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Chapter 20

PUTTING T-INVERTIBILITY TO USE

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0. INTRODUCTION

This article gives a survey of how the notion of t-invertibility has, in recent years, been used to develop new concepts that enhance our understanding of the multiplicative structure of commutative integral domains. The concept of t-invertibility arises in the context of star operations. However, in general terms a (fractional) ideal A, of an integral domain D, is t-invertible if there is a finitely generated (fractional) ideal $F \subseteq A$ and a finitely generated fractional ideal $G \subseteq A^{-1}$ such that $(FG)^{-1} = D$. In a more specialized context the notion of t-invertibility has to do with the t-operation which is one of the so called star operations. There seems to be no book other than Gilmer's [Gil] that treats star operations purely from a ring theoretic view point. But a lot has changed since Gilmer's book was published. So I have devoted a part of section 1. to an introduction to star operations, $*$-invertibility in general, and t-invertibility in particular.

While invertible ideals fare rather admirably in ring extensions, the behavior of t-invertible ideals is a bit "iffy". The search for extensions $D \subseteq R$, where D is a subring of the integral domain R, such that t-invertible ideals of D extend to t-invertible ideals of R showed first that if R is a quotient ring of D, then we have the desired property. This led to t-linked extensions of [DHLZ] and to extensions with the desired property where R is not necessarily an *overring.* (An overring of D is a ring between D and K.) The remainder of section 1. and the tail end of section 2. are devoted to such extensions.

In the earlier part of section 2., various ways of ascertaining the t-invertibility of an ideal are discussed. Various notions that arise mainly from t-invertibility or some other form of $*$-invertibility of some or all of their nonzero ideals are also discussed in section 2.. Of these, the most important are the Prüfer v-multiplication domains (PVMD's for short) and the notion of the t-class group. An integral domain whose nonzero finitely generated ideals are t-invertible is called a PVMD. For a general integral domain D, the set $Inv_t(D)$ of t-invertible t-idealsof D is a group, under what may be called t-product. It so happens that $Inv_t(D)$ contains as a subgroup the set $P(D)$ of principal nonzero fractional ideals of D. The *t-class group* is defined as the quotient group $Cl_t(D) = Inv_t(D)/P(D)$. The only close resemblance that the t-class group has with the usual divisor class group is that the members of $Inv_t(D)$ are all divisorial ideals. Otherwise, the divisor class group can be defined only for a restricted class of integral domains.

In sections 3. and 4., we show what t-invertibility can really do. To get the reader interested, a bit of introduction is in order. Traditionally the star operations were linked with intersections of valuation domains of valuations over a field. The reason: in a valuation domain a finitely generated ideal is principal. The study of valuations basically lets you do analysis in your algebra. That is why, in my opinion, Krull defined what later came to be known as a *Krull domain* as an integral domain D that is a locally finite intersection of discrete rank one valuation rings. A current and more polished definition of the same concept is: D is a Krull domain if (K1) D is a locally finite intersection of localizations at height one prime ideals and (K2) the localization at each height one prime is a discrete valuation ring (DVR). Let us consider a domain that satisfies (K1) alone and call it a *weakly Krull domain*(WKD). Then a natural set of questions is: (a) Does a WKD have any Krull like properties? (b) Are there any WKD's of interest? The answer to (b) is yes; Noetherian domains whose grade one primes are of height one are WKD's, and they are considered to be of general interest. Outlined in section 3. are characterizations of WKD's. These characterizations indicate that WKD's are strikingly similar to Krull domains (this answers (a)). So much so that the t-class group of a WKD has the same tight grip on the factorization properties of a WKD as the divisor class group has on those of a Krull domain. On the other hand, section 4. is a collection of instances where something less or something not quite related causes t-invertibility. This section also includes a discussion of a newly introduced star operation, the w-operation. It also includes a discussion of the domains that are called UMT domains [HZ 1], and of what are called H domains in [GV]. Finally, an interested reader would find, scattered here and there, some problems which to my knowledge are untouched yet. The notation and terminology are standard as in [Gil].

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1. STAR OPERATIONS

Let D be an integral domain with quotient field K, let $F(D)$ be the set of nonzero *fractional ideals* of D, and let $f(D)$ be the set of nonzero finitely generated fractional ideals of D. For the purposes of this article, we shall often refer to a fractional ideal as an "ideal" and we shall call a fractional ideal contained in D an *integral ideal.* As the title of the article indicates, we shall use the theory of star operations on an integral domain. For the sake of completeness, we include a description of star operations, and some related notions, below. For a more detailed study of star operations the reader may consult Gilmer [Gil], Griffin [Gri 1], Jaffard [Jaf], and [H-K] in that order. The reader may also note that Jaffard's treatment of ideal systems is in the frame-work of partially ordered groups and that of Halter-Koch's is via cancellative monoids and divisibility in them. We will mention results from [Gil] without any reference, and will provide necessary reference or proof for material that may not be found in [Gil].

A star operation is a function $A \mapsto A^*$ on $F(D)$ with the following properties:

If $A, B \in F(D)$ and $a \in K\backslash\{0\}$, then

(i) $(a)^* = (a)$ and $(aA)^* = aA^*$.

(ii) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

(iii) $(A^*)^* = A^*$.

We shall call A^* the $*$-*image* (or $*$-*envelope*) of A. An ideal A is said to be a $*$-*ideal* if $A^* = A$. Thus A^* is a $*$-ideal and of course every principal fractional ideal, including $D = (1)$, is a $*$- ideal for any star operation $*$. For all $A, B \in F(D)$ and for each star operation $*$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations define what is called $*$-*multiplication (or* $*$-*product).* If $\{A_\alpha\}$ is a subset of $F(D)$ such that $\cap A_\alpha \neq (0)$, then $\cap(A_\alpha)^* = (\cap(A_\alpha)^*)^*$. We may call this property the *intersection property.* Also, if $\{A_\alpha\}$ is a subset of $F(D)$ such that $\sum A_\alpha$ is a fractional ideal, then $(\sum A_\alpha)^* = (\sum A_\alpha^*)^*$; this may be called the sum property. Now, by the intersection property any nontrivial intersection of $*$- ideals is again a $*$-ideal. Thus for $a, b \neq 0$, $(a) : b = \{x \in D|\ xb \subseteq (a)\}$ is a $*$-ideal for any star operation $*$ because $(a) : b = (\frac{a}{b}) \cap D$ and so, for each $A \in F(D)$, is $A^{-1} = \{x \in K|xA \subseteq D\}$ $(= \cap_{a \in A-\{0\}}(\frac{1}{a}))$. On the particular side, if $A \in F(D)$ is a $*$-ideal for some star operation $*$ and $B \in F(D)$, then so is $A :_K B = \bigcap_{b \in B-\{0\}} \frac{1}{b}A$.

Define $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup\{F_v|\ 0 \neq F$ is a finitely generated subideal of $A\}$. The functions $A \mapsto A_v$ and $A \mapsto A_t$ on $F(D)$ are more familiar examples of star operations defined on an integral domain. A v-ideal is better known as a *divisorial* ideal. The identity function d on $F(D)$, defined by $A \mapsto A$, is another example of a star operation. There are of course many more star operations that can be defined on an integral domain D. But for any star operation $*$ and for any $A \in F(D)$, $A^* \subseteq A_v$. Some other useful relations are: For any $A \in F(D)$, $(A^{-1})^* = A^{-1} = (A^*)^{-1}$ and so, $(A_v)^* = A_v = (A^*)_v$. In fact, if μ and ν are two star operations such that for all $A \in F(D)$ $A^\mu \subseteq A^\nu$, then ν is said to be *coarser* than μ , denoted by $\mu \leq \nu$, and it is easy to establish that $(A^\nu)^\mu = A^\nu = (A^\mu)^\nu$. In this terminology then, the v-operation is the coarsest of all star operations on D. The reason for v being the coarsest is the fact that for each $A \in F(D)$, $A_v = \bigcap xD$ where $x \in K$ such that $A \subseteq xD$. The following observation often proves useful.

Observation A. Given that A is a nonzero integral ideal of D, then $A_v \neq D$ if, and only if, there exist $a, b \in D\backslash\{0\}$ such that $A \subseteq \frac{a}{b}D$, and $a \nmid b$.

The relation $\leq$ defined above is a partial order on the set $\mathcal{S}(D)$ of all star operations on D. A reader interested in this aspect of star operations may consult [AA]. Further, as $D = (1)^* = (1)$, we conclude that D is a $*$-ideal for every star operation $*$. Consequently, if $*$ is a star operation, then $A \in F(D)$ is an integral ideal if and only if A^* is.

A star operation $*$ is said to be of *finite character* if, for each $A \in F(D)$, $A^* = \bigcup\{F^* \mid 0 \neq F$ is a finitely generated subideal of $A\}$. Thus a star operation $*$ is of finite character if, and only if, for every member x of A^*, $x \in F^*$, where F is a nonzero finitely generated subideal of A. Obviously the t operation and the d operation are of finite character. In fact to each star operation $*$ we can associate a star operation $*_s$, such that for all $A \in F(D)$, $A^{*_s} = \bigcup\{F^* \mid 0 \neq F$ is a finitely generated subideal of $A\}$. Clearly, $A^{*_s} \subseteq A^*$ and if A is finitely generated, then $A^* = A^{*_s}$. By definition, $t = v_s$ and if A is finitely generated then $A_t = A_v$. It is easy to see that if $*$ is of finite character, then $A^* \subseteq A_t$ for all $A \in F(D)$. We may call $*_s$ the *$*$-companion of finite character*.

Given that $*$ is a star operation of finite character, an easy Zorn's Lemma argument shows that each proper integral $*$-ideal is contained in a maximal proper integral $*$-ideal and that such a maximal $*$-ideal is prime. Let us denote by $* - Max(D)$, the set of all maximal $*$-ideals of D. Then for each $A \in F(D)$, $A^* = \bigcap_{P \in *-Max(D)} A^* D_P$ [Gri 1]. Using the fact that D is a $*$-ideal for every star operation $*$, we conclude that $D = \bigcap_{P \in *-Max(D)} D_P$. Using Observation A, we make the following observation.

Observation B. If $*$ is a finite character star operation and if $P \in * - Max(D)$ is divisorial, then $P = (a) : b$ for some $a, b \in D\backslash\{0\}$.

As already mentioned, one classical method of getting star operations is via defining families of overrings of D. Here, by a defining family we mean a family $\{R_\alpha\}$ of overrings of D such that $D = \bigcap_\alpha R_\alpha$. Now if $\{R_\alpha\}$ is a defining family of overrings of D, the function $A \mapsto A^* = \bigcap_\alpha AR_\alpha$ on $F(D)$ is a star operation, which is said to be *induced* by $\{R_\alpha\}$. Indeed, if $*$ is induced by $\{R_\alpha\}$, then $A^*R_\alpha = AR_\alpha$. Traditionally, the overrings in the defining families were taken to be valuation domains. These days star operations induced by quotient rings, preferably localizations at primes, are in vogue. An interested reader may want to see Dan Anderson's useful paper [And] and references there. Every now and then new star operations are born. Of these, one is the so called w-operation introduced by Wang and McCasland [WM]. This operation can be defined as $A \mapsto A^w = \bigcap_{P \in t-Max(D)} AD_P$, and it is a finite character star operation. Anderson and S. Cook [AC] have recently shown that given any finite character star operation $*$ we can define $*_w$ by $A \mapsto A^{*_w} = \bigcap_{P \in *-Max(D)} AD_P$, that $*_w$ is of finite character, and that $*_w - Max(D) = * - Max(D)$. We shall revisit the w-operation in a later section. A defining family $\{R_\alpha\}$ of overrings of D is said to be of *finite character* if every nonzero nonunit of D is a nonunit in at most a finite number of R_α. A defining family of finite character induces a finite character star operation, see [And, Theorem 2] for an extremely general result.

A $*$- ideal A is said to be of *finite type* (or *$*$-finite*) if there is a finitely generated ideal B such that $A = B^*$ and $A \in F(D)$ is *strictly $*$-finite* if there is a finitely generated ideal $B \subseteq A$ such that $A^* = B^*$. This term was introduced in [Zaf 7], where it was indicated that confusing $*$-finite with strictly $*$-finite could cause problems. Strictly v-finite was called *quasi fini* by Querre in [Que 1].

Given $A \in F(D)$, we say that A is *$*$-invertible*, for some star operation $*$, if there is a $B \in F(D)$ such that $(AB)^* = D$. Multiplying this last equation by A^{-1} and applying $*$ to both sides, we get $A^{-1} = (A^{-1})^* = ((AB)^*A^{-1})^* = ((AB)A^{-1})^* = (AA^{-1}B)^* \subseteq B^*$. Next, as $AB^* \subseteq (AB^*)^* = (AB)^* = D$, we conclude that $B^* \subseteq A^{-1}$. Combining the two inclusions, we have that if for some star operation $*$, $(AB)^* = D$, then $B^* = A^{-1}$. Now if A is $*$-invertible, then so is A^{-1}, and consequently $A^* = A_v$. Now as $D = (AA^{-1})^* \subseteq (AA^{-1})_v \subseteq D$, we conclude that $*$-invertible implies v-invertible. In fact if μ is coarser than ν, then ν-invertible implies μ-invertible. Consequently an invertible $I \in F(D)$ is $*$-invertible for every star operation $*$.

If $*$ is a star operation of finite character and if $A \in F(D)$ is $*$-invertible, then $D = (AA^{-1})^* = \cup\{F^* | 0 \neq F$ is a finitely generated subideal of $AA^{-1}\}$, and so $D = (1) \subseteq F^*$ for some finitely generated $F \subseteq AA^{-1}$. Now it is easy to see that $F \subseteq GH$, where G is a finitely generated subideal of A, and H is a finitely generated subideal of A^{-1}. Next $D = F^* \subseteq (GH)^* \subseteq (AA^{-1})^* = D$ and from this it is easy to deduce that $G^* = A^*$ and $H^* = A^{-1}$. Consequently if $*$ is of finite character, then $(AA^{-1})^* = D$ implies that there is a finitely generated ideal $G \subseteq A$ and a finitely generated ideal $H \subseteq A^{-1}$ such that $G^* = A^*$ and $H^* = A^{-1}$.

Now suppose that $* = \rho_s$ for some star operation ρ. Then $A^{\rho_s} = G^{\rho_s} = G^{\rho}$ and applying ρ to these equations we get $A^{\rho} = G^{\rho}$ and similarly $H^{\rho} = A^{-1}$. Thus if $*$ is a ρ-companion of finite character and $A \in F(D)$ is $*$-invertible, then A is strictly ρ-finite and so is A^{-1}. Now as ρ is coarser than $*$, $(AA^{-1})^* = D$ implies that $(AA^{-1})^{\rho} = D$. Conversely, it is easy to see that if $(AA^{-1})^{\rho} = D$ and A and A^{-1} are strictly ρ-finite, then $(AA^{-1})^{\rho_s} = D$. The above considerations can be summed up in the following theorem.

Theorem 1.1. *Let $*$ be a star operation and let $A \in F(D)$. Then the following hold:*

(a). If A is $*$-invertible, then $A^* = A_v$ and A is v-invertible.

(b). If $*$ is of finite character, then A is $*$-invertible if and only if there are finitely generated ideals $G \subseteq A$, $H \subseteq A^{-1}$ such that $G^* = A^* = A_v$, $H^* = A^{-1}$, and $(GH)^* = D$. In particular, if $* = d$ and A is d-invertible, then we have the usual conclusion that A is a finitely generated v-ideal and the inverse is also a finitely generated v-ideal.

(c). An ideal A is $*_s$ invertible if, and only if, there are finitely generated ideals $G \subseteq A$ and $H \subseteq A^{-1}$ such that $(GH)^* = D$. In particular A is t-invertible if, and only if, there are finitely generated ideals $G \subseteq A$ and $H \subseteq A^{-1}$ such that $(GH)^{-1} = D$.

(d) A is $*_s$-invertible if and only if A is strictly v-finite, A^{-1} is v-finite and $(AA^{-1})^* = D$.

(e). If A is $*_s$-invertible, then A is t-invertible. Consequently, if $*$ is of finite character and A is $*$-invertible, then A is t-invertible.

Observation C. Every $*$-invertible $*$-ideal is divisorial.

There are several useful properties of t-invertible ideals. Of these I include a few that are more relevant to the topics discussed here.

Proposition 1.2. *Let $A \in F(D)$ be a t-invertible ideal. Then the following hold.*

(1). For any $B, C \in F(D)$, $AB \subseteq AC$ implies that $B_t \subseteq C_t$.

(2). If $B \in F(D)$ is such that $B \subseteq A_t$ then there is an integral ideal C such that $B_t = (AC)_t$. Conversely, if there is such an integral ideal C with $B_t = (AC)_t$, then $B \subseteq A_t$.

(3). If $B \in F(D)$ is t-invertible and for some $C \in F(D)$ $B_t = (AC)_t$, then C is t-invertible.

(4). If $B \in F(D)$ is t-invertible, then so is AB with inverse $(A^{-1}B^{-1})_t$.

(5). If $B \in F(D)$ is t-invertible, then there exists a t-invertible ideal C such that $C \subseteq A_t, B_t$.

(6). If $B \in F(D)$ is t-invertible and $A + B$ is t-invertible, then so is $A_t \cap B_t$.

(7). If $B \in F(D)$ is
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(2). $(AD_S)_v = (A_vD_S$

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(3). If A_v is of finite

(7). If $B \in F(D)$ is t-invertible and $A_t \cap B_t$ is t-invertible, then so is $A + B$.

Proof. (1)-(4) are straight-forward and so are left to the reader. For (5), find $d \in D \backslash \{0\}$ such that $dA, dB \subseteq D$. Then $C = dAB$ would do. For (6) and (7), note that $A_t = A_v$, $B_t = B_v$, and that A and B are t-invertible. Now $(A^{-1}B^{-1}(A+B))_v = (A^{-1}AB^{-1} + A^{-1}BB^{-1})_v = (B^{-1} + A^{-1})_v$, (because A and B are t-invertible) $= (A_v \cap B_v)^{-1}$. That is $(A_v \cap B_v) = (A^{-1}B^{-1}(A+B))^{-1}$. Now using (4) we have (6) and using (3) we have (7). □

Let us note that while in (6) t can be replaced by any star operation $*$, it is generally not the case for (7). For example, take $* = d$, $A = aD, B = bD$, where $a, b \in D \backslash \{0\}$ and D is a non-Bezout GCD domain. Then $aD \cap bD$ is (d-) invertible, being principal, whereas $aD + bD$ need not be invertible.

Let, as we have already done, $Inv_t(D)$ denote the set of all t-invertible t-ideals of D. Using (4) of the above proposition and the usual properties of fractional ideals we can show that $Inv_t(D)$ is a group under t-multiplication. Using (2), we can define a partial order $\leq$ on $Inv_t(D)$ by $A \leq B \Leftrightarrow$ there is an integral ideal $C \in Inv_t(D)$ such that $(AC)_t = B$, or equivalently $A \leq B$ if and only if $B \subseteq A$. Thus $(Inv_t(D), \leq, \star_t)$, where $\star_t$ denotes the t-multiplication, is a partially ordered group with the set $Integinv_t(D)$ of integral t-invertible t-ideals its positive cone. Next, recall that a partially ordered group $(G, \leq)$ is directed, if for each pair $a, b \in G$ there is an element $c \in G$ such that $a, b \leq c$. Using (5) above, and our definition of order, we sum up the above considerations in the following statement.

Corollary 1.3. *$(Inv_t(D), \leq, \star_t)$ is a directed partially ordered group.*

The notion of ideal systems, as introduced by Prüfer and Krull, had not taken any real shape when it was hijacked by Lorenzen [Lor] into partially ordered groups, where it really took its pre-Griffin shape. Griffin's work [Gri 1], [Gri 2], brought rings, essentially integral domains, into the picture. With rings came their usual questions, and one of them was that of localization. Aubert [Aub] produced a very general and very brief description of how an ideal system would fare under localization. Then appeared, as Lemma 4, in [Zaf 2], the following result.

Lemma 1.4. *Let A be a nonzero ideal of D and let S be a multiplicative set in D. If A is finitely generated, then*
(1). $(AD_S)^{-1} = A^{-1}D_S$.
(2). $(AD_S)_v = (A_vD_S)_v$.

Part(2) of this lemma related the v-operation in D_S with the v-operation in D. It was later improved in the proof of Corollary 1.6 of [MMZ] to:
(3). *If A_v is of finite type, then $(AD_S)_v = (A_vD_S)_v$.*

Then came Lemma 3.4 of [Kan]:

(4). *For any nonzero ideal A of D, $(AD_S)_t = (A_tD_S)_t$.*

We note that (4) is a decided improvement on (2), but it does not show the link between the v-operations of D and D_S. But this can be easily remedied by noting that the v-operation is coarser than the t-operation. Thus we have for any $A \in F(D)$

(5). $(AD_S)_v = (A_tD_S)_v$.

Some of the consequences of these results are: (i). If $\mathfrak{A}$ is a t-ideal of D_S, then so is $\mathfrak{A} \cap D$. (The proof consists of taking a finitely generated $A \subseteq \mathfrak{A} \cap D$ and noting that $A_vD_S \subseteq (AD_S)_v \subseteq \mathfrak{A}$) (ii). If A is strictly v- finite such that A^{-1} is of finite type also, then $A_vD_S = (AD_S)_v$ [BZ 2]. This result has two direct consequences:

(a) If A is a t-invertible t-ideal of D, then $((AA^{-1})D_S)_t = ((AA^{-1})_tD_S)_t = D_S$, by (4)above. Moreover, A and A^{-1} being of finite type, $AD_S = (AD_S)_v$. This means that every t-invertible t-ideal of D extends to a t-invertible t-ideal of D_S. Using this it can be shown that there is a homomorphism from $Cl_t(D)$ to $Cl_t(D_S)$ defined by $[A] \longrightarrow [AD_S]$ [AR, Proposition 2.2].

(b) If D is a *quasi-coherent* domain, i.e., a domain such that for every non-zero finitely generated ideal I, I^{-1} is of finite type, then the finite type v-ideals of D extend to, finite type v-ideals of D_S, and by Kang's result, (4) above, t-ideals of D extend to t-ideals of D_S. Quasi-coherent domains include *Mori* domains, integral domains with ACC on integral divisorial ideals (see [Rai] and [Nis 2]), *coherent* domains, and hence *Noetherian* domains.

The formulas, as stated in (1), (2), and (3) of Lemma 1.4, are deadly traps if you are not very careful. Needless to say that I was the first one to stumble (and of course I was not the last!). I thought that from $(AD_S)_v = (A_vD_S)_v$, for finitely generated ideals A, it followed that if P were a prime t-ideal of D then PD_S would be a prime t-ideal of D_S, and drew a conclusion that was pointed out to me as false by Joe Mott, from whom I have learned so much. This led to the construction of examples, in [Zaf 4] and in [Zaf 8], of integral domains R with prime t-ideals M such that MR_M is not a t-ideal, see also the paper by Gabelli and Roitman [GR]. Luckily, for most of the integral domains of interest such as Noetherian, coherent and quasi-coherent, the prime t-ideals extend to prime t-ideals of their rings of fractions. Domains with this property are called *well behaved* and they are characterized in [Zaf 8].

The rather sad aspect of part (2) of Lemma 1.4 is that, with some hindsight, it can be derived from Exercise 20, page 432 of [Gil]. However, it was the formula character of the statements in (1),(2),(3) that caught the fancy of researchers in the area. (For the purposes of organization, we shall consider statements (3)- (5) above as parts of Lemma 1.4). The result is that there are similar formulae being employed when the overring is flat [Ryk,

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Lemma 2.6] or a generalized quotient ring[Kan, Lemma 3.4], [Gab 2]. Then results similar to (1) and (2) of Lemma 1.4, but in totally different contexts, started appearing along with consequences similar to the ones mentioned above. The result is that a unified approach to these results seems to be necessary. This we shall do in a separate paper. For the present, we include in the next section a brief description of what is to come.

2. T-INVERTIBILITY, V-INVERTIBILITY AND CONCEPTS ARISING FROM THEM

We have already seen the definition of t-invertibility. Now we tackle the question of characterizing t-invertibility. That is, we shall look into characteristic confirmatory tests of t-invertibility. The first that comes to mind arises from the observation that if A is t-invertible, then $(AA^{-1})_t$ $(= D)$ is not a "proper integral" ideal, and so is not in any proper maximal t-ideal. Conversely, if the integral ideal AA^{-1} is not contained in any maximal t-ideal, then as $AA^{-1} \subseteq (AA^{-1})_t$, $(AA^{-1})_t$ must be improper and hence must be D. We have proved a special case of the following more general result.

Theorem 2.1. *Let $*$ be a finite character star operation. Then $A \in F(D)$ is $*$-invertible if and only if AA^{-1} is not contained in any maximal $*$-ideal.*

We have seen that if $*$ is a star operation and $A \in F(D)$, then $(AA^{-1})^{*_s} = D$ implies $(AA^{-1})^* = D$, and either of these implies $(AA^{-1})_v = D$. So a study of v-invertibility is important in that if $A \in F(D)$ is not v-invertible, then A is not $*$-invertible for any star operation $*$. Similarly, if A is not t-invertible, then A is not $*$-invertible for any $*$ of finite character. So the first test of invertibility of any kind involves checking if $B_v = D$ for an integral ideal B. In this connection, some results have proved useful time and again. To give the results a somewhat more general flavor let us introduce some terminology.

Call a subset X of $Spec(D)$ a defining family of primes of D if $D = \bigcap_{P \in X} D_P$. Let $*_X$ be the star operation defined by $A \mapsto \bigcap_{P \in X} AD_P$. It can be easily established that X is a defining family of primes if, and only if, each proper ideal of the form $(a) : b$ is contained in some member of X [Gil, Exercise 22 p. 52]. The best known defining families of primes are: $Max(D)$, the set of maximal ideals of D, $* - Max(D)$ for any finite character star operation $*$, and $\mathfrak{P} = \{P \in Spec(D) : P$ is minimal over a proper nonzero ideal of the form $(a) : b\}$. (The members of $\mathfrak{P}$ were shown to be prime t-ideals in [Zaf 2].) The members of $\mathfrak{P}$ are called associated primes of principal ideals by J. Brewer and W. Heinzer in [BH]. (A prime ideal minimal over the proper ideal $(a) : b$ is in Bourbaki terminology a weakly associated prime of the ideal (a).) Let us call an ideal A *transportable* through a defining family X if $(AD_P)^{-1} = A^{-1}D_P$ for all $P \in X$. Because every ring of fractions of

D is a flat D module, every finitely generated ideal is transportable through every defining family. The same, of course, can be shown for every strictly $*$-finite ideal for any $*$ because $A^* = B^*$ implies $A^{-1} = B^{-1}$.

Recall also that a prime ideal P is called *essential* if D_P is a valuation domain, and that a domain D is called *essential* if there is a defining family $X \subseteq Spec(D)$ such that each member of X is essential. Borrowing notation from the usual Zariski Topology, we have for I an integral ideal of D, $V(I) = \{P \in Spec(D) : P \supseteq I\}$ and $D(I) = \{P \in Spec(D) : P \not\supseteq I\}$. Now we are in a position to sum up in the following theorem the numerous useful results, that have been proved over the years, about v-invertibility and t-invertibility.

Theorem 2.2. *Let A be a nonzero integral ideal of D and let X be a defining family of primes of D.*

(1). If $A \not\subseteq P$ for all $P \in X$ (or equivalently $D(A) \supseteq X$), then $A^{*_X} = D$, and hence $A_v = A^{-1} = D$.

(2). If $A^{-1} \neq D$, then A is contained in some member of X (or equivalently $V(A) \cap X \neq \phi$).

(3). If A is finitely generated and $X = \mathfrak{P}$, then $A_v = D$ if and only if A is not contained in any member of $\mathfrak{P}$ (or equivalently, if and only if $V(A) \cap \mathfrak{P} = \phi$).

(4). If A is finitely generated and $X = \mathfrak{P}$, then $A_v \neq D$ if and only if A is contained in at least one member of $\mathfrak{P}$ (or equivalently, if and only if $V(A) \cap \mathfrak{P} \neq \phi$)

(5). If A is transportable through X and if AD_P is principal for every $P \in X$, then A is $*_X$-invertible (and hence is v-invertible).

(6). If A and A^{-1} are both strictly $*_X$-finite, then A is t-invertible if and only if AD_P is principal for each $P \in X$.

(7). If A is transportable through $t - Max(D)$, then A is t-invertible if and only if AD_P is principal for every $P \in t - Max(D)$.

(8). If $*$ is a finite character star operation and if A is strictly $*$-finite, then A is $*$-invertible if and only if AD_P is principal for every $P \in * - Max(D)$. Consequently if A is strictly v-finite, then A is t-invertible if and only if AD_P is principal for every $P \in t - Max(D)$.

(9). If A is finitely generated and every member of X is essential, then A is v-invertible and A is t-invertible if and only if A^{-1} is of finite type.

Comments and references. (1) and (2) are contrapositives of each other and (2) appeared in [Zaf 6] as Theorem 1. The conclusions of (3) and (4), being contrapositives of each other, are equivalent. For the proof, note that because A is finitely generated $A_v = A_t$. So if $A_v = D$, then A is not contained in any proper prime t-ideal and all the members of $\mathfrak{P}$ are prime t-ideals. For the converse, it is enough to note that $\mathfrak{P}$ is a defining family of primes of D, for then (1) applies. This result has appeared in a

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number of papers, see Tang [Tan] or [Zaf 5, Lemma 6]. For (5), we note that for every $P \in X$, $(AD_P)^{-1} = A^{-1}D_P$, and as AD_P is principal, we have $AA^{-1}D_P = D$. This gives $AA^{-1} \not\subseteq P$ for each $P \in X$, which leads to $(AA^{-1})^{*_X} = D$. We can apply (5) and Theorem 1.1 to prove (6). (7) was proved in [Bou, Proposition 1]; the proof runs along essentially the same lines as (5), but in this case we end up with $AA^{-1} \not\subseteq P$ for each $P \in t-Max(D)$ which means that $(AA^{-1})_t = D$. The converse is straightforward. (8) appeared for $* = t$ in [MMZ, Lemma 1.5 and Corollary 1.6] and then in [Kan] and it follows directly from (7) once we note that for a strictly $*$-finite A, $(AD_S)^{-1} = A^{-1}D_S$. Finally, (9) can be proved using (5) and (6) and the fact that every finitely generated ideal in a valuation domain is principal.(See also [Zaf 2, Lemma 8]).

Remark 2.3. For (1)-(4) of the above theorem, A must be an integral ideal. For otherwise, taking a nonzero nonunit a, $A = (\frac{1}{a})$ would cause problems . For (3) and (4), the ideal A can be taken to be strictly v-finite. For (5), we can take A such that A is strictly $*_X$ finite, and A need not be an integral ideal. Finally, (6), (7), (8) and (9), also hold for A fractional.

Let us now bring in some definitions and results relating certain kinds of "invertibilities" with certain domains. This will make it easier to explain the applications of t-invertibility. An integral domain D is called a *v-domain* if every finitely generated ideal of D is v-invertible. Note that by *every finitely generated ideal is ...* we mean *every nonzero finitely generated ideal is ....* There are other ways of defining v-domains and we refer the reader to [Gil]. Using (8) of Theorem 2.2, we note that an essential domain is a v-domain. Later we will see that a v-domain is not necessarily essential.

An integral domain D is called a *Prüfer v-multiplication domain* (*PVMD* for short), if every finitely generated ideal of D is t-invertible. PVMD's may also be called *t-Prüfer* (or *pseudo-Prüfer*) partly because of the fact that the definitions are alike: D is Prüfer if every finitely generated ideal of D is invertible, and partly because of the fact that the theory of PVMD's runs along lines more or less parallel to that of Prüfer domains. For example, D is Prüfer (PVMD) if and only if for each maximal ideal (maximal t-ideal) M of D, D_M is a valuation domain. Another characterization of Prüfer domains (resp., PVMD's) is that D is Prüfer (resp., PVMD) if, and only if, every two generated ideal of D is invertible (resp., t-invertible). These characterizations for PVMD's were established by Griffin in [Gri 1]. For further development on PVMD's, the reader may consult [Kan], [MZ], [HMM], and [Hou]. Another characterization of PVMD's comes from the study of $Inv_t(D)$. The following can be established using Proposition 1.2 and Corollary 1.3: D is a PVMD if and only if for every pair $A, B \in Inv_t(D)$, $A \cap B \in Inv_t(D)$, if and only if for every pair $A, B \in Inv_t(D)$ $(A+B)_t \in Inv_t(D)$. Using the definition of

$\leq$ on $Inv_t(D)$ given prior to Corollary 1.3 we can define for $A, B \in Inv_t(D)$, $A \vee B = \sup(A, B) = A \cap B$ and $A \wedge B = \inf(A, B) = (A, B)_v$. Thus we have the following statement.

Proposition 2.4. *An integral domain D is a PVMD if and only if $(Inv_t(D), \leq, \star_t)$ is a lattice ordered group.*

A related, yet more easily accessible, concept is that of a *GCD domain*, i.e., an integral domain in which every pair of elements a, b, such that at least one is nonzero, has a greatest common divisor. A GCD domain D is characterized by the fact that I_v is principal for all $I \in f(D)$. Indeed, a GCD domain is a somewhat special PVMD. D.D. Anderson has written a detailed survey article on GCD domains [A 1] for this collection. Established in [A 1] are the necessary and sufficient conditions under which a PVMD is a GCD domain. It is my experience that if a statement is true for GCD domains, then very often some form of that statement can also be proved for PVMD's. Now in [Pic], G. Picavet studies GCD domains from a topological point of view. Some introduction to the language used in [Pic] is made prior to Theorem 2.2. It appears that some of the tools used in [Pic] can be shown, in light of Theorem 2.2, to hold more generally. It would be interesting to see if most of [Pic] can be restated for PVMD's.

A domain D such that A_v is invertible for every $A \in f(D)$, is called a Generalized GCD domain (*G-GCD domain*). Of the various characterizations of G-GCD domains known and given in [AA 1], and in [A 1], my favorite is: D is a G-GCD domain if, and only if, D is a PVMD that is also locally a GCD domain. The reason is that it follows from a result that indicates the relationship between t-invertibility and invertibility. The result is

Proposition 2.5. *Let A be a nonzero ideal of D. Then the following are equivalent:*

(1). A is invertible.

(2). A is t-invertible and AD_M is principal for each maximal ideal M of D.

(3). A is strictly v-finite and AD_M is principal for each maximal ideal M of D.

Proof. (1)$\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. For (3)$\Rightarrow$ (1), use Corollary 2 of [Zaf 9]. □

The link between this result and the above characterization of G-GCD domains is routine and we leave it to the reader.

Another concept that is related to v-invertibility is that of a completely integrally closed domain. An integral domain D is said to be *completely integrally closed* if, for $x \in K$ and for some $r \in D\backslash\{0\}$, $x^n r \in D$ for all $n \in N$

implies that $x \in D$. A completely integrally closed integral domain is also characterized by the fact that every nonzero ideal of D is v-invertible [Gil, p. 421]. Finally, it is well known that if D is an intersection of completely integrally closed overrings, then D is completely integrally closed. Nagata [Nag 1] and [Nag 2] gave an example of a completely integrally closed one-dimensional quasilocal domain that is not a valuation domain. This example can be used as an example of a v-domain that is not essential.

Finally, a whole set of v, t, and d invertibility based characterizations of several integral domains of interest have been synthesized into a chart in [AMZ 1]. In that chart the reader will also find the fact that D is a Krull domain if every nonzero ideal of D is t-invertible. (There is a minor error in the statement box of TV domains(every t-ideal is a v-ideal). An interested reader may correct it by reading "If $\forall\ A \in F(D), A_t = A_v$ and $\forall A \in f(D), A$ is" in place of what is written there.)

Another notion that arose from the notion of t-invertibility is that of the t-class group. It arose from the observation that the group $Inv_t(D)$ contains as subgroups the group $Inv(D)$ of all invertible ideals of D and the group $P(D)$ of all nonzero principal fractional ideals of D. The quotient groups $Cl_t(D) = Inv_t(D)/P(D)$ and $G(D) = Inv_t(D)/Inv(D)$ were introduced in [Bou] as the class group, and the local class group of D. We shall refer to $Cl_t(D)$ as the t-class group to differentiate it from the divisor class group $Cl(D)$, which is traditionally defined only for completely integrally closed domains as $Cl(D) = Div(D)/P(D)$, where $Div(D)$ may stand for all the divisorial ideals of a completely integrally closed D. (The usual description of $div(D)$ is in terms of divisor classes.) Now the divisor class group had proved to be very useful in determining how far a Krull domain was from being a UFD and had shown some remarkable functorial properties (see Fossum [Fos]), and it was hoped that $Cl_t(D)$ would show the same nice properties for PVMD's because of the abundance of t-invertible t-ideals in them.

What made $Cl_t(D)$ interesting, at that time, was the fact that if D is Krull, then $Cl_t(D)$ turns out to be the divisor class group, and when D is Prüfer $Cl_t(D)$ is the usual ideal class group. Moreover, when D is a PVMD, $Cl_t(D)$ seems to behave just like the divisor class group, for instance (D PVMD $\wedge Cl_t(D) = 0$) $\Rightarrow$ D is GCD [Bou]. Then in [Zaf 3], it was indicated that (D PVMD $\wedge Cl_t(D)$ torsion)$\Rightarrow$ D is almost GCD, i.e., for every pair x, y of nonzero elements of D there is $n = n(x, y) \in N$ such that $(x^n) \cap (y^n)$ is principal. Later, Anderson and Rykaert [AR] provided some other similarities the t-class group of a PVMD had with the divisor class group of a Krull domain. To top it all, the t-class group was defined for a general integral domain. On the other hand, Gabelli [Gab 1] came up with a devastatingly beautiful result that $Cl_t(D)$ is isomorphic to $Cl_t(D[X])$ if and only if D is integrally closed. Then David Anderson wrote a very exhaustive article [An 1] on the topic indicating that, in fact, for any star operation $*$ a $*$-class group can be defined. The details of how the concept of the t-class group developed and where it stands are included in a survey article by David Anderson [An 2] in this collection. We shall briefly touch upon it from time to time.

It was indicated in section 1. that if S is a multiplicative set of D, then there is a homomorphism from $Cl_t(D)$ to $Cl_t(D_S)$ defined by $[A] \longmapsto [AD_S]$. There has been a great deal of activity in search of similar relations for ring extensions $D \subseteq R$, where R is a suitable extension, not necessarily the ring of fractions. Of crucial importance here is the knowledge that corresponding to each $A \in Inv_t(D)$, there is a $B \in Inv_t(R)$ such that some kind of homomorphism can be verified from $Cl_t(D)$ to $Cl_t(R)$. David Anderson [An 1, Section 4] came up with a suggestion of the operations on D and R being compatible. Denoting by $*_X$ a $*$ operation of the domain X he said that $*_D$ and $*_R$ are compatible on $D \subseteq R$ if for all $I \in F(D)$ $(IR)^{*_R} = (I^{*_D}R)^{*_R}$. Using this notion, he was able to prove some positive results when R was faithfully flat over D. Barucci, Gabelli, and Roitman extended the idea in a slightly different direction. With a few modifications their result [BGR, Proposition 1.1] reads as follows:

Proposition 2.6. *Let $D \subseteq R$ be an extension of integral domains. Then the following statements are equivalent:*

(1). $I_{v_D}R \subseteq (IR)_{v_R}$ for each nonzero finitely generated $I \in F(D)$.

(2). $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ for each finitely generated $I \in F(D)$.

(3). $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ for each $I \in F(D)$.

(4). $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ for each $I \in F(D)$.

(5). If A is an integral t-ideal of R such that $A \cap D \neq (0)$, then $A \cap D$ is a t-ideal of D.

(6). $I_{t_D}R \subseteq (IR)_{t_R}$ for each $I \in F(D)$.

Moreover, under these equivalent conditions, the map $[A] \longrightarrow [(AR)_{t_R}]$ is a homomorphism from $Cl_t(D)$ to $Cl_t(R)$.

If these equivalent conditions hold for an extension $D \subseteq R$, we say that $D \subseteq R$ is *t-compatible*. Now it is easy to see that $D \subseteq R$ is t-compatible in each of the following cases.

(a). When $(AR)^{-1} = A^{-1}R$ for each $A \in f(D)$.

This includes not only the cases when R is a quotient ring or a flat overring, but also a host of other cases, and in each case the t-invertible t-ideals of D extend to the t-invertible t-ideals of R. Because of the space constraints on this article, it would be difficult to mention all the cases. So we shall mention some and provide references for an interested reader to have some idea until the promised paper appears. The first case that comes to mind is when $R = D + XD_S[X]$, where X is an indeterminate over D_S, [Zaf 8, Lemma 3.1]. Recently, Anderson, Elbaghdadi, and Kabbaj have indicated in [AEK, Lemma 3.6] that the construction $T = D + X\mathfrak{D}[X]$, where X is an indeterminate over $\mathfrak{D}$, is a flat D module if and only if $\mathfrak{D}$ is a flat D module. This of course means that (a) above applies [AEK, Lemma 3.7]. This construction, which is usually referred to as the $A + XB[X]$ construction, was studied in [AAZ 1] and a look into recent literature indicates that it can do much more than the $D + XD_S[X]$ construction. For instance, $A + XB[X]$ can be used to produce examples of domains that satisfy ACC on principal ideals. A number of papers [BIK], [ANeA], [DRSS], and a host of other papers, that are either in press or in preparation, study this aspect of this construction. Coming back to the topic, (a) also applies to certain pullback situations other than the $A + XB[X]$ type, see for example [FG, Proposition 1.8].

(b). When $(AR)^{-1} = (A^{-1}R)_{v_R}$ for all $A \in f(D)$.

This would include cases when R is defined by a family $\{R_\alpha\}$of overrings such that each R_α is a flat D module. For in that case, $(AR)^{-1} = R :_L AR = (\bigcap_\alpha R_\alpha) :_L AR = \bigcap_\alpha (R_\alpha :_L AR_\alpha) = (\bigcap_\alpha A^{-1}R_\alpha) = (A^{-1}R)_v$. Here L is the quotient field of R. In this case, t-invertible t-ideals of D extend to t-invertible ideals of R. The special case when each of R_α is a localization can be witnessed in [AHZ].

(c) When $(AR)^{-1} = A^{-1}R$ for all $A \in F(D)$.

This happens when $R = D[X]$ (see [Nis 1] and [HH]) and when R is a generalized $D + M$ construction of Brewer and Rutter [BR], (see [CMZ 1], [CMZ 2], [AR], [An 1], and [BG] for various forms of the $D+M$ construction and the proof that (c) holds). In case (c) a somewhat stronger form of Proposition 2.6 can be proved.

(d) When for all $A \in F(D)$, $(AR)^{-1} = (A^{-1}R)_{v_R}$. This holds when $R = Int(D) = \{f \in K[X] : f(D) \subseteq D\}$. The reader may consult [CGH, Lemma 3.1], when it appears, to see that this holds.

We note that in each of the extensions $D \subseteq R$, discussed above, R is t-linked over D, i.e., for every $A \in f(D), A^{-1} = D$ implies $(AR)^{-1} = R$ [DHLZ]. So in each case there is a homomorphism $\theta : Cl_t(D) \longrightarrow Cl_t(R)$ defined by $\theta([A]) = [(AR)_t]$ [AHZ, Theorem 2.2]. However, if R is t-linked over D, the extension $D \subseteq R$ may not satisfy any of (a)- (d) and may not satisfy any of the equivalent conditions. (These facts will be included in a detailed account in the promised article.) Besides, the presence of any of (a) - (c) allows a better grip, especially in deciding whether the homomorphism is actually an isomorphism or not. For instance, in the proof of Theorem 4.8 of [AHZ] the presence of (b) above helped. The results on isomorphisms of $Cl_t(D)$ and $Cl_t(R)$, for various extensions $D \subseteq R$, are covered amply in [An 2] and [An 1].

It would be unfair not to mention the important special case that actually caused [BGR, Proposition 1.1]. The case of $D \subseteq R$ where R is a generalized ring of fractions of D is somewhat peculiar. The generalized ring of fractions, by the way, is defined as follows: Let $\mathfrak{S}$ be a generalized multiplicative system, i.e., a multiplicatively closed set of ideals generated by a set of nonzero ideals of D. Then the set $D_{\mathfrak{S}} = \{x \in K : xA \subseteq D \text{ for some } A \in \mathfrak{S}\}$ is a ring called the *generalized ring of fractions* w.r.t. $\mathfrak{S}$. There are two kinds of ideal extensions from D to $D_{\mathfrak{S}}$: (α) Given an integral ideal J of D we define $J_{\mathfrak{S}} = \{x \in K : xA \subseteq J \text{ for some } A \in \mathfrak{S}\}$ and (β) The usual extension $JD_{\mathfrak{S}}$. It is well known that $JD_{\mathfrak{S}} \subseteq J_{\mathfrak{S}}$. For further details the reader may consult Arnold and Brewer [AB] and references there. For a more current treatment of the topic see [FHP]. Querre in [Que 2, Lemme 1] stated that if A and B are ideals of D and if $\mathfrak{S}$ is a generalized multiplicative system, then $(A :_K B)_{\mathfrak{S}} \subseteq A_{\mathfrak{S}} :_K B_{\mathfrak{S}}$. Moreover if B is of finite type, then $(A :_K B)_{\mathfrak{S}} = A_{\mathfrak{S}} :_K B_{\mathfrak{S}} = A_{\mathfrak{S}} :_K BD_{\mathfrak{S}}$. This means that (i) $(B^{-1})_{\mathfrak{S}} = (B_{\mathfrak{S}})^{-1} = (BD_{\mathfrak{S}})^{-1}$ for every finitely generated ideal B. Kang [Kan, Lemma 3.4], extended this result to (ii) $(B_{\mathfrak{S}})_v = (BD_{\mathfrak{S}})_v = (B_{v\mathfrak{S}})_v = (B_vD_{\mathfrak{S}})_v$ when B is finitely generated, and (iii) $(BD_{\mathfrak{S}})_t = (B_tD_{\mathfrak{S}})_t$ for any $B \in F(D)$, see also [Gab 2]. So, by (ii), Proposition 2.6 applies and $D \subseteq D_{\mathfrak{S}}$ is t-compatible. Now under certain conditions, $D_{\mathfrak{S}}$ is expressible as an intersection of localizations of D. Under these conditions, as we have already seen, we have $(B^{-1}D_{\mathfrak{S}})_v = (BD_{\mathfrak{S}})^{-1}$ for all finitely generated $B \in F(D)$. This leads to the following question:

Question 2.7. Is it true that every extension of domains $D \subseteq R$, where R is a generalized ring of fractions of D, falls under one of the conditions (a) - (d)?

3. *T*-INVERTIBILITY AND LOCALLY FINITE INTERSECTIONS OF LOCALIZATIONS

In connection with invertibility (t or d), results of the following type have always fascinated me for their "roll-over" quality. The result stated here, can be found as Lemma 2.2 in [AMZ 2]. For a similar flavor, in the case of invertibility, the reader may want to see an old short note [Zaf 1].

Proposition 3.1. *Suppose that $X \subseteq Spec(D)$ is a defining family of D such that each nonzero nonunit of D belongs to at most a finite number of members of X. If $A \in F(D)$ is such that AD_P is principal for each $P \in X$, then A is $*_X$-invertible, and hence t-invertible.*

The proof essentially uses (5) of Theorem 2.2, but first it uses the finite character of $\{D_P\}_{P\in X}$ and the fact that AD_P is principal for each $P \in X$ to show that A is strictly $*_X$-finite. We shall call X a defining family of finite character if $D = \bigcap_{P\in X} D_P$ is of finite character.

The above proposition is a useful result, but let $X = Max(D)$ in the above proposition. Then we are actually getting $*_X = d$ (as in the case of [Zaf 1, Theorem 2]). If we want t-invertibility exactly, there is Lemma 2.1 of [AMZ 2]. We include it here for record.

Proposition 3.2. *Let X be a set of prime t-ideals such that no two distinct elements of X are comparable and suppose that X is a defining family of finite character. Then the following statements hold.*

(1). $*_X - Max(D) = X = t - Max(D)$.

(2). A is $*_X$-invertible $\Leftrightarrow$ A is t-invertible.

(3). If A is t-invertible, then $A^{*_X} = A_t$.

It is customary to denote by $X^{(1)}$ the set of height one primes of D. Using this notation, we state the following result which is an immediate corollary to Proposition 3.1, yet it is very important from the applications point of view.

Corollary 3.3. *[AMZ 2, Corollary 2.3] Suppose that $D = \bigcap_{P \in X^{(1)}} D_P$, where the intersection has finite character. Then for $P \in X^{(1)}$ and for $0 \neq x \in P$, $xD_P \cap D$ is a t-invertible primary t-ideal.*

Why is Corollary 3.3 interesting? Let us recall that D is a Krull domain if (K1) $D = \bigcap_{P \in X^{(1)}} D_P$, where the intersection has finite character, and (K2) D_P is a discrete valuation domain for each $P \in X^{(1)}$. A domain that satisfies (K1) only is called a weakly Krull domain (WKD). Weakly Krull domains have been around in various guises, each guise involving an extra property. For example, a generalized Krull domain (GKD) of Ribenboim [Rib] satisfied in addition to (K1) the property: (K2$'$) D_P is a valuation domain for each $P \in X^{(1)}$. Then there were the generalized UFD's that turned out to be GKD's with the GCD property (see [AAZ 2]), and then there were the weakly factorial domains (WFD's) of Anderson and Mahaney [AM]. A WFD is a domain whose nonzero nonunits are products of primary elements. It was shown in [AZ 1, Theorem], that in addition to various other properties, a WFD satisfies (K1), and for $0 \neq x \in P \in X^{(1)}$, $xD_P \cap D$ is principal. The result that is Corollary3.3 here led to a whole set of characterizations of WKD's [AMZ 2, Theorem 3.1]. In the following we include some.

Theorem 3.4. *The following are equivalent for an integral domain D.*

(1). $D = \bigcap_{P \in X^{(1)}} D_P$, where the intersection has finite character.

(2). Every proper principal ideal of D is a t-product of primary (t-) ideals.

(3). Every proper t-ideal of D is a t-product of primary (t-) ideals.

(4). If P is a prime ideal minimal over a proper principal ideal (x), then $xD_P \cap D$ is t-invertible.

(5). Every proper principal ideal of D has a primary decomposition, where all the associated prime ideals have height one.

This theorem shows that the WKD's have easy to use analogs of all the good properties of Krull domains, at least at the proper principal ideal level, including the various factorization properties. So what was special with Krull domains could be done to some extent for say Noetherian domains in which grade one primes have height one. Those interested in, not necessarily unique, factorization properties took special notice of WKD's. A reader interested in this kind of study may look up Geroldinger's work see e.g. [Ger] and the references there. For a treatment of weakly Krull monoidsand other generalizations of Krull monoids the reader may consult Chapter 22 of Halter-Koch's book [H-K]. Characterizations such as given in Theorem 3.4 also indicate that there are sufficiently many t-invertible t-ideals for the t-class group to be effective. This led to more interesting results such as the weakly Krull analog of Nagata's class group theorem in [AHZ]. Then the fact that D is weakly factorial if and only if D is weakly Krull and $Cl_t(D) = 0$, which can be traced back to [AZ 1], made the factorization properties of certain classes of Noetherian domains transparent on the one hand and aroused interest in the t-class groups on the other. An interested reader may also want to check a recent paper by Martine Picavet [P-H]. This paper deals with weakly factorial algebraic orders. A reader who is interested in the multiplicative ideal theory in cancellative monoids may want to see [H-K], Chapters 12 and 13, and in works such as [Kai] and references there for a treatment of class groups from that angle.

Now in a WKD D, for every proper principal ideal (x) and for each member P of $X^{(1)}$ that contains x we have $xD_P \cap D$ which is t-invertible and which is contained in only one member of $X^{(1)}$, that is P. Now suppose that we have a defining family $\mathcal{F}$ of mutually incomparable prime ideals of D such that, for every $P \in \mathcal{F}$ and for every $0 \neq x \in P$, we have that $xD_P \cap D$ is t-invertible and that P is the only member of $\mathcal{F}$ that contains $xD_P \cap D$. This approach led to the notion of a *P-pure* ideal in [AMZ 2] as an integral ideal A such that A is contained in a unique prime $P \in \mathcal{F}$ and later, [AZ 2], to, a *unidirectional ideal* as an ideal A that belongs to a unique member of $\mathcal{F}$ (pointing to P if $A \subseteq P$). Now let $*_{\mathcal{F}}$ be the star operation induced by $\mathcal{F}$ on D, and call two ideals A and B $*_{\mathcal{F}}$ *-comaximal* if $(A+B)^{*_{\mathcal{F}}} = D$. It is easy to see that if A and B are unidirectional ideals pointing to the same member P of F, then so are $A^{*_{\mathcal{F}}}$, $B^{*_{\mathcal{F}}}$, $(AB)^{*_{\mathcal{F}}}$ and $A^{*_{\mathcal{F}}} \cap B^{*_{\mathcal{F}}}$. Moreover, two unidirectional ideals A and B of D are $*_{\mathcal{F}}$ comaximal if, and only if, they point to different members of $\mathcal{F}$ [AZ 2, Lemma 2.5]. Noting also that if A and B are $*_{\mathcal{F}}$ comaximal, then $(AB)^{*_{\mathcal{F}}} = A^{*_{\mathcal{F}}} \cap B^{*_{\mathcal{F}}}$, we conclude as in Lemma 2.6 of [AZ 2] that if $A_1, A_2, \ldots, A_n$ are unidirectional ideals such that no two of the A_i point to the same member of $\mathcal{F}$, then $(\prod_{i=1}^{n} A_i)^{*_{\mathcal{F}}} = \bigcap_{i=1}^{n} A_i^{*_{\mathcal{F}}}$. Using these observations, we come to the conclusion that if a $*_{\mathcal{F}}$-ideal A of D is expressible as a finite $*_{\mathcal{F}}$-product of unidirectional $*_{\mathcal{F}}$-ideals, then A is expressible uniquely, up to order, as a $*_{\mathcal{F}}$-product of mutually $*_{\mathcal{F}}$-comaximal $*_{\mathcal{F}}$-ideals [AZ 2, Lemma 2.7]. Now our theory of "unique factorization" of unidirectional ideals is complete. Let us call $\mathcal{F}$ an independent family if no two distinct members P and Q of $\mathcal{F}$ contain a common nonzero prime ideal. Then, using some straight-forward arguments, we get the following statement which is a part of Proposition 2.8 of [AZ 2].

Proposition 3.5. *Let $\mathcal{F}$ be a defining family of primes of D and let $*_{\mathcal{F}}$ be the star operation induced by localizations at members of $\mathcal{F}$. Then the following are equivalent:*

(1). $\mathcal{F}$ is an independent family of finite character,

(2). every $*_{\mathcal{F}}$-ideal is uniquely expressible as a finite $*_{\mathcal{F}}$-product of unidirectional $*_{\mathcal{F}}$-ideals,

(3). every proper principal ideal is expressible as a finite $*_{\mathcal{F}}$-product of unidirectional $*_{\mathcal{F}}$-ideals,

(4). every proper principal ideal is uniquely expressible as a finite $*_{\mathcal{F}}$-product of mutually $*_{\mathcal{F}}$-comaximal unidirectional $*_{\mathcal{F}}$-ideals.

Now taking $*_{\mathcal{F}}$ of finite character, it is shown in [AZ 2] that the following are equivalent.

(1). $\mathcal{F}$ is independent of finite character,

(2). every nonzero prime ideal of D contains a unidirectional $*_{\mathcal{F}}$-invertible $*_{\mathcal{F}}$-ideal,

(3). for $P \in \mathcal{F}$ and $0 \neq x \in D$, $xD_P \cap D$ is $*_{\mathcal{F}}$-invertible and unidirectional,

(4). $\mathcal{F}$ is independent and for any nonzero ideal A of D, $A^{*_{\mathcal{F}}}$ is of finite type whenever AD_P is finitely generated for all P in $\mathcal{F}$.

This is a part of Theorem 3.4 of [AZ 2]. The above two results characterize what are called in [AZ 2] $\mathcal{F}$-Independent of Finite Character ($\mathcal{F}$-IFC, or IFC) domains. I have included them here to (a) indicate the parallelism and (b) prepare an interested reader for a set of problems in this connection. The best known examples of IFC domains are the independent rings of Krull type of Griffin [Gri 2] which are PVMD's D with $\mathcal{F} = t - Max(D)$ and $\mathcal{F}$ is independent of finite character. Then there are the semi-t-pure domains of [AMZ 3]. These domains have, again, $\mathcal{F} = t - Max(D)$, where $\mathcal{F}$ is independent of finite character, and these domains have the extra property that their t-invertible t-ideals are principal, i.e., their t-class group is trivial. Finally, the h-local domains of Matlis [Mat] are precisely the $\mathcal{F}$-IFC domains with $\mathcal{F} = Max(D)$. I do not know of an example of an $\mathcal{F}$-IFC domain D such that $\mathcal{F} \neq Max(D)$ or $\mathcal{F} \neq t - Max(D)$. Yet even if it is easy to construct such an example, it should be on the record.

Problem 3.6. Find an example of a $\mathcal{F}$-IFC domain such that $\mathcal{F} \neq Max(D)$ and $\mathcal{F} \neq t - Max(D)$.

Another problem that is close to this one can be stated as follows.

Problem 3.7. Call D a $\mathcal{G}$-finite character, or a finite character, domain if there is a family $\mathcal{G}$ of primes of D such that $D = \bigcap_{P \in \mathcal{G}} D_P$ and every nonzero nonunit of D belongs to finitely many members of $\mathcal{G}$. Characterize a $\mathcal{G}$-finite character domain using the notion of unidirectional ideals.

IFC domains being characterized by $*_{\mathcal{F}}$-invertible $*_{\mathcal{F}}$-ideals and the factorization properties of their elements may bring the notion of $*_{\mathcal{F}}$-class groups into the picture. A solution to problem 3.6 may give rise to a $*$-class group that lies properly between the t-class and the Picard group.

4. GETTING T-INVERTIBILITY INDIRECTLY

We have seen characterizations of t-invertibility and v-invertibility in terms of defining families of primes. There are other ways and situations in which t-invertibility of an ideal happens or is caused by its v-invertibility or less. This section is concerned with instances where we get t-invertibility at the price of v-invertibility or as a consequence of some event on which we seem to have no control.

Note, with reference to (1) of Theorem 2.2, that if for some ideal A, $A^{-1} = D$, A can still be contained in some associated prime of a principal ideal. In fact, A can itself be an associated prime. The simplest example is that of a non-discrete rank one valuation domain (V, M). The maximal ideal M of V is obviously an associated prime of a principal ideal, but $M^{-1} = M_v = V$. In [GV], Glaz and Vasconcelos chose to study integral domains D with the property that if for an ideal A we have $A^{-1} = D$ (or equivalently $A_v = D$), then A is strictly v-finite. An integral domain that satisfies this property is called an H domain. Houston and the author proved in [HZ 1, Proposition 2.4] the following result.

Proposition 4.1. *An integral domain D is an H domain if and only if every maximal t-ideal of D is divisorial.*

By Observation B, every maximal t-ideal of an H domain D is of the form $(a) : b$, and so is an associated prime of a principal ideal. Let $A \in F(D)$ be v-invertible. Then $(AA^{-1})_v = D$, and consequently AA^{-1} is strictly v-finite, because D is an H domain. So by (3) of Theorem 2.2, AA^{-1} is not contained in any associated prime of a principal ideal, and hence is not contained in any maximal t-ideal, which leads to $(AA^{-1})_t = D$. Thus in an H domain every v-invertible ideal is t-invertible. On the other hand, if D is such that every v-invertible ideal of D is t invertible and if there is a nonzero ideal A such that $A^{-1} = D$, then A is v-invertible, hence is t-invertible, and hence is strictly v-finite. Thus we have an interesting characterization of H domains.

Proposition 4.2. *The following are equivalent for an integral domain D.*
(1). D is an H domain.
(2). Every maximal t-ideal of D is divisorial.
(3). Every maximal t-ideal of D is of the form $(a) : b$.
(4). Every v-invertible ideal of D is t-invertible.

The somewhat restricted class of H domains includes Noetherian domains and Mori domains, i.e., integral domains in which integral divisorial ideals satisfy ACC. Now these are integral domains which are called TV domains, i.e., ones in which every t-ideal is a v-ideal [HZ 1]. Yet H domains are much more general than that. Using the $D + XK[X]$ construction of [CMZ 1] we can construct non TV H-domains with some help from [HZ 1].

Now in a v-domain, every finitely generated ideal is v-invertible. So an H domain that is a v-domain must be a PVMD. Next, if an H domain D is completely integrally closed, then every nonzero fractional and hence integral ideal of D is v-invertible, hence t-invertible, and hence strictly v-finite. This, by Lemma 1 of Nishimura [Nis 2], means that integral divisorial ideals of D satisfy ACC and that D is completely integrally closed and consequently by Theorem 2 of [Nis 2] we conclude that D is Krull. So we have the following result which was proved in [GV, 3.2 (d)].

Proposition 4.3. *A completely integrally closed H domain is a Krull domain.*

In [MMZ], it was shown rather laboriously that D is a Krull domain if and only if every associated prime of a principal ideal of D is t-invertible. This result was then improved by Kang [Kang] to: D is a Krull domain if and only if every minimal prime of a principal ideal of D is t-invertible. These results were then put to rest using the TV domain connection in [HZ 1, Theorem 2.3]. It appears that if we use the properties of t-invertible prime t-ideals, given in Proposition 1.3 of [HZ 2], a more satisfactory proof can be effected. This result is included here for reference.

Proposition 4.4. *If P is a t-invertible prime t-ideal of D, then the following hold.*
(a). P is a minimal prime of a principal ideal.
(b). P has the form $(a) : b$ for some $a, b \in D$.
(c). P is a maximal t-ideal.

The following result is immediate, but it does not seem to have found its way into the literature.

Corollary 4.5. *If $*$ is of finite character and if P is a $*$-invertible prime $*$-ideal, then the following hold.*
(a). P is a minimal prime of a principal ideal.

(b). P has the form $(a) : b$ for some $a, b \in D$.
(c). P is a maximal t-ideal.

Proof. If P is a $*$-invertible $*$-ideal, then P is strictly $*$-finite and $P = P^* = P_v$, and so P is a t-ideal. Now as t is coarser than any finite character $*$, P is a t-invertible t-ideal. Now the other conclusions are immediate. □

This means that if $* \neq t$ is of finite character we cannot in general say that if P is a $*$-invertible prime $*$-ideal, then P is a maximal $*$-ideal. For a concrete example, note that in a UFD that is not a PID, every height one prime ideal, being principal, is (d-) invertible but is not necessarily a maximal (d-) ideal. These observations indicate the pivotal position the t-operation has among the star operations. Now let us use them to provide the more "satisfying proof" that was promised.

Theorem 4.6. *[Kang] An integral domain D is Krull if and only if every minimal prime of every proper principal ideal is t-invertible.*

Proof. If D is a Krull domain, then it is well known that every nonzero ideal is t-invertible. For the converse, note that if every minimal prime P of a principal ideal is t-invertible, then by Proposition 4.4 it is a maximal t-ideal and of the form $(a) : b$. This makes every prime t-ideal maximal of height one and of finite type. So D is an H domain. Next, as P is t-invertible, PD_P is principal by (8) of Theorem 2.2. But PD_P principal of height one makes D_P a discrete rank one valuation domain. Now as $D = \bigcap_{P \in t-Max(D)} D_P$, where each of D_P is completely integrally closed, we have that D is a completely integrally closed H domain, and hence a Krull domain. □

Before we move to another topic, a general comment is in order. Given any star operation $*$, we have seen that $*$-invertible implies v-invertible, and in an H domain v-invertible implies t-invertible. So every $*$-class group of an H domain is contained in the t-class group. Now let ρ be a star operation defined by $A \longrightarrow \bigcap_{P \in \mathfrak{P}} AD_P$, where $\mathfrak{P}$ is our trusted set of associated primes of principal ideals. Now let A be a t-invertible ideal of any domain D. Because $(AA^{-1})_t = D$, we have that AA^{-1} is not contained in any maximal t-ideal, and hence not contained in any associated prime of a principal ideal. Consequently, in a general domain, t-invertible implies ρ-invertible. Moreover, as we have seen, in an H domain the converse also holds. So in an H domain the ρ-class group is precisely the t-class group and in view of (4) of Proposition 4.2, we have the following statement: For an H domain D, $Cl_t(D) = Cl_v(D) = Cl_\rho(D)$. It is easy to establish that in general v, t and ρ are distinct.

Glaz and Vasconcelos discuss in [GV] another concept that may have some far reaching effects on the theory of star operations. They introduce the notion of a *semidivisorial ideal*. An ideal I is semidivisorial if $I : J = I$ whenever J is finitely generated such that $J^{-1} = D$. This concept was briefly yet effectively touched on by Hedstrom and Houston in [HH, Proposition 3.2]. They showed that the operation $I \mapsto I^* = \cup\{I : J | J$ is a finitely generated ideal such that $J_v = D\}$ is a finite character star operation on $F(D)$. Later, Wang and McCasland [WM] introduced precisely this operation, but in a more general setting. Let us first agree to denote by $GV(D)$ the set of all finitely generated ideals J of D such that $J_v = D$. Then they define for a torsion-free module M over D the *w-envelope* $M_w = \{x \in M \otimes K | xJ \subseteq M$ for some $J \in GV(D)\}$. Clearly a *w-module* is one that is equal to its w-envelope. They prove enough results on w-modules to be able to switch over to w-envelopes of ideals and to say that every integral w-ideal is contained in a maximal w-ideal and that this maximal w-ideal is prime. They also show [WM, Proposition 2.5], that for any two submodules A and B of a torsion free module F, $A_w \cap B_w = (A \cap B)_w$ which, for ideals, means that for each nonzero fractional ideal I, $I_w = \bigcap_{P \in w-Max(D)} ID_P$ [A 1, Theorem 1]. Recently, Anderson and Cook [AC] and Wang [Wan 4] have independently shown that $t - Max(D) = w - Max(D)$ and that consequently, a nonzero ideal I is t-invertible if and only if it is w-invertible. The last result was however known, though not in those terms, to the authors of [AMZ 2]. In view of so many similarities, one may want to check if the w operation and the t operation are the same. The answer is no and that comes from a beautiful result of Kang. We include this result here for the record.

Theorem 4.7. *[Kan, Theorem 3.5] D is a PVMD if and only if D is integrally closed and $I_t = \bigcap_{P \in t-Max(D)} ID_P$ for every nonzero integral ideal I of D.*

So if D is integrally closed and not a PVMD, then there exist nonzero integral ideals J such that $J_w \subsetneq J_t$. It would be useful to have some concrete examples of ideals that have this property. Some of these examples will have to be for D integrally closed and some for D not integrally closed.

Now what is so nice about the w-operation is that it is smoother than the t-operation in that (a) given any nonzero prime ideal P of D, either P is a w-ideal or $P_w = D$ because P is a w-ideal if and only if P is contained in some maximal w-ideal, and (b) as long as the w-operation is with reference to GV(D), it works pretty smoothly by inducing its own brand of a finite character $*$-operation w_0 in a w-algebra R over D by assigning to each ideal I of R its w-envelope I_w. This star operation is subservient to the actual w-operation on R (with reference to GV(R)). So the w-operation does not have any problems similar to those of the t-operation going to ring extensions. Smoothness of this sort can only mean one thing, that you can bring in a lot more homological algebra than with other star operations, if that is what you want. (A reader who wishes to explore homological aspects of the w-operation may look up [Wan 1], [Wan 2], and [Wan 4] as well.) Then there is the notion of the w-integral closure of D [Wan 3]. I will come back to it shortly. (Some of these comments are based on unpublished manuscripts [Wan 4] and [Wan 5] of Wang. I have taken the liberty to include them here hoping to arouse interest and allay any misconceptions/fears that may come in the way of accepting this new concept.)

Our next indirect source of t-invertibility, that led to applications, comes from [HZ 2, Theorem 1.4]. Let us recall first that if $R = D[X]$, where X is an indeterminate over D, and if P is a prime ideal of R such that $P \cap D = (0)$, then P is called a *prime upper to* 0.

Proposition 4.8. *Let P be a prime upper to 0 in $D[X]$. Then the following statements are equivalent.*

(1). P is a maximal t-ideal.

(2). P is t-invertible.

(3). $(c(P))_t = D$.

Here $c(P)$ denotes the content of P, i.e., the ideal generated by the coefficients of the polynomials in P. It is easy to see ([HZ 2, remark with (3)]) that (3) above is equivalent to (4): P contains a polynomial f such that $(c(f))_t = D$. Coupling this with Proposition 4 of [Zaf 5], which says that an integrally closed D is a PVMD if and only if every prime upper to 0 contains a polynomial f such that $(c(f))_v = D$, one may deduce that an integrally closed D is a PVMD if and only if every prime upper to 0 of $D[X]$ is a maximal t-ideal. This led to the notion of a UMT domain as an integral domain D such that every prime upper to 0 is a maximal t-ideal.

It is easy to see that an integrally closed UMT domain is a PVMD; but is the integral closure of a UMT domain a PVMD? No one seems to know. There are partial results available however. These results can be extracted, along with some relevant references, from Fontana, Gabelli, and Houston's paper [FGH], where, for a very good reason, UMT domains are dubbed as "PVMD's without the integrally closed property". The results that may have a bearing on future work in that direction are the following. They show among several equivalent conditions that D is a UMT domain if and only if for every maximal t-ideal M of D, D_M has Prüfer integral closure, and then using an extremely useful tool introduced by Kang [Kan] they show that D is a UMT domain $\Leftrightarrow$ $D[X]_S$ is a UMT domain $\Leftrightarrow$ $D[X]_S$ has Prüfer integral closure, where $S = \{f \in D[X] \mid (c(f))_v = D\}$. They show that a UMT domain is well behaved, i.e., every prime t-ideal of D localizes to a prime t-ideal [Zaf 8]. In view of the above result, it is fair to conjecture that D is a UMT domain if, and only if, (1) D_M is a UMT domain for each maximal ideal M and (2) D is well behaved. UMT domains have also been studied, in a hither to unpublished paper by Wang [Wan 5]. Wang introduces the notion of a w-integral closure D^w and shows that if D has UMT, then D^w is a w_0-PVMD of [HMM], i.e., every finitely generated ideal of D^w is w_0-invertible, which makes D^w slightly more than an ordinary PVMD (see [HMM] for this remark). Here w_0 is the star operation induced by GV(D) on D^w.

Our final example of indirect t-invertibility comes from the observation [HZ 2, Proposition 4.1] that if B is an ideal of $D[X]$ with $(c(B))_t = D$, then B is t-invertible. This is equivalent to saying that there is a polynomial $f \in B$ such that $(c(f))_v = D$. This led to the question: What are the elements or ideals whose presence inside any ideal would ensure that the ideal is t-invertible? Let us call such an element/ideal a *t-invertibility element/ideal.* The answer came from [GMZ, Theorem 1.3]. As it looks like a useful result, we include it here.

Theorem 4.9. *Suppose A is a nonzero integral ideal of D. Each integral ideal of D containing A is t-invertible if, and only if, A_t is a finite t-product of maximal t-invertible t-ideals of D.*

This theorem may serve as a quick confirmatory test of t-invertibility. For example, if you spot a finite product of principal primes in an ideal, then the ideal is at least t-invertible. (In fact if you spot a product of principal primes in an integral ideal B it can be easily shown that B_t is principal.) This led further to the question of invertibility elements etc. A general answer is provided in [GMZ, Theorem 1.3'] which just amounts to replacing t by a finite character star operation $*$.

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