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SOME CHARACTERIZATIONS OF v-DOMAINS AND RELATED PROPERTIES

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Let D be a commutative integral domain with quotient field K. Let $F(D)$ denote the set of nonzero fractional ideals of D and let $f(D)$ denote the subset of finitely generated members of $F(D)$. The function on $F(D)$ defined by $A \to (A^{-1})^{-1} = A_v$ is a $*$-operation called the v-*operation*. The reader may consult [1], Sections 32 and 34, for the basic properties of $*$-operations and the v-operation. The reader is also referred there for definitions and notation not given here. At one point our notation will differ from that of [1] in that, for $A, B \in F(D)$,

$$(A:B) = (A:_K B) = \{x \in K \mid xB \subseteq A\}.$$

Let $A, B \in F(D)$. Then $AB \subseteq (A:B^{-1}) \subseteq (A_v:B^{-1})$, so

$$(AB)_v \subseteq (A:B^{-1})_v \subseteq (A_v:B^{-1})$$

since $(A_v:B^{-1})$ is a v-ideal. The purpose of this paper is to investigate how the replacement of certain inclusions by the equality in the preceding chain of inclusions relates to D being integrally closed, completely integrally closed, or a v-domain.

In [3], Malik proved that if D is integrally closed, then

$$(A:B^{-1})_v = (AB)_v \quad \text{for all } A \in f(D) \text{ and } B \in F(D).$$

Example 9 at the end of this paper shows that the converse is false. We show that D is completely integrally closed (respectively, a v-domain) if and only if $(AB)_v = (A_v:B^{-1})$ for all $A \in F(D)$ (respectively, $A \in f(D)$) and all $B \in F(D)$.

Now, for $A, B \in F(D)$, $AB^{-1} \subseteq (A:B)$, so

$$(AB^{-1})_v \subseteq (A:B)_v \subseteq (A_v:B) = (A_v:B_v).$$

We show that D is completely integrally closed (respectively, a v-domain) if and only if $(AB^{-1})_v = (A:B)_v$ for all $A \in F(D)$ (respectively, $A \in f(D)$) and all $B \in F(D)$.

In general, for $A, B \in f(D)$, we need not have

$$(A \cap B)_v = A_v \cap B_v.$$

We show that if D is essential, then

$$(A_1 \cap \ldots \cap A_n)_v = A_{1v} \cap \ldots \cap A_{nv} \quad \text{for all } A_1, \ldots, A_n \in f(D);$$

and that if D is integrally closed and this equality holds, then D is a v-domain. This is done by first proving that, for D integrally closed, D is a v-domain if and only if $(A:B)_v = (A_v:B_v)$ for all $A, B \in f(D)$. We call D a *crescent domain* if, for any collection $\{A_\alpha\} \subseteq f(D)$ with $\bigcap_\alpha A_\alpha \neq 0$,

$$\left(\bigcap_\alpha A_\alpha\right)_v = \bigcap_\alpha A_{\alpha v}.$$

We show that if D is a crescent domain, then $(A:B^{-1})_v = (A_v:B^{-1})$ for all $A \in f(D)$ and $B \in F(D)$. Hence a crescent domain satisfying $(A:B^{-1})_v = (AB)_v$ for all $A \in f(D)$ and $B \in F(D)$ is a v-domain. However, a v-domain need not be a crescent domain. In fact, by Theorem 5, a Krull domain is a crescent domain if and only if it is a Dedekind domain.

Malik's result is contained in our first theorem.

THEOREM 1. *Let $A, B \in F(D)$. If $(A:A) = D$, then $(A:B^{-1})_v = (AB)_v$. In particular, if D is integrally closed (respectively, completely integrally closed), then $(A:B^{-1})_v = (AB)_v$ for all $A \in f(D)$ (respectively, $A \in F(D)$) and $B \in F(D)$.*

Proof. Suppose that $(A:A) = D$. It suffices to show that

$$(A:B^{-1})^{-1} = (AB)^{-1}.$$

Since $AB \subseteq (A:B^{-1})$, we have $(A:B^{-1})^{-1} \subseteq (AB)^{-1}$. Let $0 \neq x \in (AB)^{-1}$. Then $xAB \subseteq D$, so $xA \subseteq B^{-1}$, and hence

$$(A:xA) \supseteq (A:B^{-1}).$$

Since $(A:xA) = x^{-1}(A:A)$, we get $x(A:B^{-1}) \subseteq (A:A) = D$, so $x \in (A:B^{-1})^{-1}$. The last two statements of the theorem follow since D is integrally closed (respectively, completely integrally closed) if and only if $(A:A) = D$ for all $A \in f(D)$ (respectively, $A \in F(D)$) ([1], 34.7 and 34.3).

Recall that D is a *v-domain* (respectively, *completely integrally closed*) if, for all $A \in f(D)$ (respectively, $A \in F(D)$), there exists a $B \in F(D)$ with $(AB)_v = D$. In this case, $B_v = A^{-1}$.

THEOREM 2. *For an integral domain D, the following conditions are equivalent:*

(1) *D is a v-domain.*

(2) *$(A_v:B) = (AB^{-1})_v$ for all $A \in f(D)$, $B \in F(D)$.*

(3) *$(A_v:B^{-1}) = (AB)_v$ for all $A \in f(D)$, $B \in F(D)$.*

(4) $(A:B_v) = (AB^{-1})_v$ *for all* $A \in f(D)$, $B \in F(D)$.

(5) *D is integrally closed and* $(A:B)_v = (A_v:B_v)$ *for all* $A \in f(D)$, $B \in F(D)$.

Proof. (1)⇒(2). We always have $(AB^{-1})_v \subseteq (A_v:B)$. Let $0 \neq x \in (A_v:B)$, so $xB \subseteq A_v$. Then we have $x^{-1}B^{-1} \supseteq (A_v)^{-1} = A^{-1}$, so $xA^{-1} \subseteq B^{-1}$. Then $xAA^{-1} \subseteq AB^{-1}$, so

$$x \in Dx = x(AA^{-1})_v \subseteq (AB^{-1})_v.$$

(2)⇒(3). Replacing B by B^{-1} gives

$$(A_v:B^{-1}) = \big(A(B^{-1})^{-1}\big)_v = (AB_v)_v = (AB)_v.$$

(3)⇒(1). Take $B = A^{-1}$; then

$$D \subseteq (A_v:A_v) = (AA^{-1})_v \subseteq D.$$

Hence $(AA^{-1})_v = D$ for all $A \in f(D)$, so D is a v-domain.

(1)⇒(4). The proof of this implication is similar to the proof that (1)⇒(2).

(4)⇒(1). Take $B = A$. Then $D \subseteq (A:A)_v = (AA^{-1})_v \subseteq D$, so

$$(AA^{-1})_v = D \quad \text{for all } A \in f(D).$$

(5)⇒(1). Take $B = A$. Since D is integrally closed, $D = (A:A)$. Hence $D = D_v = (A:A)_v = (A_v:A_v)$. Consequently, A_v has a v-inverse ([1], 34.21). (Note that we have only used the hypothesis that $(A:B)_v = (A_v:B_v)$ for $A, B \in f(D)$.)

(1)⇒(5). It is well known that a v-domain is integrally closed. Let $0 \neq x \in (A:B)$. Then $xB \subseteq A$, so $xB_v \subseteq A_v$, and hence $x \in (A_v:B_v)$. Thus

$$(A:B)_v \subseteq (A_v:B_v)_v = (A_v:B_v).$$

Suppose that $0 \neq x \in (A_v:B_v)$, so $xB_v \subseteq A_v$. Then

$$xBA^{-1} \subseteq A_vA^{-1} \subseteq D,$$

so $xBA^{-1}A \subseteq A$. Hence $xA^{-1}A \subseteq (A:B)$, so

$$x \in Dx = x(A^{-1}A)_v \subseteq (A:B)_v.$$

THEOREM 3. *For an integral domain D, the following conditions are equivalent:*

(1) *D is completely integrally closed.*

(2) $(A_v:B) = (AB^{-1})_v$ *for all* $A, B \in F(D)$.

(3) $(A_v:B^{-1}) = (AB)_v$ *for all* $A, B \in F(D)$.

(4) $(A:B)_v = (AB^{-1})_v$ *for all* $A, B \in F(D)$.

The proof is similar to the proof of Theorem 2.

It is always true that $(A+B)^{-1} = A^{-1} \cap B^{-1}$ for $A, B \in F(D)$. More generally, $(\sum_\alpha A_\alpha)^{-1} = \bigcap_\alpha A_\alpha^{-1}$ provided that $A_\alpha, \sum_\alpha A_\alpha \in F(D)$. The "dual" condition $(A \cap B)^{-1} = A^{-1} + B^{-1}$ however need not be true. (In $D = K[X, Y]$,

K a field, take $A = (X)$ and $B = (Y)$.) Now, for any collection $\{A_\alpha\} \subseteq F(D)$ with $\bigcap_\alpha A_\alpha \neq 0$ we have

$$\left(\bigcap_\alpha A_\alpha\right)^{-1} \supseteq \sum_\alpha A_\alpha^{-1},$$

and hence $(\bigcap_\alpha A_\alpha)^{-1} \supseteq (\sum_\alpha A_\alpha^{-1})_v$. It is easily seen that

$$\left(\bigcap_\alpha A_\alpha\right)^{-1} = \left(\sum_\alpha A_\alpha^{-1}\right)_v$$

if and only if

$$\left(\bigcap_\alpha A_\alpha\right)_v = \bigcap_\alpha A_{\alpha v}.$$

However, even for D Noetherian, we need not have $(A \cap B)_v = A_v \cap B_v$, as the following example due to W. Heinzer shows. Let K be a field and let

$$D = K[[X^3, X^4, X^5]].$$

Then D is a one-dimensional local domain with maximal ideal

$$M = (X^3, X^4, X^5).$$

Let $A = (X^3, X^4)$ and $B = (X^3, X^5)$. Then $A_v = B_v = M$; but $A \cap B = (X^3)$, so

$$(A \cap B)_v = (X^3) \subset M = A_v \cap B_v.$$

Other examples of domains not satisfying $(A \cap B)_v = A_v \cap B_v$ may be obtained from Theorem 7. However, on the positive side, any essential domain D has the property that

$$(A_1 \cap \ldots \cap A_n)_v = A_{1v} \cap \ldots \cap A_{nv} \quad \text{for all } A_1, \ldots, A_n \in f(D).$$

(Recall that D is an *essential domain* if there is a set of prime ideals $\{P_\alpha\}$ of D such that $D = \bigcap D_{P_\alpha}$ and each D_{P_α} is a valuation domain.)

THEOREM 4. *Let D be a crescent domain. Then, for $A \in f(D)$ and $B \in F(D)$, $(A:B)_v = (A_v:B_v)$, and hence $(A:B^{-1})_v = (A_v:B^{-1})$. In particular, if D is a crescent domain and D also satisfies $(A:B^{-1})_v = (AB)_v$ for all $A \in f(D)$ and $B \in F(D)$, then D is a v-domain.*

Proof. Let $B = \sum_\alpha Db_\alpha$, where each $0 \neq b_\alpha \in K$. Then

$$(A:B) = \left(A:\sum_\alpha Db_\alpha\right) = \bigcap_\alpha (A:Db_\alpha) = \bigcap_\alpha Ab_\alpha^{-1}.$$

Since D is a crescent domain, we have

$$(A:B)_v = \left(\bigcap_\alpha Ab_\alpha^{-1}\right)_v = \bigcap_\alpha (Ab_\alpha^{-1})_v = \bigcap_\alpha A_v b_\alpha^{-1} = \bigcap_\alpha (A_v:Db_\alpha)$$
$$= \left(A_v:\sum_\alpha Db_\alpha\right) = (A_v:B) = (A_v:B_v).$$

Replacing B by B^{-1} gives the second result. If we further assume that $(A:B^{-1})_v = (AB)_v$ for all $A \in f(D)$ and $B \in F(D)$, we get $(AB)_v = (A_v:B^{-1})$ for all $A \in f(D)$ and $B \in F(D)$, which by Theorem 2 implies that D is a v-domain.

Thus an integrally closed crescent domain is a v-domain. However, a v-domain need not be a crescent domain. Let K be a field; then $K[X, Y]$ is a Krull domain, and hence a v-domain, but, by the next result, it is not a crescent domain. For D Noetherian and I an ideal of D, let $G(I)$ denote the *grade* of I, that is, the length of the longest R-sequence in I. Recall that for D Noetherian, $I_v = D$ if and only if $G(I) > 1$.

THEOREM 5. *Let D be a crescent domain that is also a Mori domain (i.e., D has ACC on v-ideals). Then every maximal v-ideal of D is a maximal ideal of D. So if D is a Noetherian crescent domain, then $G(D) = 1$. Moreover, a Krull domain is a crescent domain if and only if it is a Dedekind domain.*

Proof. Suppose that D is both a Mori domain and a crescent domain. Let P be a maximal v-ideal of D. It is well known that P is a prime ideal. Moreover, there is a finitely generated ideal $I \subseteq P$ with $I_v = P_v = P$. For $c \in D - P$,

$$(I, c)_v = (I_v, c)_v = (P_v, c)_v = D.$$

Suppose that P is not maximal; say, $P \subsetneq M$, where M is a maximal ideal of D. Then

$$\bigcap_{c \in M-P} (I, c)_v = \bigcap D = D.$$

Let

$$z \in \bigcap_{c \in M-P} (I, c).$$

Then $z \in P + (z^2)$, so $z = p + rz^2$ for some $p \in P$ and $r \in D$. Then $z(1-rz) = p \in P$; and $1 - rz \notin P$ since $z \in M$, so $z \in P$. Hence

$$\bigcap_{c \in M-P} (I, c) \subseteq P,$$

so

$$\Big(\bigcap_{c \in M-P} (I, c)\Big)_v \subseteq P_v = P \subsetneq D = \bigcap_{c \in M-P} (I, c)_v.$$

This contradiction shows that P is maximal.

Suppose that D is a Noetherian crescent domain. If $G(D) > 1$, then there is a maximal ideal M of D with $G(M) > 1$. Let P be a maximal v-ideal that is contained in M. By the preceding paragraph, P must be maximal, so $P = M$ and $G(M) = G(P) = 1$, a contradiction.

It is easily seen that a Dedekind domain is a crescent domain. Conversely, suppose that D is a crescent Krull domain. Then each prime ideal of D of height 1, being a maximal v-ideal, must be maximal, so D is a one-dimensional Krull domain, and hence a Dedekind domain.

Obviously, a Prüfer domain D is a crescent domain since every $A \in f(D)$ is invertible, and hence divisorial. More generally, if every $A \in f(D)$ is a v-ideal, then D is a crescent domain. Such domains, called FGV-*domains*, were briefly discussed in [4]. They include one-dimensional Gorenstein domains. It follows from 34.12 in [1] or from [4] that an FGV-domain is Prüfer if and only if it is integrally closed.

It is not hard to show that a valuation domain D satisfies the property that

$$(\bigcap_\alpha A_\alpha)_v = \bigcap_\alpha A_{\alpha v}$$

for any collection $\{A_\alpha\} \subseteq F(D)$ with $\bigcap_\alpha A_\alpha \neq 0$. However, while a Prüfer domain is a crescent domain, our next example shows that a Prüfer domain need not satisfy this stronger property.

EXAMPLE 6. Let A be the ring of all algebraic integers and let K be its quotient field. Then A is a one-dimensional (and hence completely integrally closed) Bézout domain. Let $\{M_\alpha\}$ be the collection of maximal ideals of A. Each M_α is idempotent ([1], pp. 520–522), and hence $M_{\alpha v} = A$. Moreover, it is easily seen that

$$\bigcap_\alpha M_\alpha = 0$$

(since $Z \subseteq A$ is integral). Let

$$D = A + XK[[X]].$$

Then D is a Bézout domain ([1], Exercise 13, pp. 286 and 287) with quotient field $K[[X]][X^{-1}]$. Since D is a Prüfer domain, it is a crescent domain. Let

$$M_\alpha^* = M_\alpha + XK[[X]].$$

Then each $(M_\alpha^*)_v = D$, so

$$\bigcap_\alpha (M_\alpha^*)_v = D,$$

while

$$(\bigcap_\alpha M_\alpha^*)_v = (XK[[X]])_v = XK[[X]].$$

THEOREM 7. (1) *If D is an essential domain, then*

$$(A_1 \cap \ldots \cap A_n)_v = A_{1v} \cap \ldots \cap A_{nv} \quad \textit{for all } A_1, \ldots, A_n \in f(D).$$

(2) *If D is integrally closed and $(A_1 \cap \ldots \cap A_n)_v = A_{1v} \cap \ldots \cap A_{nv}$ for all $A_1, \ldots, A_n \in f(D)$, then D is a v-domain.*

Proof. (1) Let ω be the $*$-operation on D defined by

$$A_\omega = \bigcap_\alpha AD_{P_\alpha},$$

where $D = \bigcap_\alpha D_{P_\alpha}$ is a representation of D as an essential domain, i.e., each P_α is a prime ideal of D and each D_{P_α} is a valuation domain. By 44.13 in [1], ω is equivalent to the v-operation, i.e., for all $A \in f(D)$, $A_\omega = A_v$. Now, for $A_1, \ldots, A_n \in f(D)$,

$$\begin{aligned}(A_1 \cap \ldots \cap A_n)_\omega &= \bigcap_\alpha (A_1 \cap \ldots \cap A_n) D_{P_\alpha} \\ &= \bigcap_\alpha (A_1 D_{P_\alpha} \cap \ldots \cap A_n D_{P_\alpha}) = \Big(\bigcap_\alpha A_1 D_{P_\alpha}\Big) \cap \ldots \cap \Big(\bigcap_\alpha A_n D_{P_\alpha}\Big) \\ &= A_{1\omega} \cap \ldots \cap A_{n\omega} = A_{1v} \cap \ldots \cap A_{nv}.\end{aligned}$$

Hence

$$\begin{aligned}(A_1 \cap \ldots \cap A_n)_v &= ((A_1 \cap \ldots \cap A_n)_\omega)_v \\ &= (A_{1v} \cap \ldots \cap A_{nv})_v = A_{1v} \cap \ldots \cap A_{nv}.\end{aligned}$$

(2) By Theorem 2, it suffices to show that, for $A, B \in f(B)$, $(A:B)_v = (A_v:B_v)$. But this follows as in the proof of the first part of Theorem 4 where we can now take B to be finitely generated.

So, by Theorem 7 (2), an integrally closed domain that is not a v-domain does not satisfy

$$(A_1 \cap \ldots \cap A_n)_v = A_{1v} \cap \ldots \cap A_{nv} \quad \text{for some } A_1, \ldots, A_n \in f(D).$$

For an example of an integrally closed domain that is not a v-domain, see [1], Exercise 2, p. 429. This raises the natural question of whether a v-domain satisfies

$$(A_1 \cap \ldots \cap A_n)_v = A_{1v} \cap \ldots \cap A_{nv} \quad \text{for all } A_i \in f(D).$$

We have shown that an essential domain (which is of course a v-domain) has this property and that this property plus integral closure (or the crescent property plus $(A:B^{-1})_v = (AB)_v$ for $A \in f(D)$ and $B \in F(D)$) imply that D is a v-domain.

We end by giving the promised example of an integral domain D satisfying $(A:B^{-1})_v = (AB)_v$ for all $A \in f(D)$ and $B \in F(D)$, although D is not integrally closed. Our example will involve PVD's. Recall that a domain D is called a *pseudo-valuation domain* (PVD) if every prime ideal P of D is strongly prime (i.e., $x, y \in K$ with $xy \in P$ implies $x \in P$ or $y \in P$). PVD's were introduced in [2] and further studied in a number of other papers. We recall a few facts about PVD's that are given in [2]. Let D be a PVD. Then D is quasi-local, say with the maximal ideal M. If D is not a valuation domain, then $V = M^{-1}$ is a valuation overring of D with the maximal ideal M. Moreover, the nonprincipal divisorial ideals of D coincide with the nonzero ideals of V.

PROPOSITION 8. *Let D be a PVD (which is not a valuation domain) with the maximal ideal M and valuation overring $V = M^{-1}$. Suppose that*

(a) *the ideals of V are all of the form xV or xM $(x \in V)$,*

(b) *V is not a finitely generated D-module.*

Then, for all $A \in f(D)$ and $B \in F(D)$, $(A:B^{-1})_v = (AB)_v$.

Proof. Since the equality $(A:B^{-1})_v = (AB)_v$ is easily seen to hold if B_v is principal, we may assume that B_v is not principal. Since $(AB)_v = (AB_v)_v$ and $B^{-1} = (B_v)^{-1}$, we may assume that B is a nonprincipal v-ideal of D. Hence, by [2] and (a), $B = xM$ or xV for some $x \in K$. So it suffices to show that

$$(A:V)_v = (AM)_v \quad \text{and} \quad (A:M)_v = (AV)_v.$$

(1) $(A:V)_v = (AM)_v$. Now $AMV = AMM^{-1} \subseteq A$, so $AM \subseteq A:V$, and hence $(AM)_v \subseteq (A:V)_v$. Let $y \in (A:V)$. Then $yV \subseteq A \subseteq AV = aV$ for some $a \in A$ since A is finitely generated. If $yV = aV$, then $A = aV$, so $V = a^{-1}A$ is a finitely generated D-module, a contradiction. So $yV \subsetneq aV$, and hence $y/a \in M$; consequently, $y \in aM = AM$. Hence $(A:V) \subseteq AM$, so

$$(A:V)_v \subseteq (AM)_v.$$

(2) $(A:M)_v = (AV)_v$. Now $AVM \subseteq A$, so $AV \subseteq (A:M)$, and hence $(AV)_v \subseteq (A:M)_v$. Let $y \in (A:M)$; so $yM \subseteq A \subseteq AV = aV$ for some $a \in A$. If $yV \supsetneq aV$, then $yV \supsetneq aV \supseteq yM$; so $M = (a/y)V$. Thus, using (1), we get

$$(A:M)_v = (A:(a/y)V)_v = (y/a)(A:V)_v = (y/a)(AM)_v = \big(A((y/a)M)\big)_v = (AV)_v.$$

Consequently, we can assume that $yV \subseteq aV$. Then

$$y \in yV \subseteq aV = AV;$$

so $(A:M) \subseteq AV$, and hence $(A:M)_v \subseteq (AV)_v$.

EXAMPLE 9. Let K/k be an infinite algebraic field extension and take

$$V = K[[X]] \quad \text{and} \quad D = k + XK[[X]].$$

Then D is a PVD with $M^{-1} = V$, and the conditions of Proposition 8 are all satisfied. Now D is not integrally closed, but, by Proposition 8, D satisfies $(A:B^{-1})_v = (AB)_v$ for all $A \in f(D)$ and $B \in F(D)$.

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