Star operations of finite character induced by rings of fractions

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Throughout this article D is an integral domain with quotient field K. We use $F(D)$ to denote the set of nonzero fractional ideals of D. Let $\{R_\alpha\}_{\alpha\in I}$ denote a family of *overrings* of D (rings between D and K). We say that $\{R_\alpha\}_{\alpha\in I}$ *defines* D if $D = \cap_\alpha R_\alpha$. Given that $\{R_\alpha\}_{\alpha\in I}$ defines D the function $*$ on $F(D)$ defined by $A \mapsto A^* = \cap AR_\alpha$ is a star operation *induced* by $\{R_\alpha\}_{\alpha\in I}$. Recall that a star operation is a function $*$ on $F(D)$ such that for $a \in K\backslash\{0\}$ and $A,B \in F(D)$ we have (1) $(aD)^* = aD$, $(aA)^* = aA^*$ (2) $A \subseteq A^*$, $A \subseteq B$ implies $A^* \subseteq B^*$ and (3) $(A^*)^* = A^*$. An ideal $A \in F(D)$ is called a $*$-ideal if $A^* = A$ and a $*$-ideal of finite type if there is there is a finitely generated ideal $B \in F(D)$ such that $A = B^*$. A star operation $*$ is said to be of finite character if for each $A \in F(D)$, $A^* = \cup\{F^*$: where $0 \neq F$ is a finitely generated subideal of $A\}$. Star operations of finite character have some interesting properties. For instance if $*$ is of finite character then every integral $*$-ideal A of D is contained in a maximal $*$-ideal M, which is necessarily a prime ideal. The reader may consult sections 32 and 34 of Gilmer's book [Gil] for a quick review of star operations. It seems pertinent, however, to introduce the reader to some of the notions and star operations that we intend to use in this article. Two of the very important star operations are the v and t-operations and their definitions depend upon the notion of the inverse of an $A \in F(D)$, which is $A^{-1} = \{x \in K : xA \subseteq D\}$. It is easy to see that for $A \in F(D)$ we have $A^{-1}, A_v = (A^{-1})^{-1} \in F(D)$ and $A_t = \cup\{F_v : 0 \neq F$ is a finitely generated subideal of $A\} \in F(D)$. Obviously the t-operation is of finite character. An ideal $A \in F(D)$ is said to be $*$-invertible if there is $B \in F(D)$ such that $(AB)^* = D$; in this case $B^* = A^{-1}$ and if $*$ is of finite character then A^{-1} is a v-ideal of finite type. An integral domain D is called a Prufer v-multiplication domain (PVMD) if each finitely generated $A \in F(D)$ is t-invertible. It can be shown that D is a PVMD if and only if $aD \cap bD$ is t-invertible for each pair $a,b \in D^* = D\backslash\{0\}$ [Z, putting]. One aim of this note is to indicate and use some of the circumstances in which a defining family of overrings $\{R_\alpha\}$ of D induces a star operation of finite character, especially when each of R_α is a ring of fractions of D. We will focus more on two types of defining families for D: (1) $\{R_\alpha\}$ where each R_α is a ring of fractions of D and $D = \cap R_\alpha$ is of finite character and (2) For a given star operation $*$ of finite character the defining family is $\{D_M\}$ where M ranges over the maximal $*$-ideals of D. In both cases, as we shall see in the sequel the star operation induced by the defining family is of finite character. One aim of this note is to prove, study the consequences and the variations of, the following result: $(*)$ If D has a defining family $\{D_{S_\alpha}\}$ of rings of fractions of D that induces a finite character star operation on D and if each of D_{S_α} is a PVMD then so is D. It is easy to see that GCD domains and Prufer domains are all special cases of PVMDs. We shall indicate that if in $(*)$ above, we replace "PVMD" by "GCD domain" the result may still be just a PVMD. We study the conditions under which a locally finite intersection D of its GCD rings of fractions is actually a GCD domain. Call D an almost GCD domain if for each pair of nonzero elements $a,b \in D$ there is a positive integer $n = n(a,b)$ such that $a^nD \cap b^nD$ is principal, if $n = 1$ for each pair a,b then D reduces to a GCD domain. Recall that D is a Generalized GCD (GGCD) domain if for each pair $a,b \in D^*$ we have $aD \cap bD$ invertible. Let us also call D an almost GGCD domain

(Almost Generalized GCD domain) if for each pair $a,b \in D^*$ we have some natural number n such that $a^nD \cap b^nD$ is invertible. As a variation of $(*)$ we study the conditions under which a locally finite intersection D of almost GCD (almost GGCD) rings of fractions is almost GCD (resp. almost GGCD). We also study $(*)$ where each of the D_{S_α} is a localization at a maximal ideal. We also introduce and study integral domains that we prefer to call almost PVMD's (APVMD's). These are domains in which for each pair $x,y \in D^*$ there is a natural number $n = n(x,y)$ such that $x^nD \cap y^nD$ is t-invertible. We also study domains that are locally finite intersections of fraction rings that are AGCD (resp. APVMD).

1 Locally finite intersections of rings of fractions.

Let us call D a finite conductor (FC) domain if $aD \cap bD$ is a finitely generated ideal of D for each pair $a,b \in D^*$. Let us also call D quasi coherent if for each pair $a,b \in D^*$, $aD \cap bD$ is a v-ideal of finite type.

Lemma 1. Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. If each of D_{S_α} is a FC domain then D is quasi coherent.

Proof. Because each of D_{S_α} is FC and because $(aD \cap bD)D_{S_\alpha} = aD_{S_\alpha} \cap bD_{S_\alpha} = AD_{S_\alpha}$ where A can be assumed to be a finitely generated ideal contained in $aD \cap bD.$ Because of the locally finite condition $(aD \cap bD)D_{S_\alpha} \neq D_{S_\alpha}$ for only a finite number of D_{S_α}, say $D_{S_{\alpha 1}}$, $D_{S_{\alpha 2}}, \ldots,$ $D_{S_{\alpha n}}$. Setting $aD_{S_{\alpha i}} \cap bD_{S_{\alpha i}} = A_iD_{S_{\alpha i}}$ we note that $A = \cup A_i \subseteq aD \cap bD.$ Next, if $AD_{S_\alpha} \neq D_{S_\alpha}$ for some $\alpha's$ other than αi then there are only a finite such $\alpha's.$ Say $AD_{S_{\alpha j}} \neq D_{S_{\alpha j}}$ $(j = 1,2,\ldots,m).$ But then for each j there exists $h_j \in aD \cap bD$ such that $h_jD_{S_{\alpha j}} = D_{S_{\alpha j}}.$ Let $B = (h_1,h_2,\ldots h_m) \cup A.$ Then $BD_{S_\alpha} \neq D_{S_\alpha}$ precisely when $\alpha = \alpha 1,\ldots,\alpha n.$ Note that as $B \subseteq aD \cap bD$ we have $BD_{S_\alpha} \subseteq (aD \cap bD)D_{S_\alpha}$ and so $BD_{S\alpha} \subseteq (aD \cap bD)D_{S_\alpha}$for all $\alpha.$ Next as $(aD \cap bD)D_{S_{\alpha i}} = A_iD_{S_{\alpha i}} \subseteq BD_{S_{\alpha i}}$we have $(aD \cap bD)D_{S_{\alpha i}} = BD_{S_{\alpha i}}$ for all αi where $i = 1,2,\ldots,n.$ Thus we have $(aD \cap bD)D_{S_\alpha} = BD_{S_\alpha}$ for all $\alpha.$ Consequently $aD \cap bD = \bigcap_\alpha (aD \cap bD)D_{S_\alpha} = \bigcap_\alpha BD_{S_\alpha} = B^\omega$ where ω is the star operation induced by $\{D_{S_\alpha}\}.$ But since $aD \cap bD$ is a v-ideal we conclude that $aD \cap bD = (aD \cap bD)_v = (B^\omega)_v = B_v.$

We can do a bit better than the above lemma.

Lemma 2. Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. Also let $a,b \in D^*.$ If $aD_{S_\alpha} \cap bD_{S_\alpha}$ is a v-ideal of finite type for each α, then $aD \cap bD$ is a v-ideal of finite type.

Proof. Suppose that for each α, $(aD \cap bD)D_{S_\alpha} = aD_{S_\alpha} \cap bD_{S_\alpha} = (AD_{S_\alpha})_v$. We can assume A is a finitely generated ideal contained in $aD \cap bD.$ Because of the locally finite condition $(aD \cap bD)D_{S_\alpha} \neq D_{S_\alpha}$ for only a finite number of D_{S_α}, say $D_{S_{\alpha 1}}$, $D_{S_{\alpha 2}}, \ldots,$ $D_{S_{\alpha n}}$. (Since either a or b or both can be a nonunit in only a finite number of D_{S_α}.) Setting $aD_{S_{\alpha i}} \cap bD_{S_{\alpha i}} = (A_iD_{S_{\alpha i}})_v$ we note that $A = \cup A_i \subseteq aD \cap bD.$ Next, if $AD_{S_\alpha} \neq D_{S_\alpha}$ for some $\alpha's$ other than α_i then there are only a finite such $\alpha's.$ Say $AD_{S_{\alpha j}} \neq D_{S_{\alpha j}}$ $(j = 1,2,\ldots,m).$ But then for each j there exists $h_j \in aD \cap bD$ such that $h_jD_{S_{\alpha j}} = D_{S_{\alpha j}}.$ Let $B = ((h_1,h_2,\ldots h_m) \cup A).$ Then $BD_{S_\alpha} \neq D_{S_\alpha}$ precisely when $\alpha = \alpha 1,\ldots,\alpha n.$ Note that as $B \subseteq aD \cap bD$ we have $BD_{S_\alpha} \subseteq (aD \cap bD)D_{S_\alpha}$ and so $(BD_{S\alpha})_v \subseteq (aD \cap bD)D_{S_\alpha}$for all $\alpha.$ Next as $(aD \cap bD)D_{S_{\alpha i}} = (A_iD_{S_{\alpha i}})_v \subseteq (BD_{S_{\alpha i}})_v$ we have $(aD \cap bD)D_{S_{\alpha i}} = (BD_{S_{\alpha i}})_v$ for all α_i where $i = 1,2,\ldots,n.$ Thus we have $(aD \cap bD)D_{S_\alpha} = (BD_{S_\alpha})_v$ for all $\alpha.$ Now Lemma 2 of [Z, two charact.] says that if a domain R

is defined by a family $\{R_\alpha\}$ of overrings of R and if A is a fractional ideal of D then $A^{-1} = \bigcap_\alpha (R_\alpha :_K AR_\alpha) = \bigcap_\alpha (AR_\alpha)^{-1}$. This leads to the conclusion that $(aD \cap bD)^{-1} = \bigcap_\alpha ((aD \cap bD)D_{S_\alpha})^{-1} = \bigcap_\alpha ((BD_{S_\alpha})_v)^{-1} = \bigcap_\alpha (BD_{S_\alpha})^{-1} = \bigcap_\alpha B^{-1}D_{S_\alpha} = B^{-1}$. (Note here that $(BD_{S_\alpha})^{-1} = B^{-1}D_{S_\alpha}$ because B is finitely generated.) Having shown that $(aD \cap bD)^{-1} = B^{-1}$ we conclude that $(aD \cap bD) = B_v$ and that $aD \cap bD$ is a v-ideal of finite type.

Lemma 3. (a). Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$. Let $a,b \in D^*$ such that $aD \cap bD$ is of finite type then $aD \cap bD$ is t-invertible if and only if $aD_{S_\alpha} \cap bD_{S_\alpha}$ is t-invertible for each α.

(b) Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. Let $a,b \in D^*$ such that $aD_{S_\alpha} \cap bD_{S_\alpha}$ is t-invertible for each α. Then $aD \cap bD$ is t-invertible.

Proof. (a). Recall that $aD \cap bD$ is t-invertible if and only if (a,b) is t-invertible if and only if $(a,b)^{-1}$ is of finite type and $((a,b)(a,b)^{-1})^{-1} = D$. Now recall from Lemma 2 of [Z, two charact.] that $((a,b)(a,b)^{-1})^{-1} = \bigcap_\alpha ((a,b)(a,b)^{-1}D_{S_\alpha})^{-1} = \bigcap_\alpha D_{S_\alpha} = D$, because $aD_{S_\alpha} \cap bD_{S_\alpha}$ and hence $(a,b)D_{S_\alpha}$ is t-invertible.

(b). Recall that a t-invertible t-ideal is a v-ideal of finite type and that a v-ideal is a t-ideal. So by Lemma 2, $aD \cap bD = B_v$ for some finitely generated ideal B. Now use part (a).

An integral domain D is called a v-domain if for each pair $a,b \in D^*$ we have $((a,b)(a,b)^{-1})_v = D$ or equivalently $((a,b)(a,b)^{-1})^{-1} = D$. Let $*$ be a star operattion. Call D a $*$-multiplication domain if for each pair $a,b \in D^*$ we have $((a,b)(a,b)^{-1})^* = D$. Indeed as $((a,b)(a,b)^{-1})^* = D$ implies $(((a,b)(a,b)^{-1})^*)_v = D$ we conclude that for any star operation $*$ a $*$-multiplication domain is indeed a v-domain. If the star operation $*$ is of finite character the corresponding $*$-multiplication domain is known as a Prufer $*$-multiplication domain. Prufer $*$-multiplication domains which are often denoted as P$*$MD's were first studied by Houston, Malik and Mott in [HMM] where it was shown that a P$*$MD is indeed a PVMD.

Corollary 4 (to the proof of Lemma 3). Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that each D_{S_α} is a $*$-multiplication domain. Then D is a v-domain.

Proof. For each pair $a,b \in D^*$, note that $((a,b)(a,b)^{-1})^{-1} = \bigcap_\alpha ((a,b)(a,b)^{-1}D_{S_\alpha})^{-1} = \bigcap_\alpha D_{S_\alpha} = D$.

Proposition 5. Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. Then D is a PVMD if and only if each of D_{S_α} is a PVMD. Consequently if $D = D_{S_1} \cap D_{S_2} \cap \ldots \cap D_{S_n}$ then D is a PVMD if and only if each of D_{S_i} is a PVMD.

Proof. Suppose that each of D_{S_α} is a PVMD. Recall that a PVMD is a v-domain such that $aD \cap bD$ is of finite type for each pair $a,b \in D^*$. Now apply Corollary 4 to conclude that D is a v-domain and use Lemma 3 to conclude that $aD \cap bD$ is of finite type. For the converse recall that each quotient ring of a PVMD is again a PVMD.

This proposition extends a number of known results. Among the first few results that attracted this author's attention towards PVMD's was the proof by Griffin [Gr, RKT] that a ring of Krull type is a PVMD. Recall that an integral domain D is a ring of Krull type if D is a locally finite intersection of essential valuation domains. Here a valuation overring V of D is essential if $V = D_P$ for some (nonzero) prime ideal P. While the description makes it clear, it seems OK to include it as a part of a corollary for the record. Let us note that a Prufer domain (every finitely generated nonzero ideal is invertible) is obviously a PVMD and that a GCD domain is a PVMD too and so is a Bezout domain (every f.g. ideal is principal.)

Corollary 6. Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. Then D is a PVMD if each of the D_{S_α} is any of the following: Bezout, Prufer, GCD, P∗MD, valuation domain, discrete rank one valuation domain. In particular, a ring of Krull type is a PVMD.

Another useful property of locally finite families of overrings $\{R_\alpha\}$ that define a domain D is that the star operation $*$ iduced by $\{R_\alpha\}$ is of finite character [(5) Theorem 2 DDAnderson, Star Ops. Ind. overrings]. (In this article of Anderson's, the statement is not proved, but the reader may follow the procedure of the proof of (6) Theorem 1 of the same article.) Let us also recall that if $*$ is a finite character star operation then for each nonzero ideal A of D we have $A^* \subseteq A_t$. These observations lead directly to the following theorem.

Theorem 7. Let $\{R_\alpha\}$ be a defining family of overrings of D and suppose that $\{R_\alpha\}$ is of finite character. If A is a proper t-ideal of D then for at least one α, $AR_\alpha \neq R_\alpha$.

Proof. Let $*$ be the star operation induced by $\{R_\alpha\}$. Then by [(5) Theorem 2 DDA] $*$ is of finite character and so $A^* \subseteq A_t = A$.

Corollary 8. Let $D = \cap\{D_{S_\alpha}$ where S_α are multiplicative subsets of $D\}$ and suppose that the intersection is locally finite. If A is a t-ideal then $A \cap S_\alpha = \phi$ for some α.

Corollary 9. Let $D = \cap\{D_{P_\alpha}$ where $\{P_\alpha\}$ is a family of prime t-ideals of $D\}$ and suppose that the intersection is locally finite. Then every maximal t-ideal M is equal to P_α for some α. If moreover the P_α are mutually incomparable then $\{P_\alpha\}$ is precisely the set of maximal t-ideals of D.

Proof. Let P be a maximal t-ideal. Then, by Theorem 7, $P \subseteq P_\alpha$ for some α. But since P is a maximal t-ideal we have $P = P_\alpha$. Next let P_α be a member of $\{P_\alpha\}$ then P_α is contained in some maximal t-ideal M, which by the above reasoning is contained in some $P_\beta \in \{P_\alpha\}$. But since distinct members of $\{P_\alpha\}$ are mutually incomparable we must have $\alpha = \beta$.

Recall from [Z, two charact.] an integral domain D is non-t-local (has more than one maximal t-ideals) if and only if there is a set of nonunits $x_1, x_2, \ldots x_n$ such that $(x_1, x_2, \ldots x_n)_v = D$ if and only if $D = \cap D_{S_i}$ where each of the S_i is a multiplicative set generated by x_i for $i = 1, 2, \ldots, n$. Note that D has more than one maximal t-ideals if and only if there is a set of nonunits $x_1, x_2, \ldots x_n \in D$ such that $(x_1, x_2, \ldots x_n)_v = D$. Let us call such a domain non-t-local.

Corollary 10. Given that D is a non-t-local domain then D is a PVMD if and only for every multiplicative set S generated by a nonzero nonunit we have D_S a PVMD.

Now the question is: If in Corollary 8, the D_{S_α} are consistently say GCD must D be GCD? The answer is no, because there are non t-local Krull domains (that happen to a special case of the rings of Krull type) which are not UFD's and hence not GCD domains. However as shown in [Z, Res. Gilmer] in certain situations a locally finite intersection of

GCD rings of fractions may turn out to be very close to GCD domains. Here we present a slightly more general set of results than [Z, Res. Gilmer].

Proposition 11. (a). Suppose that for some $a, b \in D^*$ we have $aD \cap bD$ of finite type. Then $aD \cap bD$ is invertible if and only if $aD_M \cap bD_M$ is principal for each maximal ideal M of D.

(b). Let $\{D_{P_\alpha}\}$ be a family of localizations at primes of D such that $D = \cap D_{P_\alpha}$ is a locally finite intersection. Then for any $a, b \in D^*$ $aD \cap bD$ is invertible if and only if $aD_M \cap bD_M$ is principal for every maximal ideal M of D.

Proof. For part (a) note that locally principal is flat and flat of finite type is invertible [Z, Flatness] and for the converse note that invertible is locally principal. For part (b) use Lemma 2 to see that $aD \cap bD$ is of finite type and then use part (a).

Call D a generalized GCD (GGCD) domain if the inverse of every finitely generated nonzero ideal of D is invertible. Parts (a) and (b) of Proposition 11 lead directly to the following propositions.

Corollary 12. A domain D is a GGCD domain if and only if D a locally GCD domain and each pair $a, b \in D^*$ $aD \cap bD$ is of finite type. Consequently D is a GCD domain if and only if (a) D is locally GCD (b) for each pair $a, b \in D^*$ $aD \cap bD$ is of finite type and (c) $Pic(D) = 0$.

Proof. Use Proposition 11 part (a).

Corollary 13. A domain D is a GGCD domain if and only if D is a PVMD that is locally a GCD domain.

Proof. Exercise.

Corollary 14. Let D be a domain and let $\{D_{P_\alpha}\}$ be a family of localizations of D at primes of D such that $D = \cap D_{P_\alpha}$ is a locally finite intersection. Then D is a GGCD domain if and only if D is a locally GCD domain.

Proof. Use Proposition 11 part (b).

Corollary 15. Let D be a ring of Krull type. Then D is a GGCD domain if and only if D is locally a GCD domain.

Proof. Exercise.

Call an integral domain D an almost GCD (AGCD) domain if for each pair $a, b \in D^*$ there is a (least) natural number $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal. Clearly if for each $a, b \in D^*$ we have $n(a, b) = 1$ then we conclude that D is a GCD domain. In other words a GCD domain is an AGCD domain. These domains were introduced in [Z, almost factoriality] and studied in more detail in [AZ, Almost Bezout]. For our purposes we recall .....

Theorem 13. Let D be an integral domain and let $*$ be a finite character star operation induced by a defining family of D consisting of quotient rings of D. Then the following are equivalent:

1 For each pair $f, g \in K[X]\backslash\{0\}$ $(A_{fg})^* = (A_f A_g)^*$, where X is an indeterminate over D and $K = qf(D)$ and A_f denotes the content of f

2 D is a $P * MD$.

Proof. Let $\{D_\alpha\}_{\alpha\in I}$ be a family of quotient rings of D such that $D = \cap D_\alpha$, let $*$ be defined by $A \mapsto A^* = \cap AD_\alpha$ and suppose that $*$ is of finite character.

1 ⇒ 2 For each α, $A^*D_\alpha = AD_\alpha$. So, for each α the equation $(A_{fg})^* = (A_fA_g)^*$ becomes $A_{fg}D_\alpha = A_fA_gD_\alpha$. Since for each pair $f,g \in D[X]\backslash\{0\}$ we can have $A_{fg}D_\alpha = (A_{fg})^*D_\alpha = (A_fA_g)^*D_\alpha = A_fA_gD_\alpha$ we conclude from a well known result that D_α is a Prufer domain for each α (see 28.6 of Gilmer's Multiplicative Ideal Theory, Marcel Dekker, 1972). Now, to show that D is a $P*MD$ we show that for each finitely generated nonzero fractional ideal A we have $(AA^{-1})^* = D$. This is easy since $(AA^{-1})^* = \cap AA^{-1}D_\alpha = \cap AD_\alpha A^{-1}D_\alpha = \cap AD_\alpha(AD_\alpha)^{-1}$ (because D_α is a quotient ring of D and A is finitely generated). Further since D_α is Prufer for each α and A is finitely generated we have, for each α, $AD_\alpha(AD_\alpha)^{-1} = D_\alpha$. But then $(AA^{-1})^* = \cap AA^{-1}D_\alpha = \cap D_\alpha = D$. Now as $*$ is of finite type and $(AA^{-1})^* = D$ for each finitely generated ideal A we conclude that D is a $P*MD$.

2 ⇒ 1 Suppose that D is a $P*MD$ then for each finitely generated nonzero fractional ideal A of D we have $A^* = A_v$. (Since every $*$-invertible $*$-ideal is a v-ideal.) Now for any $*$-operation of finite character a $P*MD$ is a PVMD and so integrally closed. Thus for each pair $f,g \in K[X]\backslash\{0\}$ we have $(A_{fg})_v = (A_fA_g)_v$ (see 34.8 of Gilmer's Multiplicative Ideal Theory) but since both A_{fg} and A_fA_g are finitely generated we have $(A_{fg})_v = (A_{fg})^*$ and $(A_fA_g)_v = ((A_fA_g)^*$ and this gives the required equality.

Remark 14. The proof of 2 ⇒ 1 shows that for D any $P*MD$ and for any $f,g \in K[X]\backslash\{0\}$ we have $(A_{fg})^* = (A_fA_g)^*$.

Now the following corollary shows that being a finer star operation the w-operation is more potent than the t-operation.

Corollary 15. An integral domain D is a PVMD if and only if for each pair $f,g \in D[X]\backslash\{0\}$ we have $(A_{fg})_w = (A_fA_g)_w$.

The proof depends upon the fact that w is a finite character star operation induced by the defining family $\{D_M\}$ of D where M ranges over the maximal t-ideals of D. (Each maximal t-ideal of D is indeed a maximal w-ideal of D and vice versa.)

On the other hand, because for each finitely generated non-zero ideal A we have $A_t = A_v$, $(A_{fg})_t = (A_fA_g)_t$, for all $f,g \in D[X]\backslash\{0\}$ if and only if D is integrally closed.

This raises the question: Why can't we do away with the t-operation altogether? The answer is simple. Being the coarsest star operation of finite character the t-operation has its position fixed in the the realm of star operations. But we should look for ways in which we can efficiently use both the w- and the t-operations without the confusion about which to use where. I recall that Wang Fanggui actually wrote a whole article on $PwMD$'s only to find that a $PwMD$ is precisely a PVMD.

The following material is my redo of a part of Anderson and Cook's paper [AC]

Now here is a question: Let $Z(D) = \{J \in F(D) : J^{-1} = D\} = \{J \in F(D) : J_v = D\}$ and define $\overline{w}$ on $F(D)$ by $A \mapsto A_{\overline{w}} = \{x \in K : \exists_{J\in Z(D)}(xJ \subseteq A)\}$. Is $\overline{w}$ a star operation? If it is how close to the v-operation is the $\overline{w}$-operation?

Proof that $\overline{w}$ is a bonafide star operation distinct from the w and the t-operation.

1. $Z(D)$ is a monoid with unit D. Obvious

2. $A_{\overline{w}}$ is a fractional ideal for each $A \in F(D)$.

Let $x, y \in A_{\overline{w}}$. Then there are $J, K \in Z(D)$ such that $xJ \subseteq A$ and $yK \subseteq A$. Now $(x+y)JK \subseteq xJK + yJK \subseteq A$. Obviously $-yK \subseteq A$. Next let $d \in D$. Then $dx \in A_{\overline{w}}$ because $dxJ = d(xJ)$ where $xJ \subseteq A$ and so is $dxJ \subseteq A$.

3. $\overline{w}$ is a star operation.

Let $x \in K\backslash\{0\}$ and let $A, B \in F(D)$.

(i). $(x)_{\overline{w}} = (x)$ and $(xA)_{\overline{w}} = xA_{\overline{w}}$.

Obviously as $xD \subseteq (x)$ we have $(x) \subseteq (x)_{\overline{w}}$. Next since $y \in (x)_{\overline{w}} \Rightarrow \exists_{J \in Z(D)}(yJ \subseteq (x))$. But then $(yJ)_v \subseteq (x)$ which forces $y \in (x)$ and so $(x)_{\overline{w}} \subseteq (x)$. Now consider
$(xA)_{\overline{w}} = \{y \in K : yJ \subseteq xA$ for some $J \in Z(D)\}$
$= \{y \in K : \frac{y}{x} \in A_{\overline{w}}\} = \{y \in K : y \in xA_{\overline{w}}\} = xA_{\overline{w}}$. So, $(xA)_{\overline{w}} = xA_{\overline{w}}$.

(ii). $A \subseteq A_{\overline{w}}$ and $A \subseteq B$ implies that $A_{\overline{w}} \subseteq B_{\overline{w}}$.

That $A \subseteq A_{\overline{w}}$ follows from the fact that $D \in Z(D)$. Let $A \subseteq B$ and let $x \in A_{\overline{w}}$ then there exists $J \in Z(D)$ such that $xJ \subseteq A \subseteq B$. Thus $x \in A_{\overline{w}}$ implies that $x \in B_{\overline{w}}$.

(iii). $A_{\overline{w}} = (A_{\overline{w}})_{\overline{w}}$.

That $A_{\overline{w}} \subseteq (A_{\overline{w}})_{\overline{w}}$ follows from (ii). For the reverse containment let $x \in (A_{\overline{w}})_{\overline{w}}$. Then there exists $J \in Z(D)$ such that $xJ \subseteq A_{\overline{w}}$. Now for each $y \in J$ we have $xy \in A_{\overline{w}}$. So, there exists for each $y \in J$ a J_y such that $xyJ_y \subseteq A$. Now let $B = \sum_{y \in J} yJ_y$ where $J_y \in Z(D)$ such that $xyJ_y \subseteq A$. Then since $y \in J \subseteq D$ and since $J_y \subseteq D$ we conclude that $\sum_{y \in J} yJ_y$ is a nonzero ideal of D. Next $(\sum_{y \in J} yJ_y)_v = (\sum_{y \in J} y(J_y)_v)_v = (\sum_{y \in J} y)_v = J_v = D$. But $xB = x\sum_{y \in J} yJ_y = \sum_{y \in J} xyJ_y \subseteq A$. So for each $x \in (A_{\overline{w}})_{\overline{w}}$ we have a $B \in Z(D)$ such that $xB \subseteq A$ and this forces $(A_{\overline{w}})_{\overline{w}} \subseteq A_{\overline{w}}$.

4. For each $J \in Z(D)$ we have $J_{\overline{w}} = D$.

Note that by 3(i) $D_{\overline{w}} = D$. (Independently $D \subseteq D_{\overline{w}}$ is obvious. Now $x \in D_{\overline{w}}$ implies that there is $J \in Z(D)$ such that $xJ \subseteq D$. But then $(xJ)_v \subseteq D$ which forces $x \in D$). Next since $J \subseteq D$ we have $J_{\overline{w}} \subseteq D$. Next let $x \in D$ then as J is an ideal $xJ \subseteq J$ and this forces $x \in J_{\overline{w}}$.

5. $\overline{w}$ is not a finite type star operation and so $\overline{w} \neq w$

Let (V, M) be a non-discrete rank one valuation domain. Then $M^{-1} = V$ and $Z(V) = \{V, M\}$. If $\overline{w}$ were of finite type then $\overline{w}$ must have a maximal (nonzero) $\overline{w}$-ideal which must be prime. But there is only one nonzero prime M which is in $Z(V)$ and so $M_{\overline{w}} = D$.

6. $\overline{w}$ is not the v-operation

Note that over a Noetherian domain the $\overline{w}$-operation and the w-operation coincide. Now let (D, M) be a one-dimensional local (Noetherian) domain in which every ideal is not a v-ideal, i.e., that is not a Gorensterin domain. Note that every ideal in D is a $\overline{w}$-ideal but not every nonzero ideal of D is a v-ideal.

Now, the $\overline{w}$-operation is known and can be found in Anderson and Cook's paper, though their proofs may be a bit involved. But my purpose here is to check out the domains in which every ideal is a $\overline{w}$-ideal. Clearly w is finer than $\overline{w}$ which in turn is finer than the v-operation. A dw-domain (Mimouni's terminology) would be interesting.

If you are intrigued by "Theorem 13" let me know. When I got this idea, I looked if there was something close enough on my computer. I found a piece in which I had discussed domains that are locally finite intersections of rings of fractions. It has some interesting results but linking the two would be a big job. (Wrote this when I sent this material to Chang.)

References

[DDA] D.D. Anderson, Star Operations induced by overrings, Comm. Algebra 16(1988) 2535-2553.

[AA] D.D. Anderson and D.F. Anderson,”Generalized GCD domains” Comment. Math. Univ. St. Pauli 28(1979) 215-221.

[AC] D.D. Anderson and S. Cook, Two star-operations and their induced lattices. Comm. Algebra 28 (2000), no. 5, 2461–2475.

[AZ, Almost Bezout], D.D. Anderson and M. Zafrullah, Almost Bezout domains,

[FJS] M. Fontana, P. Jara and E. Santos, Prüfer $*$-multiplication domains and semistar operations. J. Algebra Appl. 2 (2003), no. 1, 21–50.

[FM] Wang Funggui, and R. McCasland, On w- modules over strong Mori domains, Comm. Algebra 25 (1997), no. 4, 1285–1306

[G] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, 1972.

[Gr, RKT] M. Griffin, Rings of Krull type,

[Z, Res. Gilmer] M. Zafrullah, On a result of Gilmer,

[Z, almost factoriality]

[Z, Flatness] M. Zafrullah, Flatness and invertibility,

[Z, two charact.] M. Zafrullah, Two characterizations of Mori domains, Math Japonica 33 (4)(1988) 645-652.

[Z, putting], M. Zafrullah, Putting t-invertibility to use, in