonzero principal ideal of R is contained in at most a finite number of ideals which are maximal with respect to being principal. In terms of the group of divisibility of R, $G(R)$, this may be recast as each positive element of $G(R)$ is greater than at most a finite number of minimal positive elements.

These domains were introduced in [23] by Grams and Warner in relationship to a problem in Bourbaki [5, p.87, Exercise 25] (cf. also [22]). Among other things, they show that if $R = \bigcap V_\alpha$ is a domain of finite character and each valuation domain V_α, except possibly one of them, is rank one discrete, then R is an idf-domain [23, Proposition 1]. In particular, a Krull domain is an idf-domain. We pause to give two nontrivial examples of integral domains which are *not* idf-domains.

Example 4.1. (a) Let $R = \mathbb{R} + X\mathbb{C}[X]$. Then R is Noetherian, but R is not an idf-domain. In fact, $\{(r+\mathrm{i})X \mid r \in \mathbb{R}\}$ is an infinite family of nonassociate irreducible divisors of X^2. We have already observed that R is a HFD, so a HFD or a BFD need not be an idf-domain.

(b) Let k be a field and $T = \{q \in \mathbb{Q} \mid q \geq 1\} \cup \{0\}$ an additive submonoid of $\mathbb{Q}^+$. Then $R = k[X; T]$ is not an idf-domain since X^{1+q} are nonassociate irreducible divisors of X^3 for each $q \in \mathbb{Q}$ with $0 \leq q < 1$. In Example 2.7(a), we showed that R is actually a BFD. (Again, we could use $T = \{r \in \mathbb{R} \mid r \geq 1\} \cup \{0\}$.)

We have been mainly interested in atomic domains in this paper. By the above examples, an atomic domain (in fact, a Noetherian domain) need not be an idf-domain. We next show how to construct examples of idf-domains using the $D + M$ construction. Example 4.1(a) above is a special case of part (a) of our next proposition.

Proposition 4.2. *Let T be an integral domain of the form $K + M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let k be a subfield of K and $R = k + M$. Then*:

(a) *Suppose that M contains an irreducible element. Then R is an idf-domain if and only if T is an idf-domain and the multiplicative group K^*/k^* is finite.*

(b) *Suppose that M contains no irreducible elements. Then R is an idf-domain if and only if T is an idf-domain. In particular, if T is quasilocal, then both T and R are idf-domains.*

Proof. (a) We first note that an element of M is irreducible in R if and only if it is irreducible in T. Let $m \in M$ be irreducible. First suppose that R is an idf-domain. Then $\alpha m \mid m^2$ for all $\alpha \in K^*$. Note that αm and βm are irreducible in both R and T, and that they are associates in R if and only if α and β lie in the same coset in K^*/k^*. Hence K^*/k^* is finite. Let $x \in T$. By multiplying by a suitable $\alpha \in K^*$, we may assume that $x \in R$. Let $x_1, \ldots, x_n$ be the distinct nonassociate irreducible divisors of x

in R. It is easily verified that any irreducible divisor of x in T is associated to one of the x_i's. Thus T is also an idf-domain. Conversely, suppose that T is an idf-domain and that K^*/k^* is finite. Let $x \in R$. Let $x_1, \ldots, x_r$ be a complete set of non-associate irreducible divisors of x in T, which we may assume are all in R, and let $\alpha_1, \ldots, \alpha_s$ be a set of coset representatives of K^*/k^*. Then any irreducible divisor of x in R is an associate of some $\alpha_i x_j$. Hence R is an idf-domain.

(b) Since M has no irreducible elements, an irreducible element in T (resp., in R) has the form $\alpha + m$ for some $\alpha \in K^*$ (resp., $\alpha \in k^*$) and $m \in M$. Hence, up to associates, each has the form $1 + m$ for some $m \in M$. It is then easily verified that $\{1 + m_1, \ldots, 1 + m_n\}$ is a complete set of nonassociate irreducible divisors of a given element with respect to R if and only if it is a complete set of nonassociate irreducible divisors with respect to T. For the 'in particular' statement, note that in this case each $1 + m$ is a unit, so neither R nor T has any irreducible elements. $\square$

We can make several interesting observations from this proposition. Unlike our earlier results (Propositions 1.2, 2.8, and 3.1), T may be an idf-domain while R is not an idf-domain. The answer also may depend upon the maximal ideal M. Note that in part (a), the multiplicative group K^*/k^* is finite (even finitely generated) if and only if either $K = k$ or K is finite (Brandis' Theorem [7]). Thus for suitable choices of K and k, as for example in Example 4.1(a), R may be Noetherian, atomic, or a HFD, and yet not be an idf-domain.

Also, unlike our earlier results, R may be an idf-domain when D is not a field (for example, when D is semilocal PID or D has no irreducible elements), and this is independent of T. However, such an R is never atomic. We state this as the following proposition:

Proposition 4.3. *Let T be a quasilocal integral domain of the form $K + M$, where M is the nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. If D is not a field, then R is an idf-domain if and only if D has only a finite number of nonassociate irreducible elements.*

Proof. Let d be a nonzero nonunit of D. Then $m = d(m/d)$ shows that no element of M is irreducible and d divides each element of M. Also, $x = d + m = d(1 + m/d)$ and $1 + m/d \in U(R)$ (since T is quasilocal) shows that x is irreducible in R if and only if d is irreducible in D. Thus R is an idf-domain if and only if D has only a finite number of nonassociate irreducible elements. $\square$

We next consider ascent and descent of the idf-property. In general, the idf-property is not very stable. Propositions 4.2 and 4.3 may be combined to show that a localization of an idf-domain need not be an idf-domain. Let $T = \mathbb{R}[\![X]\!] = \mathbb{R} + M$, where $M = X\mathbb{R}[\![X]\!]$. Then $R = \mathbb{Z}_{(2)} + M$ is an idf-domain by Proposition 4.3. For $S = \mathbb{Z} - \{0\}$, $R_S = \mathbb{Q} + M$ is not an idf-domain by Proposition 4.2 since $\mathbb{R}^*/\mathbb{Q}^*$ is infinite. If $R[X]$ is an idf-domain, then clearly R is also an idf-domain. It would be interesting to know if the converse is also true. We ask the following:

Question 2. Is $R[X]$ an idf-domain when R is an idf-domain?

One case in which the answer is positive is when R is a valuation domain. Then $R[X] = R(X) \cap K[X]$ ($= R(X) \cap (\bigcap K[X]_{(f)})$), so $R[X]$ has finite character since $R(X)$ is a valuation domain, and hence $R[X]$ is an idf-domain [23, Proposition 1]. This observation also answers the question raised in [23, p.275], and subsequently answered by Gilmer and Warner [23, added in proof], for an example of a GCD-domain R of finite-character that is not a Bézout domain or a UFD; just let $R = V[X]$, where V is a nondiscrete valuation domain. (Another example of a GCD-domain of finite-character which is not a Bézout domain or a UFD is given in [31].) In the next section, we show that Question 2 also has a positive answer when R is an atomic idf-domain.

5. FFD's

We recall that R is a finite factorization domain (FFD) if each nonzero nonunit of R has only a finite number of nonassociate divisors and hence, only a finite number of factorizations up to order and associates. Thus a FFD is both a BFD and an idf-domain. (In Theorem 5.1, we show that FFD's are precisely the atomic idf-domains.) For example, any UFD is a FFD. Other examples include the atomic integral domains mentioned in Section 1 which have only a finite number of non-associate irreducible elements. A less trivial example of a FFD is any subring R of $k[\{X_\alpha\}]$, where k is either a finite field or $\mathbb{Z}$, and $\{X_\alpha\}$ is any family of indeterminates. (In fact, each element of R has only a finite number of divisors.) At the other extreme, a domain with no irreducible elements is vacuously an idf-domain, but not a FFD. Also, a FFD need not be an HFD; consider $R = k[X^2, X^3]$, where k is a field. Conversely, the HFD $R = \mathbb{R} + X\mathbb{C}[X]$ is not a FFD since it is not an idf-domain (Example 4.1(a)). Note that an integral domain R is a FFD if and only if each nonzero principal ideal of R is contained in only a finite number of principal ideals. In $G(R)$, this translates as each positive element has only a finite number of positive summands.

Our next theorem gives the expected result that an atomic idf-domain is actually a FFD. Since a Krull domain is both atomic and an idf-domain, this theorem also gives another proof of the fact that a Krull domain is a BFD (Proposition 2.2). (Note, however, that a Noetherian domain need not be a FFD.)

Theorem 5.1. *Let R be an integral domain. Then R is a FFD if and only if R is an atomic idf-domain.*

Proof. Clearly a FFD is an atomic idf-domain. Conversely, suppose that R is an atomic idf-domain. Let $x \in R$ be a nonzero nonunit. Suppose that $x_1, \dots, x_n$ are the nonassociate irreducible factors of x. Suppose that in a factorization of x, $x =$

$ux_1^{s_1} \cdots x_n^{s_n}$, we always have $0 \le s_i \le N_i$ for each $1 \le i \le n$. Then there is a bound on the number of nonassociate factors of x. So suppose that this is not the case. Then some s_i, say s_1, is not bounded. Thus we can write for each $k \ge 1$, $x = u_k x_1^{s_{k1}} \cdots x_n^{s_{kn}}$, where $u_k \in U(R)$ and $s_{11} < s_{21} < s_{31} < \cdots$. Suppose that in this set of factorizations, $\{s_{ki}\}$ is bounded for each i with $1 < i \le n$. Then since there are only finitely many choices for $s_{k2}, \ldots, s_{kn}$, we must have $s_{k2} = s_{j2}, \ldots, s_{kn} = s_{jn}$ for some $j > k$. But then $u_j x_1^{s_{j1}} x_2^{s_{j2}} \cdots x_n^{s_{jn}} = x = u_k x_1^{s_{k1}} x_2^{s_{k2}} \cdots x_n^{s_{kn}}$; so cancelling yields $u_j x_1^{s_{j1}} = u_k x_1^{s_{k1}}$, where $s_{j1} > s_{k1}$, a contradiction. Hence, some set $\{s_{ki}\}$ for a fixed i with $1 < i \le n$ is unbounded, say for $i = 2$. Then, by taking subsequences at each stage, we may assume that $s_{11} < s_{21} < s_{31} < \cdots$ and $s_{12} < s_{22} < s_{32} < \cdots$. Continuing in this manner, we may assume for each $1 \le i \le n$ that $s_{1i} < s_{2i} < s_{3i} < \cdots$. But then $u_1 x_1^{s_{11}} \cdots x_n^{s_{1n}} = x = u_2 x_1^{s_{21}} \cdots x_n^{s_{2n}}$, where each $s_{1i} < s_{2i}$, a contradiction. $\square$

Again, the $D + M$ construction yields more examples of FFD's. For example, our next proposition applies when either $T = K[X]$ or T is a quasilocal Krull domain of the form $K + M$, where K is a finite field.

Proposition 5.2. *Let T be an integral domain of the form $K + M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and $R = D + M$. Then R is a FFD if and only if T is a FFD, D is a field, and K^*/D^* is finite.*

Proof. This follows directly from Proposition 2.8, Proposition 4.2(a), and Theorem 5.1. $\square$

Let $F_1 \subset F_2$ be finite fields; then $R = F_1 + XF_2[X]$ (or $F_1 + XF_2[\![X]\!]$) is both a FFD and a HFD, but R is a UFD if and only if $F_1 = F_2$. Our next result also holds for any family $\{X_\alpha\}$ of indeterminates.

Proposition 5.3. *Let R be an integral domain with quotient field K. Then $R[X]$ is an FFD if and only if R is an FFD.*

Proof. Clearly R is an FFD whenever $R[X]$ is a FFD. Conversely, suppose that R is a FFD. Let $0 \ne f \in R[X]$. We show that f has only finitely many nonassociate factors. We may assume that f is nonconstant since R is a FFD. Suppose that f has an infinite number of nonassociate factors. Since in $K[X]$, f has only a finite number of nonassociate factors, there is an infinite set of nonassociate factors $\{f_n\}$ of f in $R[X]$ with $f_1 K[X] = f_n K[X]$ for each $n \ge 1$. Let $f = f_n g_n$. Since the leading coefficient of f is the product of the leading coefficients of f_n and g_n, this leads to a factorization of the leading coefficient of f. Hence an infinite number of the f_n's have associate leading coefficients, and thus we may assume that all the f_n's have the same leading coefficient. But if f_1 and f_n have the same leading coefficient and $f_1 K[X] = f_n K[X]$, then $f_1 = f_n$, a contradiction. $\square$

An example of Grams [21, Example 2] may be used to show that a localization of a FFD need not be an FFD. She constructs a Prüfer domain R which satisfies ACCP and each of its localizations but one is a DVR, with the other one being a rank one nondiscrete valuation domain. Hence R has finite character and thus is an idf-domain. By Theorem 5.1, R is also a FFD. But that one localization is not even atomic (but it is an idf-domain), and hence is not a FFD. (This gives another example which shows that the classes of atomic domains, domains satisfying ACCP, and BFD's are not closed under localization.) A much more elementary example is the following:

Example 5.4. Let k be a field and $T = \{n + i/n! \mid 0 \le i \le n! - 1, n = 0, 1, \dots\}$ an additive submonoid of $\mathbb{Q}^+$. Then the monoid domain $R = k[X; T]$ is a one-dimensional domain which is a FFD, but not a HFD. R is a BFD since each nonconstant $f \in R$ has $\deg f \ge 1$. Also, R is an idf-domain, and hence a FFD, since any factorization of an $f \in R$ takes place in some polynomial ring $k[X^{1/n!}]$. However, R is not a HFD since $X^5 = X^{5/2} X^{5/2}$, and X and $X^{5/2}$ are each irreducible. Let $S = \{X^t \mid t \in T\}$. Then $R_S = k[X; \mathbb{Q}]$ is not even atomic. Also, $R' = k[X; \mathbb{Q}^+]$ is not atomic.

6. Overrings and subrings

In this section, we determine which of the factorization properties are preserved by ascent or descent for certain extensions of integral domains. We have already investigated how these properties behave with respect to polynomial extensions and localizations. Of course, in general, not much can be said. Here, we will be particularly interested in the case in which $R \subset T$ is an extension of integral domains with either $U(T) \cap R = U(R)$ or $U(T) \cap K = U(R)$, where K is the quotient field of R. In [21, Proposition 2.1 and Corollary], it was observed that R satisfies ACCP whenever T satisfies ACCP and $U(T) \cap R = U(R)$; in particular, R satisfies ACCP whenever R' satisfies ACCP. Similarly (as we have already observed in Section 2), if $U(T) \cap R = U(R)$, then R is a BFD whenever T is a BFD. In particular, this is the case when T is integral over R. This gives the amusing generalization of Proposition 2.2 that any integral overring T ($\subset K$) of a Noetherian domain R is a BFD (since R' is a Krull domain) or that a subring R of a Noetherian domain or Krull domain T is a BFD whenever T is integral over R.

Any extension $R \subset T$ of integral domains induces an order-preserving monoid homomorphism $\varphi : \text{Prin}(R) \to \text{Prin}(T)$ by $\varphi(xR) = xT$ for each $x \in R^*$. It may be easily verified that φ is injective if and only if $U(T) \cap K = U(R)$ and that φ is injective on chains if and only if $U(T) \cap R = U(R)$. (Similar results also hold for $\varphi' : G(R) \to G(T)$ defined by $\varphi'(xU(R)) = xU(T)$ for each $x \in K^*$.) Clearly $U(T) \cap R = U(R)$ whenever $U(T) \cap K = U(R)$. However, the converse may fail; let $R = \mathbb{R} + X\mathbb{C}[X]$ and $T = \mathbb{C}[X]$. Some cases in which $U(T) \cap R = U(R)$ are: (1) $R \subset T$ is integral (or more generally, satisfies LO); (2) R and T are quasilocal with maximal ideals M and N,

respectively, such that $N \cap R = M$; (3) $R = F_1[X; S] \subset T = F_2[X; T]$ are monoid domains with $F_1 \subset F_2$ fields and $S \subset T$ monoids with $U(T) \cap S = U(S)$; and (4) $R = T \cap L$ for L a field containing R.

We first consider an extension $R \subset T$ of integral domains which satisfies $U(T) \cap R = U(R)$. We have already observed that R satisfies ACCP (resp., is a BFD) if T satisfies ACCP (resp., is a BFD). However, T may be a UFD, FFD, or idf-domain while R is not; let $T = \mathbb{C}[X]$ and $R = \mathbb{R} + X\mathbb{C}[X]$ (Example 4.1(a)). To show that the HFD property is not preserved, let $T = k[X]$ and $R = k[X^2, X^3]$ for any field k. Note that in the last two examples $R \subset T$ is integral, in fact, $T = R'$. Finally, in contrast to the ACCP case, we show that the atomic property is not preserved.

Example 6.1. Let $T = F + M_M$ be Gram's example as in Example 1.1 (note the change in notation from Example 1.1). Let $R = F + N_N$, where $N = \{f \mid f$ has non-zero constant term$\}$ is a maximal ideal of $F[X; W]$, where W is the additive sub-monoid of $\mathbb{Q}^+$ generated by $\{1/2, 1/4, \ldots, 1/2^j, \ldots\}$. Then $U(T) \cap R = U(R)$ since T and R are quasilocal and $M_M \cap R = N_N$. Then T is atomic while R is not atomic since X has no irreducible factors. (For a pair of domains with the same quotient field, let $B = T[Y]$ and $A = R + YT[Y]$. Then $U(B) \cap A = U(A)$ and B is atomic while A is not atomic.)

It is interesting to observe that with the stronger hypothesis that $U(T) \cap K = U(R)$, R is a FFD when T is a FFD. This follows easily from the characterization of FFD's in terms of $\text{Prin}(R)$ and $\text{Prin}(T)$ since φ is injective.

Conversely, we next investigate which factorization properties ascend from R to T when $U(T) \cap R = U(R)$. Let k be a field, $S = \{q \in \mathbb{Q} \mid q \geq 1\} \cup \{0\}$, $R = k[X; S]$ the monoid domain as in Example 2.7(a) and $T = R' = k[X; \mathbb{Q}^+]$. Then R is a BFD and $U(T) \cap R = U(R)$, but T is not even atomic. (This answers a question raised in [21, p.325] as to whether R' satisfies ACCP when R satisfies ACCP. Also, note that the integral closure A' of Grams' example A (Example 1.1) is not atomic.) Thus the atomic, ACCP, and BFD properties need not ascend from R to T. Example 5.4 shows that the above mentioned properties plus the idf-property and FFD properties do not ascend from R to R'. Also, the UFD property does not ascend. Let $R = k[X^2, Y^2]$ and $T = k[X^2, XY, Y^2]$ for any field k. It is interesting to observe that in each case the success or failure of the ascent or descent of a given factorization property is exactly what one would intuitively expect from its characterization in terms of $\text{Prin}(R)$, $\text{Prin}(T)$, and the injectivity of φ.

We end the paper by considering these properties for locally-finite intersections. It is easily verified and well known that a locally-finite intersection of domains each satisfying ACCP again satisfies ACCP [32, Corollary 4]. A similar argument shows that this also holds for BFD's. (One can also see this as follows: Let $\{R_\alpha\}$ be a family of BFD's each with length function 1_α. If $R = \bigcap R_\alpha$ is a locally-finite intersection, then $1 = \sum 1_\alpha$ is a well-defined finite-valued length function for R. Hence R is a BFD.) This observation gives another proof of Proposition 2.2 since a Krull

domain is by definition a locally-finite intersection of DVR's. Thus a locally-finite intersection of HFD's or UFD's need not be a HFD or a UFD. Several examples are given in [23] to show that a locally-finite intersection of idf-domains need not be an idf-domain.

References

[1] D.D. Anderson, Some finiteness conditions on a commutative ring, Houston J. Math. 4 (1978) 289–299.

[2] D.D. Anderson, D.F. Anderson and M. Zafrullah, Rings between $D[X]$ and $K[X]$, Houston J. Math., to appear.

[3] D.D. Anderson and J. Mott, Cohen–Kaplansky domains: integral domains with a finite number of irreducible elements, J. Algebra, to appear.

[4] D.F. Anderson and A. Ryckaert, The class group of $D+M$, J. Pure Appl. Algebra 52 (1988) 199–212.

[5] N. Bourbaki, Éléments de Mathématique, Fasc. XXXI. Algèbre Commutative, Chap. VII: Diviseurs, Actualités Scientifiques et Industrielles 1314 (Hermann, Paris, 1965).

[6] N. Bourbaki, Commutative Algebra (Addison-Wesley, Reading, MA, 1972).

[7] A. Brandis, Über die multiplikative struktur von Körpererweiterungen, Math. Z. 87 (1965) 71–73.

[8] J. Brewer and E.A. Rutter, $D+M$ constructions with general overrings, Michigan Math. J. 23 (1976) 33–42.

[9] L. Carlitz, A characterization of algebraic number fields with class number two, Proc. Amer. Math. Soc. 11 (1960) 391–392.

[10] S. Chapman and W.W. Smith, On a characterization of algebraic number fields with class number less than three, J. Algebra 135 (1990) 381–386.

[11] S. Chapman and W.W. Smith, Factorization in Dedekind domains with finite class group, Israel J. Math. 71 (1990) 65–95.

[12] I.S. Cohen and I. Kaplansky, Rings with a finite number of primes, I, Trans. Amer. Math. Soc. 60 (1946) 468–477.

[13] P.M. Cohn, Bézout rings and their subrings, Math. Proc. Cambridge Philos. Soc. 64 (1968) 251–264.

[14] P.M. Cohn, Unique factorization domains, Amer. Math. Monthly 80 (1973) 1–18.

[15] D. Costa, J. Mott and M. Zafrullah, The construction $D + XD_S[X]$, J. Algebra 53 (1978) 423–439.

[16] D. Costa, J. Mott and M. Zafrullah, Overrings and dimensions of general $D+M$ constructions, J. Natur. Sci. Math. 26 (1986) 7–14.

[17] R.M. Fossum, The Divisor Class Group of a Krull Domain (Springer, New York, 1973).

[18] R. Gilmer, Multiplicative Ideal Theory (Dekker, New York, 1972).

[19] R. Gilmer, Commutative Semigroup Rings, Chicago Lectures in Mathematics (Univ. of Chicago Press, Chicago, IL, 1984).

[20] B. Glastad and J. Mott, Finitely generated groups of divisibilty, Contemp. Math. 8 (1982) 231–247.

[21] A. Grams, Atomic domains and the ascending chain condition for principal ideals, Math. Proc. Cambridge Philos. Soc. 75 (1974) 321–329.

[22] A. Grams, The distribution of prime ideals of a Dedekind domain, Bull. Austral. Math. Soc. 11 (1974) 429–441.

[23] A. Grams and H. Warner, Irreducible divisors in domains of finite character, Duke Math. J. 42 (1975) 271–284.

[24] W. Heinzer and D. Lantz, Commutative rings with ACC on n-generated ideals, J. Algebra 80 (1983) 261–278.

[25] E.G. Houston, T.G. Lucas and T.M. Viswanathan, Primary decomposition of divisorial ideals in Mori domains, J. Algebra 117 (1988) 327–342.

[26] I. Kaplansky, Commutative Rings (Univ. of Chicago Press, Chicago, IL, rev. ed., 1974).

[27] J. Mott and M. Zafrullah, Unruly Hilbert domains, Canad. Math. Bull. 33 (1990) 106–109.

[28] P. Samuel, Anneaux Factoriels (Soc. Mat. Sao Paulo, 1963).

[29] P. Samuel, Lectures on unique factorization domains (notes by Pavaman Murthy), Tata Institute for Fundamental Research Lectures 30 (Tata Inst. Fund. Res., Bombay, 1964).

[30] P. Samuel, Unique factorization, Amer. Math. Monthly 75 (1968) 945–952.

[31] M. Zafrullah, Rigid elements in GCD-domains, J. Natur. Sci. Math. 17 (1977) 7–14.

[32] M. Zafrullah, Two characterizations of Mori domains, Math. Japon. 33 (1988) 645–652.

[33] A. Zaks, Half-factorial domains, Bull. Amer. Math. Soc. 82 (1976) 721–724.

[34] A. Zaks, Half-factorial domains, Israel J. Math. 37 (1980) 281–302.

[35] A. Zaks, Atomic rings without a.c.c. on principal ideals, J. Algebra 74 (1982) 223–231.