THE HALF-FACTORIAL PROPERTY AND DOMAINS OF THE FORM A+XB[X]

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ABSTRACT. In this note, we use the $A+XB[X]$ and $A+XI[X]$ constructions from a new angle to construct new examples of half factorial domains. Positive results are obtained highlighting the interplay between the notions of integrally closed domain and half-factorial domain in $A + XB[X]$ constructions. It is additionally shown that constructions of the form $A + XI[X]$ rarely possess the half-factorial property.

1. INTRODUCTION

Let R be an integral domain and suppose that a nonzero nonunit of R is expressible as a product of a finite number of irreducible elements (or atoms) of R. Such a domain is called *atomic*. For a nonzero nonunit x in an integral domain R the expression $x = a_1a_2 \ldots a_n$, where each a_i is an atom (if such an expression exists), may be called an *atomic factorization* of x of *length n*. The atomic domains are precisely the integral domains, R, for which every nonzero nonunit $x \in R$ has an atomic factorization of length n for some (perhaps more than one) positive integer n. In general, the lengths of atomic factorizations of a nonzero nonunit, in an atomic integral domain, may vary. For example, if K is a field and X an indeterminate over K then the ring $K[X^2, X^3]$ is Noetherian and hence atomic. The elements X^2 and X^3 are both irreducible elements of $K[X^2, X^3]$. Hence the element X^6 has two atomic factorizations of different length, namely:

$$(1.1) \qquad X^6 = X^2X^2X^2 = X^3X^3.$$

Of interest is the case where "differing length factorizations" cannot occur. The prototype of this interesting situation is the classical *unique factorization domain* (UFD). An atomic domain such that any two atomic factorizations of

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every nonzero element involve the same set of atoms up to associates and have the same length is said to be a UFD. (More compactly, an equivalent definition of UFD is a domain where every nonzero nonunit can be expressed as a product of primes).

More recently, the concept of *half-factorial domain* (HFD) has been introduced. If R is an atomic domain such that any two atomic factorizations of a given nonzero nonunit of R have the same length then R is said to be an HFD. Half-factorial domains first showed up implicitly as Dedekind domains with class number 2 in the work of Carlitz [8], and lay in obscurity until Zaks introduced and abstracted them in [21].

Lately, examples of half-factorial domains have abounded in the literature. A large class of domains where factorization has been examined closely is the class of $A + XB[X]$ domains, where $A \subseteq B$ are integral domains. This can be regarded as a special case of the classical $D + \mathcal{M}$ construction of Gilmer.

Constructions of $A + XB[X]$ type are generally not UFDs (since $A + XB[X]$ is not completely integrally closed unless $A = B$ and A is completely integrally closed). This reduces our problem to the classical polynomial case and we see that $A + XB[X]$ is a UFD if and only if the domains A and B coincide and are UFDs. This leads to the natural question as to when the $A + XB[X]$ construction is an HFD. This problem has been addressed in [7], [14], and [6].

We look at the event of $A + XB[X]$ being an HFD in the following three situations each in a brief section of its own: (i) When A is a subfield of the integral domain B (section 2). In section 2 we show that if A is a subfield of B then $A + XB[X]$ is an HFD if and only if B is integrally closed. (ii) When A is an integral domain (section 3). In this section we find several necessary and sufficient conditions under which $A + XB[X]$ is an HFD with A not neccessarily a subfield of B. (iii) When $S = \{f \in D[X] : f(0) \neq 0\}$ is a splitting set of $A + XB[X]$. (The notion of a splitting set will be defined later.) The fourth and final section titled "odds and ends" includes answers to questions that could pop up while reading the paper and results in previous sections that need explicit statements.

Any unexplained material is standard, as in [13]

2. When A is a subfield.

In [7] it was shown that if $K \subseteq B$, where K is a field and B a UFD, then $K + XB[X]$ is an HFD. On the other hand, responding to a question in [6, Section 5] which echoed a question in [7, Remark 3.7], Kim showed in [16] that if

$K \subseteq B$, where K is a field and B is such that $B[X]$ is an HFD, then $K + XB[X]$ is an HFD. We extend both of these results with the following theorem.

Theorem 2.1. *Let $K \subseteq B$, where K is a field and B a domain. Then $K+XB[X]$ is an HFD if and only if B is integrally closed.*

To give the proof, we need some preparation. Let D be an integrally closed domain with quotient field K. By [13, Theorem 10.4], if $f, g \in K[X]$, f is monic and $fg \in D[X]$, then $g \in D[X]$. Also, by [4, Theorem 3.2], each monic nonunit $f \in D[X]$ is a product of prime elements. Our first lemma is obtained from these results by "reversing the order of the coefficients".

Lemma 2.2. *Let D be an integrally closed domain with quotient field K.*
   (a) If $f, g \in K[X]$, $f(0) = 1$ and $fg \in D[X]$, then $g \in D[X]$.
   (b) Every nonconstant polynomial $q \in D[X]$ with $q(0) = 1$ is a product of prime elements.

PROOF. (a) Let f have degree m and g have degree n. Consider the polynomials $h = fg$, $\bar{f} = f(1/X)X^m$, $\bar{g} = g(1/X)X^n$ and $\bar{h} = h(1/X)X^{m+n}$. Clearly, $\bar{f}$, $\bar{g}$, resp. $\bar{h}$ have the coefficients of f, g, resp. h in reverse order and $\bar{h} = \bar{f}\bar{g}$. Since $\bar{f}$ is monic and D is integrally closed, $\bar{g} \in D[X]$, cf. [13, Theorem 10.4]. Hence $g \in D[X]$.
   (b) Clearly, we may assume that q is irreducible in $D[X]$. Then q is irreducible in $K[X]$. Indeed, let $q = rs$ with $r, s \in K[X]$. We may assume that $r(0) = s(0) = 1$. By (a), $r, s \in D[X]$, so $r = 1$ or $s = 1$.
   As $K[X]$ is a PID, q is prime in $K[X]$. By (a), $qK[X] \cap D[X] = qD[X]$. Hence q is prime in $D[X]$. □

We retain the following consequence.

Corollary 2.3. *Let $K \subseteq B$, where K is a field and B an integrally closed domain. Then every non-prime atom of $K + XB[X]$ has order one.*

PROOF. Let f be an atom of $R = K+XB[X]$. Clearly, f has order ≤ 1, otherwise X properly divides it in R. Assume that f has order zero. As K is a field, we may assume that $f(0) = 1$. It easily follows that f is irreducible in $B[X]$. By part (b) of Lemma 2.2, f is prime in $B[X]$. Hence f is prime in R, because $fB[X] \cap R = fR$. Indeed, $fg \in R$ with $g \in B[X]$ implies $g(0) = (fg)(0) \in K$, so $g \in R$. □

The next lemma gives examples of order-one atoms.

Lemma 2.4. *Let $K \subseteq B$, where K is a field and B a domain. If $0 \neq r, s \in B$ and s does not divide r in B, then $rX^2 + sX$ is irreducible in $K + XB[X]$.*

PROOF. Deny. Then $rX^2 + sX = (aX)(1 + bX)$ for some $a, b \in B \setminus \{0\}$. We get $a = s$ and $r = sb$, a contradiction.   □

Proof of Theorem 2.1. As K is a field, the ring $R = K + XB[X]$ is atomic, cf. [7, Proposition 1.1]. Assume that B is integrally closed. It is well-known and easy to see (e.g. [1, Theorem 2.1]), that the prime elements play no role in verifying the HFD property. So, it suffices to consider an equality

$$f_1 f_2 \cdots f_m = g_1 g_2 \cdots g_n$$

with $f_1, ..., g_n$ non-prime atoms in R and show that $m = n$. By Corollary 2.3, all atoms $f_1, ..., g_n$ above have order one. Counting the orders, we get $m = n$.

Conversely, suppose that R is an HFD and B is not integrally closed. We adapt the proof of the main result of [11]. Choose $r, s \in B^* = B \setminus \{0\}$ such that r/s is integral over B and $r/s \notin B$. Let $p \in B[X]$ be a monic irreducible polynomial with $p(r/s) = 0$. If p has degree n, then

$$s^n p = (sX - r)q$$

for some $q \in B[X]$. Replacing X with $1/X$ and multiplying by X^n, we get

$$s^n f = (s - rX)g$$

where $f = p(1/X)X^n$ and $g = q(1/X)X^{n-1}$ are in $g \in B[X]$. As p is irreducible in $B[X]$, so is f. Since $f(0) = 1$, it easily follows that f is irreducible in R. Next, multiplying by X^2 we get

$$(s^n X)Xf = (sX - rX^2)(Xg).$$

The left hand side of this equality is a product of three atoms in R and, as R is an HFD, so has to be the right hand side. By Lemma 2.4, $sX - rX^2$ is an atom. So Xg must be reducible in R, that is, $Xg = (Xk)(1 + Xh)$ for some nonzero $k, h \in B[X]$. We get

$$s^n f = (s - rX)k(1 + Xh).$$

The nonconstant polynomial $m = (s - rX)k/s^n$ belongs to $L[X]$, where L is the quotient field of B. As $f = m(1 + Xh) \in B[X]$, [17, Lemma] shows that $m \in B[X]$. So, f is reducible in $B[X]$, a contradiction. □

By [11], if $B[X]$ is an HFD, then B is integrally closed. So, we get the following corollary which answers the question posed in [7, Remark 3.7].

Corollary 2.5. *Let $K \subseteq B$, where K is a field and B a domain. If $B[X]$ is an HFD, then $K + XB[X]$ is an HFD.*

The proof of Theorem 2.1 suggests the following connection. Let $K \subseteq B$, where K is a field and B a domain, and consider the set $S = \{f \in R; \; f(0) \neq 0\}$ where $R = K + XB[X]$. Clearly, S is a saturated multiplicative set of R. Now assume that B is integrally closed. As remarked in the proof of Theorem 2.1, R is an atomic domain and every nonconstant polynomial of S is a product of primes. Thus S is a typical example of a splitting set [3, Corollary 1.7].

Recall that, according to [3], a saturated multiplicative subset S of a domain D is called a *splitting multiplicative set* if every $0 \neq a \in D$ is expressible as $a = st$ where $s \in S$ and t is v-coprime with every element of S. Here two nonzero elements x and y are said to be *v-coprime* if $xD \cap yD = xyD$. It may be noted that being v-coprime is considerably more than being coprime. It may also be noted that for x, y nonzero $xD \cap yD = xyD$ if and only if $x \mid yz$ implies $x \mid z$ for all $z \in D$. The *m-complement* T of S is the set of nonzero elements in D which are v-coprime to every $s \in S$. It is easy to see that T is also a splitting set. As shown in [3, Corollary 1.4], the factorization $a = st$ above is unique up to associates. Call s the *S-part* and t the *m-complement part* of a. For a study of splitting sets the reader may consult [3]. We can give the following extension of Theorem 2.1.

Proposition 2.6. *Let $K \subseteq B$, where K is a field and B a domain, $R = K + XB[X]$ and $S = \{f \in R; \; f(0) \neq 0\}$. Then R is an HFD if and only if S is a splitting set of R.*

PROOF. The "if" part was argued above. Conversely, assume that S is a splitting set of R. By Theorem 2.1, it suffices to show that B is integrally closed. Let $0 \neq b, c \in B$ such that c/b is integral over B. As in the proof of Theorem 2.1, $af = (b - cX)g$ for some $f, g \in B[X]$ with $f(0) = 1$ and some $0 \neq a \in B$. Then

$$aX^2 f = (bX - cX^2)(gX)$$

is an equality in R. Since S is a splitting set of R, we can equate the m-complement parts with respect to S. As $f \in S$, the m-complement part of the left hand side is aX^2. Then the m-complement part of $bX - cX^2$ is a degree-one monomial. Hence $bX - cX^2 = eX(1 - dX)$ for some $e, d \in B$. Thus $c/b = d \in B$. □

Let C be a domain and T a splitting multiplicative set of C such that C_T is an HFD. By [3, Theorem 3.3], C is an HFD if and only if each element of T is a product of atoms and whenever $s_1 s_2 ... s_m = t_1 ... t_n$ with $s_i, t_j \in T$ atoms, then $m = n$. In particular, if T is a splitting set generated by primes, then C is an

HFD if and only if C_T is an HFD. Now let $A \subseteq B$ be an extension of domains. We intend to apply this result for $T = A^* = A \setminus \{0\}$ and $C = A + XB[X]$. This will reduce the study of the HFD property for $A + XB[X]$ to the case when A is a field, a case covered by Theorem 2.1. To this end, we need to know when A^* is a splitting multiplicative set of $A + XB[X]$. Recall that in [5, Theorem 2.2], it was shown that A^* is a splitting set of $A[X]$ if and only if A is a GCD domain.

Proposition 2.7. *Let $A \subseteq B$ be an extension of domains with A atomic. Then A^* is a splitting set of $A + XB[X]$ if and only if*

(a) *A is a UFD,*

(b) *$aB \cap A = aA$ for each $a \in A$,*

(c) *every prime element of A is prime in B, and*

(d) *for any infinite sequence of (not necessarily distinct) primes $\{p_n\}$ of A,* $\bigcap_n p_1 \cdots p_n B = 0$.

In particular, if A^* is a splitting set of $R = A + XB[X]$, then R is atomic, A^* is generated by primes in R and the saturation of A^* in B is a splitting set of B generated by primes.

PROOF. Set $R = A + XB[X]$. Assume that conditions (a)-(d) hold. Let p be a prime element of A. As $pB \cap A = pA$ and p is prime in B, $A \subseteq B$ induces the domain extension $A/pA \subseteq B/pB$. Since $R/pR = A/pA + X(B/pB)[X]$, p is prime in R. So, A^* is generated by primes in R. As (d) holds, R is atomic, cf. [12, Proposition 1.2]. Thus A^* is a splitting set of R, cf. [3, Corollary 1.7].

Conversely, assume that A^* is a splitting set of R. Clearly, the m-complement T of A^* consists of all $0 \neq f \in R$ which have no nonunit factor $d \in A$.

To prove (b), let $a, b \in A^*$ such that $a \mid_B b$. Then $a \mid_R b(a + X)$. Since A^* is a splitting set of R and $a + X \in T$, we get $a \mid_R b$. Hence $a \mid_A b$.

As A is atomic, to prove (a) it suffices to show that A is a GCD domain. Let $a, b \in A^*$. We look for their LCM adapting the argument in [5, Theorem 2.2]. Since A^* is a splitting set, $a + bX$ can be written as $c(d + eX)$ with $c \in A$ and $d + eX \in T$. As $cB \cap A = cA$, $e \in A$. Similarly $d \in A$.

Replacing the pair a, b by d, e, we may assume that a, b have no proper common divisor in A. Now, let m be a common multiple of a, b in A. Then $m = aa' = bb'$ for some $a', b' \in A$. Hence $a'b'(a + bX) = m(b' + a'X)$. As A^* is a splitting set and $a + bX \in T$, we get $b' + a'X = n(a + bX)$ for some $n \in A$. Then $m = aa' = abn$. Consequently, ab is an LCM of a and b in A; that is, a and b are v-coprime.

For (c), let $p \in A$ be a prime element and $r, s \in B$ such that $p \mid_B rs$. Write $rX = r'(tX)$ and $sX = s'(uX)$ with $r', s' \in A^*$ and $tX, uX \in T$. Then $p \mid_R (rX)(sX) = r's'(tX)(uX)$. Since A^* is a splitting set, we get $p \mid_R r's'$, so

$p \mid_A r's'$. As p is prime in A, $p \mid_A r'$ or $p \mid_A s'$. Hence $p \mid_B r$ or $p \mid_B s'$; that is, p is prime in B.

For (d), consider an infinite sequence of (not necessarily distinct) primes $\{p_n\}$ of A and suppose that $0 \neq b \in \bigcap_n p_1 \cdots p_n B$. Then $bX \in \bigcap_n p_1 \cdots p_n R$, contradicting the fact that A^* is a splitting set of R, cf. [3, Proposition 1.6].

The "in particular" part has been shown.   □

Recall that an extension of domains $A \subseteq B$ is called *inert*, if whenever $a = bc$ with $0 \neq a \in A$ and $b, c \in B$, there exists $u \in U(B)$ such that $bu, cu^{-1} \in A$.

Remark 2.8. Let $A \subseteq B$ be an extension of domains, with A a UFD.

(i) Condition (b) of Proposition 2.7 is equivalent to $pB \cap A = pA$ for each prime $p \in A$.

(ii) If condition (c) of Proposition 2.7 holds, then the extension $A \subseteq B$ is inert. Indeed, let $0 \neq b, c \in B$ and $bc \in A$. As A is a UFD, we can express bc as a product of primes in A, $bc = p_1 \cdots p_n$. As the p_i's are also primes in B, we can assume that $b = p_1 \cdots p_m u$ and $c = p_{m+1} \cdots p_n u^{-1}$ for some unit u of B. Thus $bu^{-1}, cu \in A$.

Corollary 2.9. *Let A be an atomic domain and $A \subseteq B$ an extension of domains which satisfies conditions (a)-(d) of Proposition 2.7. Then $A + XB[X]$ is an HFD if and only if B is integrally closed.*

PROOF. Let K be the quotient field of A. By Proposition 2.7, A^* is a splitting set of R generated by primes. So, by the paragraph preceding Proposition 2.7, $A + XB[X]$ is an HFD if and only if its quotient ring $Q = (A + XB[X])_{A^*}$ is an HFD. Clearly, $Q = K + XB_{A^*}[X]$. By Theorem 2.1, Q is an HFD if and only if B_{A^*} is integrally closed. By Proposition 2.7, the saturation of A^* in B is a splitting set of B generated by primes. Then [3, Proposition 4.2] shows that B_{A^*} is integrally closed if and only if B is integrally closed.   □

Example 2.10. (a) If $\mathbb{Z}$ is the ring of integers and $B = \mathbb{Z}[t] + Y\mathbb{Z}[t, t^{-1}][Y]$, it is easy to see that $\mathbb{Z} \subseteq B$ satisfies the hypotheses of Corollary 2.9. So $\mathbb{Z} + XB[X]$ is an HFD.

(b) Let D be a UFD. Then D is completely integrally closed, hence so is the power series ring $D[[Y]]$. By Corollary 2.9, $D + XD[[Y]][X]$ is an HFD. Recall that there exist UFDs, D, for which $D[[Y]]$ is not a UFD.

We pause to extend a result from [14]. Let $A \subseteq B$ be an extension of domains. In [14, Proposition 1.8], it was shown that $A + XB[X]$ is an HFD, provided $A \subseteq B$ is inert (see paragraph before Remark 2.8), B is a UFD and $U(B) \cap A = U(A)$. We extend this result from one to many variables.

Proposition 2.11. *Let $A \subseteq B$ be an inert extension of domains such that B is a UFD and $U(B) \cap A = U(A)$. Then for every $n \geq 1$, $A + (X_1, ..., X_n) B[X_1, ..., X_n]$ is an HFD.*

The result is a consequence of the next lemma applied for $C = A + (X_1, ..., X_n) B[X_1, ..., X_n]$, $D = B[X_1, ..., X_n]$ and $M = (X_1, ..., X_n)D$.

Lemma 2.12. *Let $C \subseteq D$ be an extension of domains having a common prime ideal M. Suppose that D is a UFD, $U(D) \cap C = U(C)$ and whenever $a = b_1 b_2 \cdots b_n$ with $a \in C \setminus M$ and $b_1, ..., b_n \in D$, there exist $u_1, ..., u_n \in U(D)$ such that $u_1 \cdots u_n = 1$ and $b_i u_i \in C$, $i = 1, ..., n$. Then C is an HFD.*

PROOF. By [3, Proposition 1.2], C is atomic. Call an element $b \in D$ *special* if $bu \in C$ for some $u \in U(D)$. Consider a nonunit element $c \in C$ and an atomic factorization of c in C, $c = a_1 \cdots a_m$. It suffices to show that every atom a of C has exactly one special irreducible factor in its (unique) decomposition as a product of atoms of D, for then m is the number of special atoms in the decomposition of c in D. Let a be an atom of C and $a = b_1 \cdots b_n$ its decomposition in D. If $a \notin M$, then, by our hypothesis, we may assume that each b_i is in C. So $n = 1$; that is, a is irreducible in D. Assume that $a \in M$. Since M is a prime ideal of D (and C), some b_i, say b_1, lies in M. Suppose that another b_i, say b_n, is special and let $u \in U(D)$ such that $u b_n \in C$. Then $a' = b_1 \cdots b_{n-1} u^{-1} \in M \subseteq C$ and a is a product of two nonunits of C, $a = a'(u b_n)$, a contradiction. $\square$

3. THE SPLITTING SET CONNECTION.

In this section, we investigate the relationship between $A + XB[X]$ being an HFD and $S = \{f \in A + XB[X] \mid f(0) \neq 0\}$ being a splitting set. A first connection is given in:

Proposition 3.1. *Let $A \subseteq B$ be a domain extension and $R := A + XB[X]$. Suppose that $S = \{f \in R \mid f(0) \neq 0\}$ is a splitting set of R and that each element of S has all atomic factorizations of fixed length. Then R is an HFD.*

PROOF. Let f be a nonzero nonunit of R. Then there exists $h \in S$ and g (v-coprime to every element of S) such that $f = gh$ uniquely (up to associates). Now clearly, if g is a nonunit, then $g(0) = 0$ and g has no nonunit factors from S. As in the proof of Theorem 2.1, we can show that all the atomic factorizations of g have length precisely the order of g. $\square$

Lemma 3.2. *Let $A \subseteq B$ be a domain extension and $R := A + XB[X]$. If $S = \{f \in R \mid f(0) \neq 0\}$ is a splitting set of R, then $aB \cap A = aA$ for each $a \in A$.*

PROOF. We first note that no nonunit member of S divides X in R. For if it were so, then the only elements of S that can divide X in R would have to come from A. But then, as S is a splitting set, we must have $X = a(X/a)$ where $a \in S$ and X/a is v-coprime to every element of S and in particular to a. But $X/a \in R$ implies that $1/a \in B$. That is, a is a unit in B. But then $X/a^n \in R$ for all n and so X/a being v-coprime to a is possible only if a is a unit in A. Indeed from these considerations it follows that no nonunit of A is a unit of B.

Now to see that $aB \cap A = aA$, let $a \in A$ be a nonzero nonunit (the property is clear for $a = 0$ or a is a unit of A). Since, as shown above, no nonunit member of S divides X in R, X must be v-coprime to a. Consequently, if $0 \neq b \in B$ and $ab = c \in A$, then $a(bX) = cX$, so $X \mid a(bX)$ and that implies $X \mid bX$ because X and a are v-coprime in R. But then $bX/X = b \in A$. □

When every $b \in B$ has an associate in A, the converse holds as well.

Proposition 3.3. *Let $A \subseteq B$ be a domain extension such that for every $b \in B$ there exists $u \in U(B)$ with $ub \in A$, and let $R := A + XB[X]$. Then $S = \{f \in R \mid f(0) \neq 0\}$ is a splitting set of R if and only if $aB \cap A = aA$ for each $a \in A$.*

PROOF. It suffices to prove the "if" part. Let $0 \neq g \in R \setminus S$. Then $g = X^r h$ for some $h \in B[X]$ with $b = h(0) \neq 0$. Let $u \in U(B)$ such that $ub = a \in A$. Then $g = (uh)(u^{-1}X^r)$ with $uh \in S$. It remains to show that for each $v \in U(B)$, vX is v-coprime to every $s \in S$. Fix such vX and s, and let $p, r \in R$ with $(vX)p = sr$. If r has order ≥ 2, then vX divides r in R. Otherwise, r has order one, say, $r = Xq$ with $0 \neq q \in B[X]$. Then $vp(0) = s(0)q(0)$, so $s(0)q(0)u^{-1} = p(0) \in A \cap s(0)B = s(0)A$. We get $q(0)u^{-1} \in A$, hence $r/vX = qv^{-1} \in R$. □

Apart from this case, S is a splitting set implies A is a UFD (provided A is atomic). Let $A \subseteq B$ be an extension of domains. For brevity, call an element $z \in B$ *proper* if $uz \notin A$ for every $u \in U(B)$.

Proposition 3.4. *Let $A \subseteq B$ be an extension of domains with A atomic and assume B has proper elements. Set $R = A + XB[X]$ and $S = \{f \in R \mid f(0) \neq 0\}$. If S is a splitting set of R, then*
   (a) A is a UFD,
   (b) $aB \cap A = aA$ for each $a \in A$, and
   (c) every prime element of A is prime in B.

PROOF. Condition (b) was proved in Lemma 3.2 and (c) can be proved as the corresponding part of Proposition 2.7.

We prepare the way for proving (a). Let T be the m-complement of S. If $0 \neq b \in B$, we can write $bX = a(cX)$ with $a \in A$ and $cX \in T$. Call such an element $c \in B$ with $cX \in T$ a *clean* element. So $b = ac$ with $a \in A$ and $c \in B$ clean. Note that c is v-coprime in B to every nonzero element of A. Indeed, if $0 \neq d \in A$ and $x, y \in B$ such that $dx = cy$, then $d(xX^2) = (cX)(yX)$. As cX is v-coprime to d in R, we get $xX^2/cX = (x/c)X \in R$, hence $x/c \in B$.

As A is atomic, to prove (a) it suffices to show that every two non-associated atoms of A are v-coprime (see the proof of [19, Proposition 6.4]). Let a_0, a_1 be non-associated atoms of A and $z \in B$ a proper element. Clearly, z may be assumed to be clean. We investigate the S-splitting factorization of $f = a_0 z X + a_1 X^2 \in R$. Assume that $f = (c_0 + c_1 X) dX$ with $c_0 + c_1 X \in S$ and $dX \in T$. So d is clean. Then $a_0 z = c_0 d$ and $a_1 = c_1 d$. Being clean, d is v-coprime to a_1, hence $d \in U(B)$. So $a_0(zd^{-1}) = c_0 \in A$. By (b), $zd^{-1} \in A$, that is, z is not proper, a contradiction. Thus the S-part of f (see paragraph before Proposition 2.6) is constant, say equal to $e \in A$. Then e divides $a_0 z$ and a_1 in B. As z is clean, $e \mid_B a_0$. By (b), e divides a_0 and a_1 in A, so e is a unit of A, because a_0 and a_1 are non-associated atoms. Consequently, $f \in T$.

To prove that a_0, a_1 are v-coprime in A, let $r, s \in A$ such that $r \mid_A sa_0$ and $r \mid sa_1$, and prove that $r \mid_A s$. We get $r \mid_R fs$. As $f \in T$, r is v-coprime to f. So $r \mid_R s$, hence $r \mid_A s$. □

Corollary 3.5. *Let $A \subseteq B$ be an extension of domains such that B has proper elements and $R = A + XB[X]$ is an atomic domain. The following assertions are equivalent:*

(a) $S = \{f \in R \mid f(0) \neq 0\}$ *is a splitting set of R.*
(b) A^* *is a splitting set of R and R is an HFD.*
(c) A^* *is a splitting set of R and B is integrally closed.*

PROOF. Since R is atomic, so is A. By Propositions 3.4 and 2.7, we may assume that A^* is a splitting set of R generated by primes and its saturation in B is a splitting set of B generated by primes. The equivalence of (b) and (c) was given in Corollary 2.9.

By [3, Corollary 1.3], a saturated multiplicative subset V of a domain D is splitting if and only if each principal ideal of D_V contracts to a principal ideal of D. Using this description and the tower $R \subseteq R_{A^*} \subseteq R_S$, it is easy to see that S is a splitting set of R if and only if the saturation of S in R_{A^*} is a splitting set of R_{A^*}. Also, by [3, Proposition 4.2], B_{A^*} is integrally closed if and only if B is integrally closed. So, for proving the equivalence of (a) and (c), we may

replace R by R_{A^*}, that is, we may assume that A is a field. Now Proposition 2.6 applies. $\qquad\square$

4. ODDS AND ENDS.

In the paper [15] domains of the form $A + XI[X]$ were investigated (in this construction, A was assumed to be a UFD and I is a nonzero proper ideal of A). One of the most interesting results in this paper is the result that R is an HFD if and only if I is a prime ideal of A. This result, coupled with some of our earlier results, inspires the following question:

Question 4.1. If $K \subseteq B$ is an extension of domains where K is a field and B a UFD, is $K + XP[X]$ an HFD if P is a prime ideal of B?

We answer this question in a more general setting with the following theorem which shows that the construction $A + XI[X]$ (with I an ideal of a ring containing A) is rarely an HFD.

Theorem 4.2. *Let $A \subseteq B$ be an extension of domains with A a UFD and $I \subseteq B$ a proper ideal. If I contains an irreducible element $\pi \in B$ such that π^2 has no nontrivial factor in A, then $R = A + XI[X]$ is not an HFD.*

PROOF. Suppose that $\pi \in I$ is an irreducible element of B. It is easy to see in the domain $A + XI[X]$ that the elements $\pi X, \pi X^2$ are irreducible. The assumption that π^2 has no nontrivial factors in A guarantees that the element $\pi^2 X$ is also an irreducible element of R. We now have the following factorizations in R:

$$(4.1) \qquad (\pi X^2)(\pi^2 X) = (\pi X)(\pi X)(\pi X)$$

and so R is not an HFD. $\qquad\square$

Corollary 4.3. *Suppose that $K \subseteq B$ with K a field and $I \subseteq B$ a proper ideal. If I contains an irreducible element of B, then $R = K + XI[X]$ is not an HFD.*

PROOF. If $\pi \in I$ is an irreducible element of B, then π^2 has no nontrivial factors in K, since K is a field. Apply the previous theorem. $\qquad\square$

We will now use this result to show that the result of Gonzalez, Pellerin, and Robert [15] mentioned above does not generalize to our situation.

Example 4.4. Let F be any field and let $V = F[[t]]$. V is a Noetherian valuation domain and the ideal $\mathfrak{P} = tV$ is its unique nonzero prime ideal. We consider the

construction $A + XP[X] = F + X\mathfrak{P}[X]$. Certainly we have that A is a field and $B\ (= V)$ is a UFD (as per the question above). However, the above results show that $F + X\mathfrak{P}[X]$ is not an HFD.

There is a result implicit in the previous sections which could be of independent interest. Let $F(D)$ be the set of fractional ideals of a domain D. Recall that the *v-operation* is the mapping $I \mapsto I_v = (I^{-1})^{-1}$ of $F(D)$ into $F(D)$.

Proposition 4.5. *Let D be an integral domain and K the quotient field of D. The following assertions are equivalent.*

(1) D is integrally closed,

(2) for each $f \in K[X]$ such that $(A_f)_v \supseteq D$, $fg \in D[X]$ with $g \in K[X]$ implies that $g \in D[X]$.

PROOF. (1) $\Rightarrow$ (2) Note that $(A_f)_v \supseteq D$ implies that $A_f^{-1} \subseteq D$ and that D is integrally closed if and only if for $f, g \in K[X]$, $A_{fg} \subseteq D$ implies $A_f A_g \subseteq D$, cf. [18, Theorem 1.5]. Now, $fg \in D[X]$ means that $A_{fg} \subseteq D$ which combined with the fact that D is integrally closed gives $A_f A_g \subseteq D$, which in turn forces $A_g \subseteq A_f^{-1}$ and consequently $g \in D[X]$.

(2) $\Rightarrow$ (1). Let $\alpha \in K$ be integral over D. Let $h = X^n + a_{n-1}X^n + ... + a_1 X + a_0$ be a monic over D satisfied by α. Then $h = (X - \alpha)k$ are both monics and so the contents of both contain D and their product is in $D[X]$. So by (2) both $(X - \alpha)$ and g belong to $D[X]$, because in each case we have $(A_f)_v \supseteq D$. But then $\alpha \in D$ and thus D is integrally closed. $\square$

Remark 4.6. There are several ways of proving the fact that if f is a polynomial in $D[X]$ such that $(A_f)_v = D$ and if f is irreducible in $K[X]$ then f is a prime in $D[X]$. One way of doing it is via a pretty result due to Querre [20] that says that D is integrally closed if and only if for each $f \in K[X]$, $fK[X] \cap D[X] = fA_f^{-1}D[X]$. Clearly if $(A_f)_v = D$ then $fK[X] \cap D[X] = fD[X]$ which is a prime if and only if $fK[X]$ is a prime.

As a necessary tool for our next result, we introduce an involution on the multiplicative semigroup $T = \{f \in K[X] | f$ is nonconstant with $f(0) \neq 0\}$ by defining for each $f = \sum_{i=0}^{i=n} a_i X^i \in T$ of degree n, the polynomial $\widetilde{f} = \sum_{i=0}^{i=n} a_{n-i}X^i$. Clearly $\widetilde{f} = X^n f(1/X)$. Note that if $g = \sum_{i=0}^{i=m} b_i X^i$ then $\widetilde{f}\widetilde{g} = X^{m+n}f(1/X)g(1/X) = \widetilde{fg}$ because the coefficients of $(1/X)^i$ in $f(1/X)g(1/X)$ are the same as coefficients of X^i in fg. Consequently if $f \in T$ is reducible (or irreducible) then so is $\widetilde{f}$. Indeed this involution can also be defined, with the same properties, on the multiplicative semigroup of nonconstant polynomials in $D[X]$ with nonzero constant term.

Proposition 4.7. *The following assertions are equivalent.*

(1) *D is integrally closed,*

(2) *Every irreducible polynomial over D with a unit constant is prime.*

PROOF. $(1) \Rightarrow (2)$. Let D be integrally closed and let f be an irreducible polynomial over D such that $f(0) = 1$ and suppose that f is not a prime. Then $f = gh$ in $K[X]$. Then as $g(0)h(0) = 1$ each of $g(0)$, $h(0)$ is a unit. So $A_g, A_h \supseteq D$ and by Proposition 4.5 we have that both g and h are in $D[X]$, contradicting the irreducibility of f. So f must be irreducible in $K[X]$ and hence a prime by Remark 4.6.

$(2) \Rightarrow (1)$. Suppose that every irreducible polynomial $f \in D[X]$ with $f(0) = 1$ is a prime and suppose on the contrary that D is not integrally closed. Let $\alpha \in K \backslash D$ be integral over D. Then α satisfies an irreducible monic polynomial $f = X^n + a_{n-1}X^{n-1} + ... + a_0$. But then $\widetilde{f} = a_0 X^n + a_1 X^{n-1} + ... + a_{n-1}X + 1$ is irreducible and hence a prime in $D[X]$ and irreducible in $K[X]$. Yet if α is a root of f then $f = (X - \alpha)g$. But then $\widetilde{f} = (-\alpha X + 1)\widetilde{g}$ is reducible in $K[X]$ a contradiction. $\square$

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