COMMUNICATIONS IN ALGEBRA, 18(7), 2151-2158 (1990)

FLATNESS AND INVERTIBILITY OF AN IDEAL

Muhammad Zafrullah

Department of Mathematics
Winthrop College
Rock Hill, SC 29733

It is well known that a finitely generated flat ideal of an integral domain is invertible. In this note we record the consequences of the following result.

PROPOSITION 1. Let A be a non-zero fractionary ideal of an integral domain D. Suppose that there exist $x_1, x_2, \ldots, x_n \in A$ such that $A^{-1} = (x_1, x_2, \ldots, x_n)^{-1}$. If A is flat then A is invertible.

PROOF. Consider $AA^{-1} = A(\bigcap_{i=1}^{n} (\frac{1}{x_i}))$. Because A is flat, according to Anderson [2, Theorem 2] we have

$AA^{-1} = \bigcap_{i=1}^{n} \frac{A}{x_i}$. Now because $x_i \in A$ we have $AA^{-1} \supseteq D$. But already we have $AA^{-1} \subseteq D$.

As consequences of the above proposition we indicate that by Proposition 1, in a Krull domain a non-zero flat ideal is invertible and, so is a locally principal ideal. We also indicate

Copyright © 1990 by Marcel Dekker, Inc.

some other results which would require non-trivial proofs in the absence of Proposition 1.

It is well known that an integral domain is a Prufer domain if and only if its non-zero ideals are flat [5, Th. 4.2]. We characterize integral domains with the property that every divisorial ideal is flat. These integral domains turn out to be the generalized GCD(G-GCD) domains of [3]. Using some terminology from [8, Sections 32 and 34] we provide a characterization of G-GCD domains which also includes Prufer domains.

Throughout this note the letter D will denote an integral domain, with quotient field K. A function $*: F(D) \longrightarrow F(D)$ is called a <u>star operation</u> if for all $A,B \in F(D)$ and for all $a \in K - \{0\}$. (i) $(a)^* = (a)$ and $(aA)^* = aA^*$, (ii) $A \subseteq A^*$ and $A \subseteq B$ implies $A^* \subseteq B^*$ and (iii) $(A^*)^* = A^*$.

Let * be a star operation on D. An ideal $A \in F(D)$ is called a *-ideal if $A = A^*$ and A is called a *-ideal of finite type if there is a finitely generated $B \in F(D)$ such that $A = B^*$. Following [14] we call $A \in F(D)$ <u>strictly *-finite</u> if there exists a finitely generated ideal $B \subseteq A$ such that $A^* = B^*$. The function defined on F(D) by $A \longrightarrow (A^{-1})^{-1} = A_v$ is a star operation called the v-operation. It is easy to see that for any star operation *, $A^{-1} = (A^*)^{-1} = (A^{-1})^*$. So if $A^* = B^*$ for any star operation then $A^{-1} = B^{-1}$ and $A_v = B_v$. Indeed the function defined by $A \longrightarrow A$ on F(D) is another star operation called the <u>d-operation</u>. Finally a v-ideal is also called <u>divisorial</u>.

An integral domain which satisfies ACC on integral v-ideals is called a Mori-domain [6]. According to Querre [10, Th.1] an integral domain D is a Mori domain if, and only if, each A ϵ F(D) is strictly v-finite.

Given the above introduction, Proposition 1 proves that a strictly v-finite (or a v-ideal of finite type) which is also flat is invertible. In view of this observation the following results are straightforward.

COROLLARY 2. (c.f.[1] Theorem 2.1). Let A be a non-zero locally principal ideal. If A is strictly v-finite then A is invertible.

PROOF. Locally flat is flat.

COROLLARY 3. (c.f.[9] Theorem 2.4). In a Krull domain every non-zero flat ideal is invertible.

PROOF. A Krull domain is Mori and so every non-zero ideal in it is strictly v-finite.

In fact a statement better than Corollary 3 can be made.

COROLLARY 4. In a Mori domain every non-zero flat ideal is invertible.

COROLLARY 5. (c.f. [10] Corollary 2, p. 342) A Mori domain which is also Prufer is a Dedekind domain.

PROOF. In a Prufer domain every ideal is flat [5, Th. 4.2].

An integral domain D is called a generalized GCD (G-GCD) domain if for every finitely generated A ϵ F(D); A_v is invertible [3].

COROLLARY 6. For an integral domain D the following statements are equivalent.

(1) D is a G-GCD domain.

(2) For every finitely generated $A \in F(D)$ there is some star operation $*$-such that A^* is invertible.

(3) For every finitely generated $A \in F(D)$, there is some star operation $*$ such that A^* is flat.

PROOF. (1) $\Rightarrow$ (2). Trivial. (2) $\Rightarrow$ (3). Obvious because projective is flat.

(3) $\Rightarrow$ (1). Let $A = (x_1, \ldots, x_n)$. Then the flatness of A^* gives

$$A^*A^{-1} = A^*(\cap (\frac{1}{x_i})) = \cap \frac{A^*}{x_i}.$$

Now use the argument used in the proof of Proposition 1, to establish that A^* is invertible. It is easy to see that an invertible ideal is divisorial so $A^* = (A^*)_v = A_v$ is invertible.

The notion of a locally factorial Krull domain is connected with the invertibility of divisorial ideals. Some lists of equivalent conditions characterizing locally factorial Krull domains can be found in [1], and [12]. One of the conditions being: D is Krull and every divisorial ideal is invertible. In the following we add a few more equivalent conditions that characterize locally factorial Krull domains.

COROLLARY 7. The following are equivalent in an integral domain.

1. D is Krull and locally factorial

2. D is Krull and every divisorial ideal of D is flat.

3. D is Mori and every divisorial ideal is flat.

4. D is Mori and every divisorial ideal is invertible.

5. D is Mori and every divisorial ideal is locally principal.

6. D is Mori and G-GCD.

PROOF. (1) $\Leftrightarrow$ (6), follows from [12, Theorem 1.10] and (4) $\Leftrightarrow$ (6) by Corollary 6. (3) $\Rightarrow$ (4) is a consequence of Corollary 4. (4) $\Rightarrow$ (5) because an invertible ideal is locally prinicpal. (5) $\Rightarrow$ (3) because locally free is flat. (2) $\Rightarrow$ (4) because Krull is Mori and in a Krull domain every non-zero flat ideal is invertible. (4) $\Rightarrow$ (2) because by the equivalence of (1) and (3) D is Krull. *flat see attached explanation.

Now let us call D *-Prufer if every *-ideal of D is flat. Then obviously a d-Prufer domain is Prufer. Moreover, in view of Corollary 6, for a general star operation *, a *-Prufer domain is at least a G-GCD domain.

To make the study of these concepts simpler we recall that to each star operations * on D can be associated another star operation $*_s$ defined by $A^{*_s} = \cup F^*$; where F ranges over finitely generated D-submodules of A. Obviously for any $A \in F(D)$ A^{*_s} is a direct limit of its star ideals of finite type. If each of the *-ideals of finite type is flat then A^{*_s} is also flat [11, Th 3.30]. Once we note that for $A \in F(D)$ finitely generated, $A^{*_s} = A^*$ we have the following proposition.

PROPOSITION 8. If D is $*$-Prufer then D is $*_s$-Prufer.

The v_s-operation is normally called the <u>t-operation</u>. This leads to the following result.

PROPOSITION 9. If D is $*$-Prufer then $*_s = t$.

PROOF. If D is $*$-Prufer then for every finitely generated $A \in F(D)$; $A^* = A_v$ (see the proof of (3) $\Rightarrow$ (1) of Corollary 6).

Indeed we do not need any new arguments to establish that a generalized GCD-domain is t-Prufer.

PROPOSITION 10. An integral domain D is a G-GCD domain if, and only if, D is t-Prufer. Moreover a G-GCD domain is a Prufer domain if, and only if, every ideal of D is a t-ideal.

It may be noted that the invertibility criteria discussed in [1], [2] and [4], allow a property weaker than flatness to cause invertibility. An integral domain D is said to have property $*$ if for all $a_1, \ldots, a_m, b_1, \ldots, b_n \in D - \{0\}$; $(\cap a_i)(\cap b_j) = \cap a_i b_j$ [13]. So if, for A finitely generated in a $*$-domain D; A^{-1} is of finite type then A_v is flat and hence invertible. An interested reader may consult [13] for some interesting characterizations of G-GCD domains. Indeed Proposition 3.9 of [13] may be considered as a fore-runner of Proposition 1 of this note, and for this the author thanks the anonymous benefactor who refereed [13].

REMARK 11. Recall that if D is a G-GCD domain and X an indeterminate over D then D[X] is G-GCD. This is remarkable in that if D were Prufer, D[X] ceases to be Prufer yet it keeps on functioning as t-Prufer.

ACKNOWLEDGEMENT

Professor David E. Dobbs was exposed to Proposition 1 and some of its consequences. I gratefully acknowledge the fact that he not only appreciated the result but also encouraged me to write up the material for publication.

REFERENCES

[1] D.D. Anderson, Globalization of some local properties in Krull domains, Proc. Amer. Math. Soc. 85(2)(1982), 141-145.

[2] _______________, On the ideal equation $I(B \cap C) = IB \cap IC$, Canad Math. Bull. 26(1983), 331-332.

[3] D.D. Anderson and D.F. Anderson, Generalized GCD domains, Comment. Math. Univ. St. Pauli, 23(1979), 213-221.

[4] D.D. Anderson and D.E. Dobbs, Flatness, LCM-stability and related module theoretic properties, J. Algebra 112(1)(1988), 139-150.

[5] S.U. Chase, Direct products of modules, Trans. Amer. Math. Soc. 97(1960), 457-473.

[6] N. Dessagnes (Raillard), Sur les anneaux de Mori, Thesis, Paris VI(1976).

[7] D.E. Dobbs, On the criteria of D.D. Anderson for invertible and flat ideals, Canad. Math. Bull. 29(1)(1986), 25-32.

[8] R.W. Gilmer, Multiplicative Ideal Theory, Dekker, New York(1972).

[9] S. Glaz and W. Vasconcelos, Flat ideals II, Manuscripta Math. 22(1977), 325-341.

[10] J. Querre, Sur une propriete des anneaux de Krull, Bull. Sc. Math. 2^c serie 95(1971), 341-354.

[11] J.J. Rotman, Notes on Homological Algebra, van Nostrand-Reinhold 1970.

[12] M. Zafrullah, Generalized Dedekind domains, Mathematika, 33(1986), 285-295.

[13] ______________, On a property of pre-Schreier domains, Comm. Algebra, 15(9)(1987), 1895-1920.

[14] ______________, Ascending chain conditions and star operations, Comm. Algebra 17(6)(1989), 1523-1533.

Received: February 1989
Revised: August 1989

Change $*$-Prufer mentioned in this paper to $*$-Flat

I recommend calling a domain whose nonzero $*$-ideals are flat, a $*$-flat domain and not a $*$-Prufer domain. The main reasons are: (1) There is literature, that appeared later than this paper, on $*$-Prufer monoids (for finite type star operations) see e.g. Chapter 17 of [Halter-Koch, Ideal Systems, Dekker, 1998], (2) Recently, following in Halter-Koch's footsteps, Anderson, Anderson, Fontana and myself have have written a paper "On v-domains and star operations" that calls $*$-Prufer, for a star operation $*$, an integral domain in which every nonzero finitely generated ideal is $*$-invertible. The above mentioned paper is to appear in Comm. Algebra.

Muhammad Zafrullah