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ON FINITE CONDUCTOR DOMAINS

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An integral domain D is a FC domain if for all a,b in D, $aD\cap bD$ is finitely generated. Using a set of very general and useful lemmas, we show that an integrally closed FC domain is a Prüfer v-multiplication domain (PVMD). We use this result to improve some results which were originally proved for integrally closed FC domains (or for coherent domains) to results on PVMD's. Finally we provide examples of integrally closed integral domains which are not FC domains.

Throughout this note the letters D and K denote a commutative integral domain and its quotient field respectively.

Let $a/b \in K$. The ideal I of D such that for $x \in I$, $x(a/b) \in D$ is called the conductor of a/b in D. An integral domain D is called a finite conductor (FC) domain if the conductor of every $x \in K$ is finitely generated (fg). Clearly D is FC if and only if $aD\cap bD$ is fg for all a,b in D.

It appears that there is no general method available, at present, to decide whether a certain integral domain has this FC property or not. Consequently, examples of integral domains

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without FC property keep appearing in current literature. One among the aims of this note is to show that a certain class of integrally closed integral domains do not have the FC property. We recall that D is coherent if every finitely generated ideal of D is finitely presented. According to Chase [1] D is coherent if and only if the intersection of any two fg ideals is again fg Thus by showing that a certain class of integral domains does not have the FC property we actually show that no coherent ring can be found in that class.

We prove that an integrally closed FC domain is a, so called, Prüfer v-multiplication domain (PVMD). Thus an integrally closed integral domain which is not a PVMD is not a FC domain, and hence cannot be a coherent domain. The above mentioned result will also be used to modify certain known results and to supply alternative proofs of others. This note is split into two sections. In the first section we prove a set of useful lemmas from which it follows that an integrally closed FC domain is a PVMD. In the second section we apply our knowledge of PVMD's to some known results either to modify them or to improve (simplify) their proofs. Included in this section are also examples of non FC domains.

PRUFER V-MULTIPLICATION DOMAINS

PVMD's are based on the notion of *-operations. For a detailed study of *-operations we refer the reader to sections 32 and 34 of Gilmer [5]. For a quicker definition and for other purposes we note the following.

Let F(D) be the set of fractional ideals of D. The operation $A \mapsto (A^{-1})^{-1} = A_v$, on F(D), is called the v-operation. An ideal $A \in F(D)$ is said to be a v-ideal if $A = A_v$. Moreover A is a v-ideal of finite type if $A = B_v$ for some finitely generated B in F(D). If A, B $\in$ F(D) then (1) $A \subseteq A_v$; $(A_v)_v = A_v$; $D = D_v$ (2) $A \subseteq B$ implies $A_v \subseteq B_v$ (3) if A is principal then $(AB)_v = AB_v$.

DEFINITION 1. An integral domain D is a PVMD if the set H(D) of v-ideals of finite type, of D, forms a group under the v-multiplication $(AB)_v = (AB_v) = (A_v B_v)_v$.

Finally an ideal A of D is called a t-ideal if for every fg ideal $B \subseteq A$, $B_v \subseteq A$. Prime t-ideals and maximal t-ideals are defined in the usual manner. Griffin ([6] Proposition 4) showed that $D = \bigcap D_P$ where P ranges over maximal t-ideals. He also proved (loc cit) that D is a PVMD if and only if D_P is a valuation domain for every maximal t-ideal P of D.

THEOREM 2. An integrally closed FC domain D is a PVMD.

REMARK 3. We note that this theorem appears as an exercise in [5] (cf Ex.21 p.432). But since our proof is designed to draw some extra benefits the repetition seems to be in order.

For our proof of THEOREM 2 we need a string of lemmas each of which is an independent result in its own right.

LEMMA 4. Let A be a finitely generated ideal of D and let S be a multiplicative set in D. Then

(i) $A^{-1}D_S = (AD_S)^{-1}$

(ii) $(A_v D_S)_v = (AD_S)_v$.

PROOF. (i). We note that if $A = (x_1, \ldots, x_n)$ then
$A^{-1} = ((1/x_1)D \cap (1/x_2)D \cap \ldots \cap (1/x_n)D)D_S =$
$(1/x_1)D_S \cap \ldots \cap (1/x_n)D_S = ((x_1, \ldots, x_n)D_S)^{-1} = (AD_S)^{-1}$.
(ii). Clearly $A \subseteq A_v$ and so $AD_S \subseteq A_vD_S$. Whence $(AD_S)_v$
$\subseteq (A_vD_S)_v$. Conversely, because A is fg $A^{-1}D_S = (AD_S)^{-1}$.
Now $(AD_S)_v = ((AD_S)^{-1})^{-1} = (A^{-1}D_S)^{-1} \supseteq (A^{-1})^{-1}D_S =$
$= A_v D_S$. So $(AD_S)_v \supseteq (A_vD_S)_v$ and hence $(AD_S)_v = (A_vD_S)_v$.

LEMMA 5. Let D be an integrally closed integral domain and let P be a prime ideal of D such that PD_P is a maximal t-ideal of D_P and D_P is a FC domain. Then D_P is a valuation domain.

PROOF. From the hypothesis we know that if P is a maximal t-ideal of D then PD_P is a maximal t-ideal of D_P. Now let $a,b \in PD_P$. Since D_P is FC $(a,b)^{-1} = (1/ab)(aD_P \cap bD_P)$ is fg. Further since D_P is integrally closed we have $((a,b)(a,b)^{-1})^{-1}$ $= ((a,b)(1/ab)(aD_P \cap bD_P))^{-1} = D_P$ (cf Kaplansky [7] Ex 39, p.45). We claim that $(a,b)((1/ab)(aD_P \cap bD_P)) = D_P$. For if not $(a,b)((1/ab)(aD_P \cap bD_P))$ is a fg ideal of D_P and so $((a,b)((1/ab)(aD_P \cap bD_P)))_v \subseteq PD_P$. Whence $D_P \subseteq PD_P$ a contradiction. Now (a,b) is an invertible ideal in a quasi-local domain and hence is principal. That D_P is a valuation domain is now easy to establish.

LEMMA 6. Let D be any commutative integral domain and let $a,b \in D$ such that $aD : bD \neq D$. Then every minimal prime P of $aD : bD$ is such that PD_P is a t-ideal.

PROOF. We recall that if $D = \bigcap D_P$ where P ranges over a set of prime ideals then $aD : bD$ is contained in at least one of the prime ideals (cf [5] Ex. 22, p. 52). Now we assume that P is a minimal prime of $aD : bD \neq D$. Then PD_P is minimal over $(aD : bD)D_P$ $= aD_P : bD_P$. Suppose that PD_P is not a t-ideal. Then, since $D_P = \bigcap (D_P)_M$ where M ranges over maximal t-ideals of D_P, $aD_P : bD_P$ is contained in at least one of these M's. This contradicts the minimality of P over $aD : bD$.

Now we note that if D is a FC domain then for every multiplicative set S, D_S is a FC domain. Using LEMMA's 5 and 6 and the above observation we have the following lemma.

LEMMA 7. Let D be an integrally closed FC domain. Then for every $a,b \in D$ with $aD : bD \neq D$ each minimal prime P of $aD : bD$ is such that D_P is a valuation domain.

Recall that a ring between D and K is called an overring of D. A valuation overring V of D is called essential if V is a quotient ring of D. The centre of an essential valuation overring of D is called an essential prime. Moreover an integral domain D is called essential if it is expressible as an intersection of essential valuation overrings of D. Finally let $D = \bigcap V$, where $\{V_\alpha\}$ is a family of valuation rings. Then $A \mapsto \bigcap V_\alpha A$, where $A \in F(D)$, is called the w-operation induced by the family $\{V_\alpha\}$

of valuation domains. For this operation too the reader is referred to sections 32 and 34 of [5]. For the purposes of this note we need only to know the definition.

Now by LEMMA 7 above and by Theorem E of Tang [10], an integrally closed FC domain is an essential domain. So now we are left with essential FC domains. Recall from [5](Proposition 44.13) that if D is defined by a family F of essential valuation domains then the w-operation induced by F is equivalent to the v-operation. That is, for every fg ideal A of D $A_v = A_w$. Now with the help of these remarks we state a some-what stronger result. But first we note that the intersection of every two v-ideals is again a v-ideal.

LEMMA 8. *Let D be an essential domain. Then the following are equivalent.*

(1). *The intersection of any two principal ideals is a v-ideal of finite type.*

(2). *The intersection of a finite number of principal ideals is a v-ideal of finite type.*

(3). *The inverse of every finitely generated ideal is a v-ideal of finite type.*

(4) *D is a PVMD.*

PROOF. Let $\{D_{P_\alpha}\}$ be the family of valuation domains defining D and let A be a finitely generated ideal. Then according to the remark made prior to LEMMA 8 $A_v = A_w = \bigcap A D_{P_\alpha}$.

(1) $\Rightarrow$ (2). Let A be a v-ideal of finite type. Then $A = (x_1, x_2, \ldots, x_n)_v$ for some natural number n. Consider

$\Sigma (x_i) \cap (a)$ where $a \in D$. Let $B_i = (x_i) \cap (a)$. By (1) B_i is of finite type. That is $B_i = (b_i)_v$ where b_i is finitely generated. Now from the definition of the *-operations, one can deduce that,

$(\Sigma b_i)_w = (\Sigma (b_i)_v)_w = (\Sigma B_i)_w = \cap (\Sigma (x_i) \cap (a)) D_P$

$= \cap (\Sigma (((x_i) \cap (a)) D_P))$ (since D_P is a valuation domain and a quotient ring of D)

$= \cap (\Sigma ((x_i) D_P \cap aD_P)) = \cap ((\Sigma x_i) D_P \cap aD_P) =$

$(x_1, \ldots, x_n)_v \cap (a) = A \cap (a)$.

Now since Σb_i is finitely generated, $(\Sigma b_i)_w = (\Sigma b_i)_v$ and hence $A \cap (a)$ is a v-ideal of finite type. Now, we know that the intersection of two principal ideals is of finite type, assuming that the intersection A of k principal ideals is of finite type we have shown that the intersection of k+1 principal ideals is also of finite type. Thus the result follows from induction.

(2) $\Rightarrow$ (3). This follows from the fact that if $A = (x_1, \ldots, x_n)$ then $A^{-1} = \cap (1/x_i)$. By simple manipulation A^{-1} can be given the form $(1/d) ((d_1) \cap \ldots \cap (d_n))$.

(3) $\Rightarrow$ (4). That every finitely generated ideal should have a v-inverse of finite type is a requirement for D to be a PVMD. Let $A = (x_1, \ldots, x_n)$. To see that $(AA^{-1})_v = D$ we can verify that $(AA^{-1})_w = \cap (AA^{-1}) D_P = D$.

(4) $\Rightarrow$ (1). From LEMMA 4, for every pair $a, b \in D$, $(a,b)^{-1} = \frac{(a) \cap (b)}{ab}$. That $(a) \cap (b)$ is of finite type, follows from the fact that D is a PVMD.

THE PROOF OF THEOREM 2. The proof follows immediately from the above string of Lemmas.

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There do exist, in current literature, examples of essential rings which are not PVMD's. This situation naturally demands necessary and sufficient conditions for an essential domain to be a PVMD. LEMMA 8 meets that demand. Moreover this lemma points out that in a non-PVMD essential domain D, there exists at least one pair of elements a and b such that $aD \cap bD$ is not of finite type.

We call a prime t-ideal P of D *stable* if for every multiplicative set S with $P \cap S = \phi$, PD_S is a t-ideal. Using this notion and LEMMA's 4 and 5 we can prove the following result.

COROLLARY 9. *Let D be an integrally closed integral domain. Then D is a PVMD if and only if (a) every maximal t-ideal of D is stable and (b) D_P is a FC domain for every maximal t-ideal P of D.*

Recall that D is a *Prüfer* domain if D_M is a valuation domain for every maximal ideal M.

COROLLARY 10. *Let D be integrally closed. Then D is a Prüfer domain if and only if (i) D is a FC domain (ii) every prime ideal of D is a stable t-prime.*

In the next section we shall need the following two results.

COROLLARY 11. *Every quotient ring of a PVMD is again a PVMD.*

PROOF. The proof of this corollary is a straight-forward application of LEMMA 4.

COROLLARY 12. *Let D be a PVMD and let a, b be an arbitrary* pair of elements of D such that $aD : bD \neq D$. Then every minimal prime of $aD:bD$ is an essential prime.

PROOF. The proof follows from LEMMA 6 and the fact that every maximal t-ideal of a PVMD is essential.

From the fact mentioned in the above proof, or from the corollary itself (and Theorem E of [10]), it follows that a PVMD is essential.

REMARK 13. The absolute converse of COROLLARY 12 is false. A counter example to that effect, and some partial converses, will appear in a joint paper with Professor J. L. Mott.

APPLICATIONS

Most of the results of this section follow from the results and observations made in the previous section.

COROLLARY 14 (cf Theorem 2 of [8]). Let D be an integrally closed quasi-local domain whose primes are linearly ordered by inclusion. Suppose that D has the FC property. Then D is a valuation domain.

PROOF. To show that D is a valuation domain it is sufficient to show that at least one prime ideal P of D is essential. Now, by THEOREM 2, D is a PVMD and hence is essential.

Now we proceed to show that some of the known results, stated for integrally closed FC domains, can be proved for PVMD's without restricting the PVMD's to be FC domains. Recall that two commutative integral domains $D \subseteq T$ are said to satisfy going down (GD) if, whenever $P \subseteq P'$ are two prime ideals in D and a prime ideal Q' in T with $Q' \cap D = P'$ there is a prime ideal Q in T with $Q \cap D = P$.

Lemma 2 of Dobbs [4] was stated for quasi-local domains which are either Krull domains of dimension ≥ 2, or integrally closed FC domains which are not valuation rings. We state the above mentioned lemma as follows.

COROLLARY 15. *Let D be a quasi-local PVMD which is not a valuation domain. Then there exists $u \in K$ such that $D \subseteq D[u]$ does not satisfy GD.*

PROOF. Note that a Krull domain is a PVMD in which every rank one prime ideal is a maximal t-ideal. Dobbs (loc cit), uses rank one primes of a Krull domain to avoid the FC condition. In the same way we can use the maximal t-ideals of the PVMD's to avoid the FC condition. The proof then follows the same lines as that of (a) Lemma 2 of [4].

COROLLARY 16 (cf [4] Theorem 3). *Let D be a PVMD which is not Prüfer. Then there exists $u \in K$ such that $D \subseteq D[u]$ does not satisfy GD.*

PROOF. Since D is not Prüfer, there exists a maximal ideal M such that D_M is not a valuation domain. We note if $D \subseteq D[u]$ satisfy GD then so should $D_M \subseteq D[u]_{D-M} = D_M[u]$. Now, since D_M is a PVMD by COROLLARY 11, we apply COROLLARY 15 to get the result.

A noetherian domain being an FC domain we have,in the light of the above results, a simpler proof of the following well known result.

COROLLARY 17. *An integrally closed noetherian domain* D *is a Krull domain.*

PROOF. We start with the knowledge that D is a PVMD. By COROLLARY 14, and in view of the fact that D is noetherian, D_P is a discrete rank one valuation domain for every rank one prime ideal P of D. Further, since a rank 2 valuation domain is not noetherian, no rank 2 prime ideal of D is essential. Whence every rank one prime ideal P, of D, is a maximal t-ideal and so $D = \bigcap D_P$ where P ranges over rank one prime ideals. According to COROLLARY 12, every minimal prime of a principal ideal, of a PVMD, is essential. Further, D being noetherian, every principal ideal of D has only a finite number of minimal primes (cf Kaplansky [7] Theorem 88). Thus have we verified all the requirements for D to be a Krull domain.

An overring T of D is said to be *minimal* if there exists no ring between D and T. Papick [9] shows that if D is quasi-local, integrally closed and if it has a minimal overring T then $T = D_N$ where N is a prime ideal of D such that $N = ND_N$ and all non maximal prime ideals of D are contained in N ([9] Lemma 2.7).

COROLLARY 18. *Let* D *be a quasi-local PVMD. If* D *has a minimal overring* T *then* D *is a valuation ring.*

PROOF. In view of Lemma 2.7 of [9], $T = D_N$ where every non maximal prime ideal of D is contained in N. Consider $a \in M-N$ where M is the maximal ideal of D. Clearly M is minimal over aD and hence, by COROLLARY 12, is essential.

As an interesting special case of COROLLARY 18 we cite Theorem 2.8 of Papick [9] which states: Let D be coherent, quasi-local and integrally closed. If D has a minimal overring T, then D is a valuation ring.

It is interesting to note here that this result follows without any appeal to finite presentation of finitely generated ideals.

We conclude this note with some specific examples of integrally closed non FC domains. It must be said that the literature on multiplicative ideal theory abounds in examples of integrally closed integral domains which are not PVMD's. For example, rings of finite character which do not have any defining family consisting of essential valuation overrings. Moreover if D is integrally closed and not a PVMD then D[X] is integrally closed and it is not a PVMD.

Another set of examples comes from the notion of Schreier rings which were introduced by Cohn in [2]. Recall from [2] that, in D, an element x is primal if $x \mid ab$ in D implies $x = cd$ where $c \mid a$ and $d \mid b$. An integrally closed integral domain is a Schreier domain if each of its elements is primal. We note that in a Schreier domain an irreducible element is prime. In view of this, it is easy to establish that a Schreier PVMD is a GCD domain (cf [5] Ex. 7, p. 430). Thus a Schreier ring which is not a GCD domain is an example of a non FC integrally closed integral domain. Finally, to this we may add the following corollary.

COROLLARY 19. *A Schreier ring D is a GCD domain if and only if it is a FC domain.*

In the following we give an example of a Schreier domain which is not a GCD domain.

EXAMPLE 20. Let D be a valuation domain of rank greater than 1 and let $S = \{ s^i \}_{i=0}^{\infty}$ where $D[s^{-1}] \neq K$. Then according to [3], $D + XD_S[X] = \{ v_0 + \sum_{i=1} v_i X^i \mid v_0 \in D \text{ and } v_i \in D_S \}$ is a Schreier domain. Further, in view of Example 1.4 of [3], $D + XD_S[X]$ is not a GCD domain. In particular, if $p \in D$ such that pD_S is a proper principal ideal then p and X do not have a GCD and it is easy to show that $(p) \cap (X)$ is generated by $\{ pX/t \mid t \in S \}$.

REMARK 21. Having seen that an integrally closed FC domain is a PVMD, one may ask if the integral closure of a FC domain should be a PVMD. This question remains open.

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