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SOME REMARKS ON STAR-OPERATIONS

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Introduction

In this paper we study certain star-operations on an integral domain. The first section contains the pertinent definitions and elementary facts. In the second section we introduce a star-operation, which we call the t_2-operation, and we ask whether the t_2-operation is the same as the t-operation studied by Jafford and Griffin. Using I_v to denote $(I^{-1})^{-1}$, this question can be recast as follows. If I is an ideal of a domain D such that $(a, b)_v \subseteq I$ whenever $a, b \in I$, then is $J_v \subseteq I$ for every finitely generated ideal $J \subseteq I$?

The third section defines and studies two star-operations which give rise to the F-ideals introduced by H. Adams and the semi-divisorial ideals introduced by S. Glaz and W. Vasconcelos. We then ask whether F-ideals and semi-divisorial ideals are the same, or, rephrasing in terms of the v-operation: If I is an ideal in a domain D such that $I : J = I$ whenever J is a two-generated ideal with $J_v = D$, then can the same be said for all finitely generated J with $J_v = D$? We note that this question can be easily answered affirmatively if D is Noetherian.

The major results of the paper are in the fourth section. In this section we show that in $D[x]$, where D is a domain and x is an indeterminate, F-ideals and semi-divisorial ideals are the same. This follows from the following lemma, which is interesting in its own right: If A is a finitely generated ideal of $D[x]$ with $A_v = D[x]$, then A contains a two-generated ideal B with $B_v = D[x]$. We also show that, with appropriate restrictions on $D[x]$, every t_2-prime ideal is a t-ideal.

1. Star operations of finite type

Throughout this paper we shall use D to denote an integral domain with quotient field K. Also, $\mathscr{I}(D)$ and $\mathscr{F}(D)$ will denote, respectively, the sets of nonzero integral and fractional ideals of D.

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Definition. *A star-operation on D* is a mapping $I \to I^*$ of $\mathscr{F}(D)$ into $\mathscr{F}(D)$ which satisfies, for each $a \neq 0$ in K and each $I, J \in \mathscr{F}(D)$, the following conditions:

(1) $(a)^* = (a)$, and $aI^* = (aI)^*$.

(2) $I \subseteq I^*$, and $I^* \subseteq J^*$ whenever $I \subseteq J$.

(3) $(I^*)^* = I^*$.

An ideal I of $\mathscr{F}(D)$ is called a $*$-*ideal* if $I = I^*$. A star-operation $*$ is said to be of *finite type* if $I^* = \bigcup\{J^* : J$ is a finitely generated ideal contained in $I\}$ for each $I \in \mathscr{F}(D)$.

Remark. This definition, as well as many elementary properties of star-operations, can be found in [2, Section 32], where it is pointed out that a mapping $I \to I^*$ *of* $\mathscr{I}(D)$ into $\mathscr{I}(D)$ satisfying conditions (1), (2), and (3) above has a unique extension to a star-operation on D. Thus, for the most part, we shall concern ourselves only with integral ideals of D, and we shall use the word "ideal" to mean "integral ideal."

We collect for easy reference some of the facts about star-operations which we shall use:

Proposition 1.1. *Let $I \to I^*$ denote a star-operation on D. Then*

(1) $(\sum I_\alpha)^* = (\sum I_\alpha^*)^*$ *for every subset $\{I_\alpha\}$ of $\mathscr{F}(D)$ for which $\sum I_\alpha \in \mathscr{F}(D)$.*

(2) $\bigcap I_\alpha^* = (\bigcap I_\alpha^*)^*$ *for every subset $\{I_\alpha\}$ of $\mathscr{F}(D)$ for which $\bigcap I_\alpha^* \neq 0$.*

(3) $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$ *for every pair $I, J \in \mathscr{F}(D)$.*

(4) *$I : J$ is a $*$-ideal whenever $I, J \in \mathscr{F}(D)$ and I is a $*$-ideal.*

(5) *If $*$ is of finite type and P is a prime ideal minimal over a $*$-ideal I, then P is a $*$-ideal.*

Remark. Parts (1), (2), (3) constitute Proposition 32.2 of [2], and part (4) is Exercise 1 of [2, Section 32]. Part (5) follows from [6, Theorem 9, p. 30]. We offer an alternate proof of (5): Let J be a finitely generated ideal of D contained in P. We shall show that $J^* \subseteq P$. Since P is minimal over I, $PD_p = \operatorname{rad}(ID_p)$ in D_p, and there is a positive integer n such that $J^n D_p \subseteq ID_p$. Thus there is an element $s \in D - P$ with $sJ^n \subseteq I$. It follows from part (3) above that

$$s(J^*)^n \subseteq s((J^*)^n)^* = s(J^n)^* = (sJ^n)^* \subseteq I^* = I \subseteq P.$$

Therefore, since $s \notin P$, we have $J^* \subseteq P$, as desired.

2. The v- and t-operations

One of the best known examples of a star-operation is the v-operation. For $I \in \mathscr{F}(D)$ I_v is defined by $I_v = (I^{-1})^{-1} = \bigcap\{J : J$ is a principal fractional ideal containing $I\}$. The v-operation is not in general of finite type. However, [2, Exercise 3, Section 32] shows that to every star-operation $*$, we may associate a star-operation $*_s$ of finite type by defining $I^{*_s} = \bigcup\{J^* : J$ is a non-zero finitely generated fractional ideal contained in $I\}$. The v_s-operation is called the t-operation (see [4] and [5]). We introduce a closely related star-operation of finite type:

Proposition 2.1. *For each subset J of D, let $J' = \bigcup\{(a, b)_v : a, b \in J\}$. For each ideal I of D, let $I_0 = I$ and $I_n = (I_{n-1})'$ for $n \geq 1$. Define a map $*$ by $I^* = \bigcup_{k=0}^{\infty} I_k$. Then $*$ is a star-operation of finite type.*

Proof. We first show that I^* is an ideal of D. If $x, y \in I^*$ then $x, y \in I_n$ for some n and $x - y \in (x, y) \subseteq (x, y)_v$, whence $x - y \in I_{n+1} \subseteq I^*$ and I^* is closed under subtraction. An easy induction argument shows that each I_n is closed under D-multiplication, so that I^* is an ideal of D. To verify that $*$ is a star-operation, we first note that $(a)^* = (a)$ follows easily from the fact that $(a)_v = (a)$. Now

$$aI' = a\bigcup\{(c, d)_v : c, d \in I\} = \bigcup\{(ac, ad)_v : c, d \in I\}$$
$$= \bigcup\{(x, y)_v : x, y \in aI\} = (aI)'.$$

It follows that $aI^* = (aI)^*$. It is clear that $I \subseteq I^*$ and that $I^* \subseteq J^*$ whenever $I \subseteq J$. Finally,

$$(I^*)' = \bigcup\{(a, b)_v : a, b \in I^*\}$$
$$= \bigcup\{(a, b)_v : a, b \in I_n \text{ for some } n\} \subseteq I^*.$$

Thus $(I^*)^* = I^*$.

We have left to show that $*$ is of finite type. Let I be an ideal of D and let $\bar{I} = \bigcup\{C^* : C$ is a finitely generated ideal of D contained in $I\}$. Clearly $I \subseteq \bar{I}$. Inductively, assume $I_{n-1} \subseteq \bar{I}$. If $x \in I_n$, then $x \in (a, b)_v$ for some $a, b \in I_{n-1}$. Hence $a \in A^*$ and $b \in B^*$, where A and B are finitely generated ideals contained in I. Thus

$$x \in (A^* + B^*)' \subseteq (A^* + B^*)^* = (A + B)^* \subseteq \bar{I}.$$

By induction $I^* \subseteq \bar{I}$. The reverse containment being evident, the result is proved.

Remark. We shall refer to the above star-operation as the t_2-operation. One may define, for each integer $n > 2$, an analogous t_n-operation, merely by changing the definition of J' above to $J' = \bigcup\{(a_1, \ldots, a_n)_v : a_1, \ldots, a_n \in J\}$. One then verifies easily that an ideal I is a t-ideal if and only if I is t_n for each $n > 2$. We do not pursue this here, mainly because it is conceivable that the t-operation and the t_2-operation are the same. Thus we raise the following

Question. If an ideal I of D has the property that $(a, b)_v \subseteq I$ whenever $a, b \in I$, then as I necessarily a t-ideal?

We are able to give an affirmative answer in case I is prime and D is either a coherent or an integrally closed polynomial ring. This is postponed until Section 4.

3. *F*-operations

In [1] H. Adams defines an F-operation in a manner equivalent to the following. For each subset J of D, let

$$J' = \{x \in D: xa, ab \in J \text{ for some } a, b \in D \text{ with } (a, b)_v = D\}.$$

For each ideal I set $I_0 = I$ and $I_n = (I_{n-1})'$ for $n \geqslant 1$. Finally, let $I_F = \bigcup_{k=0}^{\infty} I_k$.

The details required to show that the map $I \to I_F$ is a star-operation of finite type on D are routine, with the exception of the verification that I_F is an ideal. As this is done in [1, Lemmas 2.3 and 2.4] we state without proof:

Proposition 3.1. *The map $I \to I_F$ defines a star-operation of finite type on D.*

Many results in [1] follow easily from known facts about star-operations. In particular the fact that $D = \bigcap\{D_p: P \text{ is an } F\text{-prime ideal of } D\}$ ([1, Theorem 2.14]) follows from [4, Proposition 4].

Remark. Since the ideal J' above is defined in terms of pairs of elements a, b, we could call the F-operation the F_2-operation. One can then define, for each $n \geqslant 3$, an analogous F_n-operation, merely by changing the definition of J' accordingly.

In [3] Glaz and Vasconcelos call an ideal I of D semi-divisorial if $I:J = I$ whenever J is a finitely generated ideal of D with $J_v = D$. It is easy to see that I is semi-divisorial if and only if I is F_n for each $n = 2, 3, \ldots$. In this context one could call a semi-divisorial ideal an F_∞-ideal.

Proposition 3.2. *For each ideal I of D define I^* by $I^* = \bigcup\{I: J \mid J$ is a finitely generated ideal of D with $J_v = D\}$. Then $*$ is a star-operation of finite type.*

Proof. Let I be an ideal of D. We first show that I^* is an ideal. To this end let $x, y \in I^*$ with $xA \subseteq I$, $yB \subseteq I$, where A, B are finitely generated ideals with $A_v = B_v = D$. Then $(x-y)AB \subseteq I$ and $(AB)_v = (A_vB_v)_v = D$. Hence $x - y \in I^*$. One easily verifies that I^* is closed under D-multiplication, so that I^* is an ideal. Most of the details required to show that $*$ is a star-operation are also routine. We verify only that $(I^*)^* = I^*$. If $t \in (I^*)^*$, then $tA \in I^*$ with A finitely generated and $A_v = D$. If $A = (a_1, \ldots, a_n)$ then for each $i = 1, \ldots, n$, there is a finitely generated ideal B_i with $(B_i)_v = D$ and $ta_iB_i \subseteq I$. Let $B = B_1 \ldots B_n$. Then $tAB \subseteq I$, AB is finitely generated and $(AB)_v = D$. Thus $t \in I^*$, as desired. Finally, we show that $*$ has finite type. Suppose $t \in I^*$, say $tA \subseteq I$ with A finitely generated and $A_v = D$. Note that $1A \subseteq A$ implies $1 \in A^*$, so that $A^* = D$. Thus $t \in tD = tA^* = (tA)^* \subseteq \bigcup\{C^* | C \subseteq I$ and C is a finitely generated ideal of $D\}$. This completes the proof.

The natural question arises: Are the F- and F_∞-operations the same? In Section 4 we answer this question in polynomial rings. The answer in case D is Noetherian is given by

Proposition 3.3. *If D is Noetherian, then every F-ideal is an F_∞-ideal.*

Proof. Let I be an F-ideal of D. By [7, Exercise 2, p. 102] an ideal B of D satisfies $B_v = D$ if and only if B has grade at least 2. Thus, if $xA \subseteq I$ and $A_v = D$, then there are elements $a, b \in A$ with $(a, b)_v = D$. It follows that $x \in I$, since I is an F-ideal.

Remark. It is clear that every t_2-ideal is an F-ideal and that every t-ideal is an F_∞-ideal (semi-divisorial). To show that neither of the converses is true, we shall produce an example of a Noetherian domain containing an F_∞-prime ideal which is not a t_2-ideal. Another (non-Noetherian) example can be constructed essentially by adding another indeterminate to Adams' example ([1, Section 3]).

Example. Let K be a field and let $R' = K[x_1, \ldots, x_n]_S$, where the x_i are indeterminates over K and S is the complement in $K[x_1, \ldots, x_n]$ of the union of the maximal ideals $M = (x_1, \ldots, x_n)$ and $N = (x_1, \ldots, x_{n-1}, x_n + 1)$. Let $I = M \cap N$ and put $R = K + I$. By [8, E2.1, p. 204] R is a local (Noetherian) domain with maximal ideal I and integral closure R'. Since $IR' \subseteq I \subseteq R$, we have $R' \subseteq I^{-1}$ so that $I_v = I$. It follows easily that I is an F_∞-ideal of D [2, Theorem 34.1(4)]. We then claim that every ideal A of R is an F_∞-ideal. For suppose $xJ \subseteq A$ with J an (finitely generated) ideal of R and $J_v = R$. Since $1 \notin I$ we have $1 \cdot J \not\subseteq I$ so that $J = R$. Thus $x \in A$.

Now assume $n > 2$ and let Q be the prime ideal of R' generated by x_1 and x_2. Then Q is also a prime ideal of R, since $Q \subseteq I$. We shall show that Q is not a t_2-ideal of R. Since $B = (x_1, x_2)R \subseteq Q$ it suffices to show $B_v \not\subseteq Q$. As Q is contained properly in I, it is enough to show $B_v = I$, or, equivalently, $B^{-1} \subseteq I^{-1}$. Now, $B^{-1}Q = B^{-1}BR' \subseteq R'$ and $B^{-1} \subseteq Q^{-1}$, where Q^{-1} is taken with respect to R'. Since in R' Q contains the regular sequence x_1, x_2, $Q^{-1} = R'$ (as in the proof of Proposition 3.3). Thus $B^{-1} \subseteq Q^{-1} = R' \subseteq I^{-1}$, as desired.

4. Star-operations in $D[x]$

In this section we show that t-ideals and F_∞-ideals extend, respectively, to t-ideals and F_∞-ideals in $D[x]$, x an indeterminate. We further show that the F-and F_∞-operations are the same in $D[x]$. Lemma 4.1 is due to Nishimura [9, Proposition 7].

Lemma 4.1. *Let $I \in \mathcal{F}(D)$. Then $(ID[x])^{-1} = I^{-1}D[x]$.*

Proof. Note that if $u \in K(x)$ with $uI \subseteq D[x]$ then, since $I \neq 0$, $u = k(x) \in K[x]$. Let

A_k denote the fractional ideal of D generated by the coefficients of k. Then

$$k(x)\in(ID[x])^{-1}\Leftrightarrow k(x)ID[x]\subseteq D[x]\Leftrightarrow A_kI\subseteq D$$
$$\Leftrightarrow A_k\subseteq I^{-1}\Leftrightarrow k(x)\in I^{-1}D[x].$$

Lemma 4.2. *Let* $f_1(x),\ldots,f_n(x)\in D[x]$. *Then*

$$(f_1,\ldots,f_n)^{-1}\cap K[x]=D[x]\Leftrightarrow(\sum A_{f_i})^{-1}=D.$$

Proof. Suppose $k(x)\in K[x]$ and $k(x)f_i(x)\in D[x]$ for each $i=1,2,\ldots,n$. By [2, Theorem 28.1] there is a positive integer m with $A_{f_i}^{m+1}A_k=A_{f_i}^mA_{f_ik}$ for each i. Since $A_{f_ik}\subseteq D$ we have $\sum A_{f_i}^{m+1}A_k\subseteq D$, whence $A_k(\sum A_{f_i}^{m+1})_v\subseteq D$. However, assuming $(\sum A_{f_i})^{-1}=D$, we have $(\sum A_{f_i}^{m+1})_v=D$ also. Thus $A_k\subseteq D$ and $k(x)\in D[x]$. The converse is trivial.

Proposition 4.3. *If* $*$ *denotes either the* v-, *the* t-, *or the* F_∞-*operation, then* $(ID[x])^*=I^*D[x]$ *for each* $I\in\mathscr{F}(D)$.

Proof. It suffices to prove the result for integral ideals I of D. The statement for the v-operation follows immediately from Lemma 4.1. We next consider the t-operation. We claim that $I_tD[x]$ is a t-ideal. To see this let $f_1,\ldots,f_n\in I_tD[x]$. Then $\sum A_{f_i}\subseteq I_t$, whence, since I_t is a t-ideal, $(\sum A_{f_i})_v\subseteq I_t$. Thus $(f_1,\ldots,f_n)_v\subseteq((\sum A_{f_i})D[x])_v=(\sum A_{f_i})_vD[x]\subseteq I_tD[x]$, and the claim is proved. It follows that $(ID[x])_t\subseteq I_tD[x]$. Conversely, if $g(x)\in I_tD[x]$, then $A_g\subseteq I_t$, so that $A_g\subseteq B_v$ for some finitely generated ideal B contained in I. Thus $g(x)\in A_gD[x]\subseteq B_vD[x]=(BD[x])_v$. Since $BD[x]$ is a finitely generated ideal contained in $ID[x]$, $g(x)\in(ID[x])_t$, as desired.

Finally, we let $*$ denote the F_∞-operation. An argument analogous to the one just completed yields $I^*D[x]\subseteq(ID[x])^*$. To complete the proof, it suffices to show that $I^*D[x]$ is an F_∞-ideal. To this end suppose $u(f_1(x),\ldots,f_n(x))\subseteq I^*D[x]$, where $u\in K(x)$ and $f_1(x),\ldots,f_n(x)\in D[x]$ with $(f_1,\ldots,f_n)_v=D[x]$. Then $uD[x]=u(f_1,\ldots,f_n)_v\subseteq(I^*D[x])_v\subseteq D[x]$, so that $u=h(x)\in D[x]$. By Lemma 4.2 $(\sum A_{f_i})_v=D$. Again by [2, Theorem 28.1] find m with $A_{f_i}^{m+1}A_h=A_{f_i}^mA_{f_ih}\subseteq I^*$ for each $i=1\ldots,n$. Then $A_h(\sum A_{f_i}^{m+1})\subseteq I^*$. However, $(\sum A_{f_i}^{m+1})_v=D$ and I^* is an F_∞-ideal, so that $A_h\subseteq I^*$. It follows that $h(x)\in I^*D[x]$, and $I^*D[x]$ is an F_∞-ideal.

Lemma 4.4. *Let* A *be a finitely generated ideal of* $D[x]$ *with* $A^{-1}=D[x]$. *Then*

(i) $A\cap D\neq 0$,

(ii) *there is an element* $f\in A$ *with* $A_f^{-1}=D$, *and*

(iii) *if* a *is a nonzero element of* $A\cap D$ *and* $f\in A$ *with* $A_f^{-1}=D$, *then* $(a,f)^{-1}=D[x]$.

Proof. Suppose $A\cap D=0$. Then $AK[x]=k(x)K[x]$ for some $k(x)\in K[x]$ with $\deg(k)>0$. Hence $A(k(x))^{-1}$ is a finitely generated $D[x]$ submodule of $K[x]$.

Choose $c \neq 0$ in D with $cA(k(x))^{-1} \subseteq D[x]$. Then $c(k(x))^{-1} \in A^{-1} - D[x]$, proving (i). To prove (ii) assume $A = (f_1, \ldots, f_n)$. If $f(x) = \sum_{i=1}^{n} x^{(i-1)N} f_i(x)$, where $N > \max\{\deg(f_i) \mid i = 1, \ldots, n\}$, then $A_f = \sum A_{f_i}$. By Lemma 4.2 $A_f^{-1} = (\sum A_{f_i})^{-1} = D$. Finally, let a, f be as in (iii). Since $(a, f) \cap D \neq 0$, $(a, f)^{-1} = (a, f)^{-1} \cap K[x]$. Thus, since $D = (A_f)_v \subseteq (A_a + A_f)_v \subseteq D$, $(A_a + A_f)^{-1} = D$, and $(a, f)^{-1} = D[x]$, again by Lemma 4.2.

Theorem 4.5. *Every F-ideal of $D[x]$ is an F_∞-ideal.*

Proof. Let I be an F-ideal of $D[x]$, and suppose $gA \subseteq I$ with A finitely generated and $A_v = D[x]$. By Lemma 4.4 there are elements $f \in A$ and $a \in A \cap D$ such that $(a, f)_v = D[x]$. Of course $g(a, f) \subseteq I$, whence $g \in I$, as I is an F-ideal.

Proposition 4.6. *Let $I \in F(D)$. Then the following statements are equivalent*:

(1) *I is an F_∞-ideal of D,*
(2) *$ID[x]$ is an F_∞-ideal of $D[x]$,*
(3) *$ID[x]$ is an F-ideal of $D[x]$.*

Proof. The equivalence of (2) and (3) follows from Theorem 4.5, and the equivalence of (1) and (2) follows easily from Proposition 4.3.

We next characterize those primes of $D[x]$ which are F-primes.

Proposition 4.7. *A prime ideal P of $D[x]$ is an F-prime $\Leftrightarrow$ either $P \cap D = 0$ or P contains no elements f with $A_f^{-1} = D$.*

Proof. If $P \cap D = 0$, then P is minimal over a principal ideal, which implies that P is an F-prime by Proposition 1.1(5). Suppose that $A_f^{-1} \neq D$ for each $f \in P$. If P is not an F-prime, then by [1, Proposition 1.4] there are elements $f_1, f_2 \in P$ with $(f_1, f_2)_v = D[x]$. In this case we may use Lemma 4.4 to find $f \in (f_1, f_2) \subseteq P$ with $A_f^{-1} = D$, a contradiction.

Conversely, suppose $P \cap D \neq 0$ and there is an element $f \in P$ with $A_f^{-1} = D$. Then, choosing $p \neq 0$ in P, one shows easily that $(p, f)_v = D[x]$, and P is not an F-prime.

We close by giving some conditions on $D[x]$ that ensure that every t_2-prime of $D[x]$ is a t-prime.

Proposition 4.8. *Assume that D is integrally closed. Then every t_2-prime of $D[x]$ is a t-prime.*

Proof. Let P be a t_2-prime of $D[x]$. Suppose $(f_1, \ldots, f_n) \subseteq P$. Pick $f \in (f_1, \ldots, f_n)$ with $A_f = A_{f_1} + \cdots + A_{f_n}$. By Proposition 1.1(5) we may assume $P \cap D \neq 0$. Choose $p \neq 0$ in P and consider the ideal (p, f). Since P is a t_2-prime we have $(p, f)_v \subseteq P$. We shall complete the proof by showing that $(f_1, \ldots, f_n) \subseteq (p, f)_v$. For this it is enough to show that $(p,f)^{-1} \subseteq (f_1, \cdot, f_n)^{-1}$. Suppose $u \in (p,f)^{-1}$. Since $p \neq 0$ we have $u = k(x) \in K[x]$. Then $f(x)k(x) \in D[x]$. By [2, Theorem 28.1] choose m so that $A_f A_k A_f^m = A_f^{m+1} A_k = A_f^m A_{fk} \subseteq A_f^m$. Since $D[x]$ is integrally closed, we have $A_f A_k \subseteq D$, whence $A_{f_i} A_k \subseteq D$ for each $i = 1, \ldots, n$. Hence $k(x)f_i(x) \in D[x]$ for each i, and $k(x) \in (f_1, \ldots, f_n)^{-1}$, as was to be shown.

Proposition 4.9. *Assume that I^{-1} is finitely generated for every 2-generated ideal I of $D[x]$. (This occurs, for example, when $D[x]$ is coherent.) Then every t_2-prime of $D[x]$ is a t-prime.*

Proof. Let P be prime in $D[x]$ with $(f_1, \ldots, f_n) \subseteq P$. Choose p, f as in the above proof. If $k(x) \in (p,f)^{-1}$, then $A_f^{m+1} A_k = A_f^m A_{fk} \subseteq D$, where $m = \deg(k)$ ([2, *Theorem* 28.1]). *Hence for each* $i = 1, \ldots, n$, $A_{f_i}^{m+1} A_k \subseteq D$ and $f_i(x)^{m+1} k(x) \in D[x]$. If X is a finite base for $(p,f)^{-1}$ let $N > \max\{\deg(k) \mid k(x) \in X\}$. Then $f_i(x)^N k(x) \in D[x]$ for each $k(x) \in X$, and $(f_1^N, \ldots, f_n^N)(p,f)^{-1} \subseteq D[x]$. Choose M so that $(f_1, \ldots, f_n)^M \subseteq (f_1^N, \ldots, f_n^N)$. Then

$$((f_1, \ldots, f_n)_v)^M \subseteq (((f_1, \ldots, f_n)_v)^M)_v = ((f_1, \ldots, f_n)^M)_v$$
$$\subseteq (f_1^N, \ldots, f_n^N)_v \subseteq (p,f)_v \subseteq P.$$

This completes the proof.

References

[1] H. Adams, Factorization-prime ideals in integral domains, Pacific J. Math. 66 (1976) 1–8.
[2] R. Gilmer, Multiplicative Ideal Theory (Marcel Dekker, New York, 1972).
[3] S. Glaz and W. Vasconcelos, Flat ideals II, Manuscripta Mathematika 22 (1977) 325–341.
[4] M. Griffin, Some results on Prufer v-multiplication rings, Canad. J. Math. 19 (1967) 710–722.
[5] M. Griffin, Rings of Krull type, J. Reine Ange. Math. 229 (1968) 1–27.
[6] P. Jafford, Les Systemes d'Ideaux (Dunod, Paris, 1960).
[7] I. Kaplansky, Commutative Rings (Allyn and Bacon, Boston, 1970).
[8] M. Nagata, Local Rings (R. E. Kreiger, New York, 1975).
[9] T. Nishimura, On the v-ideal of an integral domain, Bull. Kyoto Gakugei Univ. (Ser. B) 17 (1961) 47–50.