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STAR-OPERATIONS INDUCED BY OVERRINGS

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Let D be a commutative integral domain. A $*$-operation on D is a mapping $F \longrightarrow F^*$ on the set of nonzero fractional ideals of D satisfying certain properties which will be given in the next paragraph. An important example of a $*$-operation is the v-operation. If $\{P_\alpha\}$ is a collection of prime ideals of D with $D = \cap D_{P_\alpha}$, then the mapping $F \longrightarrow F^* = \cap FD_{P_\alpha}$ is a $*$-operation satisfying $(A \cap B)^* = A^* \cap B^*$ for all nonzero fractional ideals A and B of D. Moreover, if A is a nonzero ideal of D with $A^* \neq D$, then A is contained in a prime ideal P of D with $P^* = P$. The purpose of this paper is to prove, conversely, that if $*$ is a $*$-operation on D with the properties that $(A \cap B)^* = A^* \cap B^*$ for all nonzero fractional ideals A and B of D and that each proper integral $*$-ideal is contained in a prime

$*$-ideal, then the $*$-operation is given by $A \longrightarrow A^*$ $= \cap AD_{P_\alpha}$ for some collection $\{P_\alpha\}$ of prime ideals of D with $D = \cap D_{P_\alpha}$. As a corollary, we obtain the interesting result that a Noetherian domain D has the property that $(A \cap B)_v = A_v \cap B_v$ for all nonzero fractional ideals A and B of D if and only if for each grade one prime ideal P of D, D_P is a (one-dimensional) Gorenstein domain.

Let D be a commutative integral domain with quotient field K. Let $F(D)$ denote the set of nonzero fractional ideals of D and let $f(D)$ denote the subset of finitely generated members of $F(D)$. A mapping $F \longrightarrow F^*$ of $F(D)$ into $F(D)$ is called a <u>$*$-operation</u> on D if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F(D)$:

(1) $(a)^* = (a)$, $(aA)^* = aA^*$;

(2) $A \subseteq A^*$, if $A \subseteq B$, then $A^* \subseteq B^*$; and

(3) $(A^*)^* = A^*$.

A fractional ideal $A \in F(D)$ is called a <u>$*$-ideal</u> if $A = A^*$.

A $*$-operation $*$ on D is said to be of <u>finite character</u> if $A^* = \cup\{J^* \mid J \subseteq A$ with $J \in f(D)\}$ for each $A \in F(D)$. If $*$ is a $*$-operation on D, then the function $F \longrightarrow F^{*_s}$ of $F(D)$ into $F(D)$ given by $F^{*_s} = \cup\{J^* \mid J \subseteq F$ with $J \in f(D)\}$ is a finite character $*$-operation on D. Clearly $A^* = A^{*_s}$ for each

$A \in f(D)$. Recall that the function on $F(D)$ defined by $A \longrightarrow A_v = (A^{-1})^{-1}$ is a $*$-operation on D called the <u>v-operation</u>. The <u>t-operation</u> on D is given by $t = v_s$, or equivalently, $A_t = \cup\{J_v \mid J \subseteq A$ with $J \in f(D)\}$. The reader may consult [2, section 32 and 34] for the basic properties of $*$-operations and the v-operation. The reader is also referred to [2] for definitions and notation not defined here.

Let $*$ be a $*$-operation on D. It is easily proved that for $\{A_\alpha\} \subseteq F(D)$ with $\Sigma A_\alpha \in F(D)$, then $(\Sigma A_\alpha)^* = (\Sigma A_\alpha^*)^*$ and for $A, B \in F(D)$, $(AB)^* = (A^*B^*)^*$. This raises the natural question of whether $(\cap A_\alpha)^* = (\cap A_\alpha^*)^*$ for each collection $\{A_\alpha\} \subseteq F(D)$ with $\cap A_\alpha \neq 0$. Since we always have $\cap A_\alpha^* = (\cap A_\alpha^*)^*$ provided $\cap A_\alpha \neq 0$, the question becomes whether $\cap A_\alpha^* = (\cap A_\alpha)^*$. As we will see, this is a very strong condition. In fact, if $*$ has finite character, $\cap A_\alpha^* = (\cap A_\alpha)^*$ for all $\{A_\alpha\} \subseteq f(D)$ with $\cap A_\alpha \neq 0$ if and only if $*$ is the identity $*$-operation $A \longrightarrow A^* = A$.

Now even for D Noetherian, we need not have $(A \cap B)_v = A_v \cap B_v$ for nonzero integral ideals A and B of D, as the following example due to W. Heinzer shows. Let k be a field and let $D = k[[X^3, X^4, X^5]]$. Then D is a one-dimensional local domain with maximal ideal $M = (X^3, X^4, X^5)$. Let $A = (X^3, X^4)$ and $B = (X^3, X^5)$. Then $A_v = B_v = M$; but $A \cap B = (X^3)$, so $(A \cap B)_v = (X^3) \subsetneq M = A_v \cap B_v$.

Suppose that $\{D_\alpha\}$ is a collection of overrings of D with $D = \cap D_\alpha$. Then the mapping $A \longrightarrow A^*$ $= \cap AD_\alpha$ is a $*$-operation on D and $AD_\alpha = A^* D_\alpha$ for each α and each $A \in F(D)$. The proof of this well-known result may be found in [2, Theorem 32.5]. We will be mostly interested in the case where each D_α $= D_{P_\alpha}$ for some prime ideal P_α of D. Some important properties of this type of $*$-operation are given in our first theorem.

THEOREM 1. Let $\mathcal{P} = \{P_\alpha\}$ be a set of nonzero prime ideals of D with $D = \cap D_{P_\alpha}$. For $A \in F(D)$, define $A^* = \cap AD_{P_\alpha}$.

(1) The mapping $A \longrightarrow A^*$ defines a $*$-operation on D. Moreover, $AD_{P_\alpha} = A^* D_{P_\alpha}$ for each α and each $A \in F(D)$.

(2) $(A \cap B)^* = A^* \cap B^*$ for $A, B \in F(D)$.

(3) $(A \underset{D}{:} B)^* = (A^* \underset{D}{:} B^*)$ for $A \in F(D)$ and $B \in f(D)$.

(4) $P_\alpha^* = P_\alpha$ for each $P_\alpha \in \mathcal{P}$.

(5) If A is an integral ideal of D with $A^* \neq D$, then $A \subseteq P_\alpha$ for some α. Hence each proper integral $*$-ideal of D is contained in a prime $*$-ideal.

(6) If $D = \cap D_{P_\alpha}$ is locally finite (i.e., each $0 \neq d \in D$ is a unit in almost all D_{P_α}), then $*$ has finite character.

Proof. The proof of (1) is a special case of the result mentioned in the paragraph preceding Theorem 1.

(2) Let $A,B \in F(D)$. Then $(A\cap B)^* = \cap(A\cap B)D_{P_\alpha}$ $= \cap(AD_{P_\alpha} \cap BD_{P_\alpha}) = (\cap AD_{P_\alpha}) \cap (\cap BD_{P_\alpha}) = A^* \cap B^*$.

(3) Let $A,B \in F(D)$ with B finitely generated. Now $(A \underset{D}{:} B)B \subseteq A$, so $(A \underset{D}{:} B)^*B^* \subseteq A^*$. Hence $(A \underset{D}{:} B)^* \subseteq (A^* \underset{D}{:} B^*)$. Conversely, let $d \in (A^* \underset{D}{:} B^*)$. Then $dB^* \subseteq A^*$ so $dBD_{P_\alpha} = dB^*D_{P_\alpha} \subseteq A^*D_{P_\alpha} = AD_{P_\alpha}$. Hence $d \in (AD_{P_\alpha} \underset{D_{P_\alpha}}{:} BD_{P_\alpha}) = (A \underset{D}{:} B)D_{P_\alpha}$, where the last equality follows since B is finitely generated. But then $d \in \cap(A \underset{D}{:} B)D_{P_\alpha} = (A \underset{D}{:} B)^*$.

(4) Now $P_\alpha = P_\alpha D_{P_\alpha} \cap D = P_\alpha^* D_{P_\alpha} \cap D \supseteq P_\alpha^* \supseteq P_\alpha$, so $P_\alpha^* = P_\alpha$.

(5) Since $A^* \neq D$, for some α, $AD_{P_\alpha} \neq D_{P_\alpha}$. Hence $A \subseteq AD_{P_\alpha} \cap D \subseteq P_\alpha D_{P_\alpha} \cap D = P_\alpha$.

(6) It suffices to prove that if A is a nonzero integral ideal of D and $0 \neq x \in A^*$, then $x \in A_0^*$ for some finitely generated ideal $A_0 \subseteq A$.

Choose $0 \neq a \in A$. Then a is a unit in each D_{P_α}, except, say for $\alpha = \alpha_1, \cdots, \alpha_n$. Now for each $i = 1, \cdots, n$, $x \in A^* D_{P_{\alpha_i}} = AD_{P_{\alpha_i}}$, so there is a finitely generated ideal $A_{\alpha_i} \subseteq A$ with $x \in A_{\alpha_i} D_{P_{\alpha_i}}$. Put $A_0 = (a) + A_{\alpha_1} + \cdots + A_{\alpha_n}$. Then $A_0 \subseteq A$ is a finitely generated ideal and $x \in A_0 D_{P_\alpha}$ for each α, so $x \in \cap A_0 D_{P_\alpha} = A_0^*$.

A variation of (5) of Theorem 1 appears in [8].

While Theorem 1 is sufficient for our needs, several of the parts of Theorem 1 may be greatly generalized. Suppose that $\{D_\alpha\}$ is a collection of overrings of D with $D = \cap D_\alpha$. Suppose that for each α, there is a $*$-operation $*_\alpha$ on D_α. Then the mapping $A \longrightarrow A^* = \cap (AD_\alpha)^{*_\alpha}$ is a $*$-operation on D. If we take each $*_\alpha$ to be the identity $*$-operation on D_α, we get the previously defined $*$-operation $A \longrightarrow A^* = \cap AD_\alpha$. Observe that we can take each $D_\alpha = D$, so that each $*_\alpha$ is a $*$-operation on D. In Theorem 2 we state without proof some properties of this $*$-operation. The proof of Theorem 2 is similar to the proof of [2, Theorem 32.5] and Theorem 1.

<u>THEOREM 2</u>. Let D be an integral domain and let

$\{D_\alpha\}$ be a collection of overrings of D with $D = \cap D_\alpha$. For each D_α, let $*_\alpha$ be a $*$-operation defined on D_α. For $A \in F(D)$, put $A^* = \cap(AD_\alpha)^{*_\alpha}$.

(1) The mapping $A \longrightarrow A^*$ induces a $*$-operation on D. Moreover, $(A^*D_\alpha)^{*_\alpha} = (AD_\alpha)^{*_\alpha}$ for each α and each $A \in F(D)$.

(2) If each D_α is a flat overring and each $*_\alpha$ satisfies $(A \cap B)^{*_\alpha} = A^{*_\alpha} \cap B^{*_\alpha}$ for all $A,B \in F(D_\alpha)$, then $(A \cap B)^* = A^* \cap B^*$ for all $A,B \in F(D)$.

(3) If each D_α is a flat overring and each $*_\alpha$ satisfies $(A :_{D_\alpha} B)^{*_\alpha} = (A^{*_\alpha} :_{D_\alpha} B^{*_\alpha})$ for all $A \in F(D_\alpha)$ and $B \in f(D_\alpha)$, then $(A :_D B)^* = (A^* :_D B^*)$ for all $A \in F(D)$ and $B \in f(D)$.

(4) If $D = \cap D_\alpha$ is locally finite and each $*_\alpha$ has finite character, then $*$ has finite character.

<u>THEOREM 3</u>. Let $*$ be a $*$-operation on D. Let $\mathcal{P} = \{P_\alpha\}$ be a set of prime ideals of D with the property that each proper integral $*$-ideal of D is contained in some P_α. Then $A \longrightarrow A' = \cap AD_{P_\alpha}$ is a $*$-operation on D and $A' \subseteq A^*$ for each $A \in F(D)$.

Proof. Let A be a proper ideal of D of the form $A = ((a) \underset{D}{:} (b))$ where $a,b \in D$. Then A is a $*$-ideal. By hypothesis $A \subseteq P_\alpha$ for some α. Hence by [2, exercise 22, page 52], $D = \cap D_{P_\alpha}$. Hence $A \longrightarrow A' = \cap AD_{P_\alpha}$ is a $*$-operation on D. Let $0 \neq x \in A'$. For $P_\alpha \in \mathcal{S}$, $A' \subseteq A'D_{P_\alpha} = AD_{P_\alpha}$, so $x = a/s$ where $a \in A$ and $s \in D - P_\alpha$. Then $s \in (A \underset{D}{:} x)$, so $(A \underset{D}{:} x) \not\subseteq P_\alpha$. By hypothesis, $(A \underset{D}{:} x)^* = D$. Hence $x \in xD = x(A \underset{D}{:} x)^* = (x(A \underset{D}{:} x))^* \subseteq A^*$.

Now in general, a proper integral $*$-ideal of D need not be contained in a prime $*$-ideal. For example, if (V,M) is a rank one nondiscrete valuation domain, then $M_v = V$. Hence V has no prime $*$-ideals. Note that $(A \cap B)_v = A_v \cap B_v$ for all $A,B \in F(D)$, but v is not induced by a collection of overrings of V. However, suppose that $*$ has finite character. Then it is well-known and easily proved (for example, see [5]) that each proper integral $*$-ideal of D is contained in a maximal proper integral $*$-ideal and that a maximal proper integral $*$-ideal is prime. We will denote the set of maximal proper integral $*$-ideals of D by $*$-Max(D).

COROLLARY 3.1. Let $*$ be a $*$-operation on D of finite character. Then $A \longrightarrow A' = \cap\{AD_P \mid P \in *\text{-Max}(D)\}$ is a $*$-operation on D and $A' \subseteq A^*$ for each $A \in F(D)$.

We are now ready to prove the main result of this paper.

THEOREM 4. Let $*$ be a $*$-operation on D. Then the following conditions on $*$ are equivalent.

(1) There is a collection $\mathcal{P} = \{P_\alpha\}$ of prime ideals of D with $D = \cap D_{P_\alpha}$ so that $A^* = \cap AD_{P_\alpha}$ for each $A \in F(D)$.

(2) (a) $(A \cap B)^* = A^* \cap B^*$ for all (integral) $A, B \in F(D)$.

(b) Each proper integral $*$-ideal of D is contained in a prime $*$-ideal.

(3) (a) $(A \underset{D}{:} B)^* = (A^* \underset{D}{:} B^*)$ for all (integral) $A \in F(D)$ and $B \in f(D)$.

(b) Each proper integral $*$-ideal of D is contained in a prime $*$-ideal.

Moreover, if either (2) or (3) is satisfied, then we can take $\mathcal{P} = \{P_\alpha\}$ to be any collection of prime $*$-ideals of D with the property that each proper integral $*$-ideal of D is contained in some P_α.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Theorem 1. Suppose that (2) or (3) holds. Let $\mathcal{P} = \{P_\alpha\}$ be a collection of prime $*$-ideals of D with the property that each proper integral $*$-ideal of D is contained in some P_α. For $A \in F(D)$, define $A' = \cap AD_{P_\alpha}$. By Theorem 3, $A \longrightarrow A'$ is a $*$-operation on D and $A' \subseteq A^*$ for each $A \in F(D)$.

(2) $\Rightarrow$ (1). Let $A \in F(D)$. Let $0 \neq x \in A^*$. Now $(x) = A^* \cap (x) = A^* \cap (x)^* = (A\cap(x))^*$ $= ((A \underset{D}{:} x)(x))^* = (A \underset{D}{:} x)^*(x)$, so $D = (A \underset{D}{:} x)^*$. Hence $(A \underset{D}{:} x) \not\subseteq P_\alpha$ for any $P_\alpha \in \mathcal{P}$. Hence $x \in AD_{P_\alpha}$, so $x \in \cap AD_{P_\alpha} = A'$. Hence $A^* = A' = \cap AD_{P_\alpha}$.

(3) $\Rightarrow$ (1). Let $0 \neq x \in A^*$. Then $D = (A^* \underset{D}{:} x)$ $= (A^* \underset{D}{:} (x)^*) = (A \underset{D}{:} x)^*$. The proof that $A^* = A'$ now is identical to the proof of (2) $\Rightarrow$ (1). We remark that if either condition 2(a) or 3(a) holds for integral ideals, then it holds for fractional ideals as well.

COROLLARY 4.1. Let $*$ be a $*$-operation on D. Suppose that each maximal ideal of D is a $*$-ideal. Then the following conditions are equivalent.

(1) $(A\cap B)^* = A^* \cap B^*$ for all nonzero integral ideals A and B of D.

(2) $(A :_D B)^* = (A^* :_D B^*)$ for all nonzero integral ideals A and B of D with B finitely generated.

(3) Every nonzero ideal of D is a $*$-ideal.

Proof. Apply Theorem 4 with $\mathcal{S}$ being the set of all maximal ideals of D.

COROLLARY 4.2. Let $*$ be a finite character $*$-operation on D. Then the following conditions on $*$ are equivalent.

(1) $A^* = \cap\{AD_P | P \in *\text{-Max}(D)\}$ for all $A \in F(D)$.

(2) $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$.

(3) $(A \cap B)^* = A^* \cap B^*$ for all finitely generated nonzero integral ideals A and B of D.

(4) $(A :_D B)^* = (A^* :_D B^*)$ for all $A \in F(D)$ and $B \in f(D)$.

(5) $(A :_D B)^* = (A^* :_D B^*)$ for all finitely generated nonzero integral ideals A and B of D.

Proof. By Theorem 4, (1), (2) and (4) are equivalent. Clearly (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5).

(3) $\Rightarrow$ (1). The proof of this implication is a slight modification of the proof that (2) $\Rightarrow$ (1) in Theorem 4. Let $A' = \cap\{AD_P | P \in *\text{-Max}(D)\}$. Then A'

$\subseteq A^*$. Let $0 \neq x \in A^*$. Then $x \in A_1^*$ where $A_1 \subseteq A$ is finitely generated. Observing that the condition $(A \cap B)^* = A^* \cap B^*$ for integral $A,B \in f(D)$ in (3) easily carries over to all $A,B \in f(D)$, we conclude as in (2) $\Rightarrow$ (1) of Theorem 4 that $D = (A_1 \underset{D}{:} x)^*$. Hence $(A_1 \underset{D}{:} x) \not\subseteq P$ for each $P \in *\text{-Max}(D)$, so $x \in \cap A_1 D_P \subseteq \cap A D_P = A'$. The proof that (5) $\Rightarrow$ (1) is a slight modification of the proof that (3) $\Rightarrow$ (1).

<u>COROLLARY 4.3</u>. Let D be an integral domain and $*$ a $*$-operation on D. Suppose that D satisfies ACC on integral $*$-ideals. If either $(A \cap B)^* = A^* \cap B^*$ for all $A,B \in f(D)$ or $(A \underset{D}{:} B)^* = (A^* \underset{D}{:} B^*)$ for all $A,B \in f(D)$, then D_P is Noetherian for each $P \in *\text{-Max}(D)$.

<u>Proof</u>. Since D satisfies ACC on integral $*$-ideals, $*$ has finite character. By Corollary 4.2, $A^* = \cap\{AD_P \mid P \in *\text{-Max}(D)\}$ for each $A \in F(D)$. Let A be a nonzero ideal of D. Since D has ACC on integral $*$-ideals, there is a finitely generated ideal $B \subseteq A$ with $B^* = A^*$. But then $AD_P = A^*D_P = B^*D_P = BD_P$ is finitely generated. Since every ideal of D_P has the form AD_P for some ideal A of D, D_P is Noetherian.

Theorem 4 may be strengthened if we restrict ourselves to the v-operation. Recall that an integral domain satisfying ACC on integral v-ideals is called a Mori domain. Examples of Mori domains are Noetherian domains and Krull domains.

THEOREM 5. Let D be a Mori domain. Then the following statements are equivalent.

(1) $A_v = \cap\{AD_P \mid P \in v\text{-Max}(D)\}$ for each $A \in F(D)$.

(2) $(A \cap B)_v = A_v \cap B_v$ for all $A, B \in F(D)$.

(3) $(A \cap B)_v = A_v \cap B_v$ for all finitely generated nonzero integral ideals A and B of D.

(4) $(A \underset{D}{:} B)_v = (A_v \underset{D}{:} B_v)$ for all $A \in F(D)$ and $B \in f(D)$.

(5) $(A \underset{D}{:} B)_v = (A_v \underset{D}{:} B_v)$ for all nonzero finitely generated integral ideals A and B of D.

(6) For each maximal v-ideal P of D, D_P is a one-dimensional Gorenstein domain.

(7) For each height one prime ideal P of D, D_P is Gorenstein and $D = \cap\{D_P \mid \text{ht } P = 1\}$.

(8) $A_v = \cap\{AD_P \mid \text{ht } P = 1\}$ for each $A \in F(D)$.

Proof. Since D is a Mori domain, v has finite character. By Corollary 4.2, (1)-(5) are all equivalent.

(1) $\Rightarrow$ (6). Let P be a maximal v-ideal of D. By Corollary 4.3, D_P is Noetherian. For each nonzero ideal A of D, $AD_P = A_vD_P$. But since D is a Mori domain, A_vD_P is a v-ideal of D_P (see, for example, [4, Proposition 1.1]). Hence D_P is a Noetherian domain in which every nonzero ideal is a v-ideal. So D_P is a one-dimensional Gorenstein domain [7, Theorem 222].

(6) $\Rightarrow$ (7). Let P be a height one prime ideal of D. It follows from [5, Théorème 9, page 30] that P is a v-ideal. Let $P \subsetneq Q$ where Q is a maximal v-ideal. By (6), D_Q is a one-dimensional Gorenstein domain. Hence $P = Q$. So the set of maximal v-ideals of D coincides with the set of height one prime ideals of D. Hence D_P is Gorenstein for each height one prime ideal of D. Also, $D = \cap\{D_P | \text{ht } P = 1\}$ since we always have $D = \cap D_P$ where P runs over the set of maximal v-ideals.

(7) $\Rightarrow$ (8). Put $A' = \cap\{AD_P | \text{ht } P = 1\}$. By Theorem 1, $A \longrightarrow A'$ is a $*$-operation on D. Hence for $A \in F(D)$, we have $A' \subseteq A_v$. Conversely, let $0 \neq x \in A_v$. Let P be a height one prime ideal of D. Now $x \in A_{1v}$ where $A_1 \subseteq A$ is finitely generated. Then $x \in (A_{1v})D_P \subseteq (A_1D_P)_v \subseteq (AD_P)_v = AD_P$ with the last equality following since each ideal of D_P

is a v-ideal. Hence $x \in \cap\{AD_P | \text{ht } P = 1\} = A'$. So $A' = A_v$.

(8) $\Rightarrow$ (2). This follows from Theorem 1.

COROLLARY 5.1. Let D be a Noetherian domain. Then the eight conditions of Theorem 5 are all equivalent. If either D is integrally closed or Gorenstein, then $(A \cap B)_v = A_v \cap B_v$ and $(A \underset{D}{:} B)_v = (A_v \underset{D}{:} B_v)$ for all $A, B \in F(D)$.

Recall that an integral domain D is called a Prüfer v-multiplication domain (PVMD) if the finite type v-ideals form a group under v-multiplication: $A*B = (AB)_v$. Equivalently, D is a PVMD if and only if D_P is a valuation domain for each maximal t-ideal P of D. For this and other results on PVMD's, the reader is referred to [3]. In [1] we showed that if D is an essential domain (i.e., $D = \cap D_Q$ where $\{Q\}$ is a set of prime ideals of D with each D_Q being a valuation domain), then $(A_1 \cap \cdots \cap A_n)_v = A_{1v} \cap \cdots \cap A_{nv}$ for all $A_1, \cdots, A_n \in f(D)$. We also showed that if D is integrally closed and $(A_1 \cap \cdots \cap A_n)_v = A_{1v} \cap \cdots \cap A_{nv}$ for all $A_1, \cdots, A_n \in f(D)$, then D is a v-domain (i.e., for each $A \in f(D)$, $(AA^{-1})_v = D$). Now a PVMD is both an essential domain and a v-domain. However, neither an essential domain nor a v-domain need be a PVMD. We have been unable to character the integrally closed domains D for which $(A\cap B)_v = A_v \cap B_v$ for all $A,B \in F(D)$. However, if we replace the v-operation by the t-operation, our next result gives a satisfactory answer. The implication (6) $\Rightarrow$ (1) of Theorem 6 is due to B. Kang [6].

THEOREM 6. Let D be an integrally closed domain. Then the following statements are equivalent.

(1) D is a PVMD.

(2) $(A\cap B)_t = A_t \cap B_t$ for all $A,B \in F(D)$.

(3) $(A\cap B)_t = A_t \cap B_t$ for all finitely generated nonzero integral ideals A and B of D.

(4) $(A \underset{D}{:} B)_t = (A_t \underset{D}{:} B_t)$ for all $A \in F(D)$ and $B \in f(D)$.

(5) $(A \underset{D}{:} B)_t = (A_t \underset{D}{:} B_t)$ for all finitely generated nonzero integral ideals A and B of D.

(6) $A_t = \cap\{AD_P \mid P \in t\text{-Max}(D)\}$.

Proof. It follows from Corollary 4.2 that (2)-(6) are equivalent.

(1) $\Rightarrow$ (6). This implication is well-known.

(6) $\Rightarrow$ (1). Let $0 \neq a,b \in D$. Since D is integrally closed, $(a^2,b^2)_t = ((a,b)^2)_t$. $((a^2,b^2)_t$

$= (a^2,b^2)_v = (A_{a^2X^2-b^2})_v = (A_{(aX+b)(aX-b)})_v$
$= (A_{aX+b}A_{aX-b})_v = ((a,b)^2)_v = ((a,b)^2)_t$ using [2, Proposition 34.8].) Hence $ab \in (a^2,b^2)_t$. So for each maximal t-ideal P, $ab \in (a^2,b^2)_t D_P = (a^2,b^2)D_P$. Since D_P is integrally closed, D_P is a valuation domain [2, Theorem 24.3]. Since D_P is a valuation domain for each maximal t-ideal P, D is a PVMD.

In [1], an integral domain D was defined to be a crescent domain if $\cap A_{\alpha v} = (\cap A_\alpha)_v$ for each collection $\{A_\alpha\} \subseteq f(D)$ with $\cap A_\alpha \neq 0$. We remarked that a Prüfer domain and a one-dimensional Gorenstein domain are crescent domains. We also showed that in a Mori crescent domain, every maximal v-ideal is a maximal ideal. A modification of the proof of this result combined with Corollary 4.1 shows that in a Mori crescent domain D, every nonzero ideal is a v-ideal; so D must be a one-dimensional Gorenstein domain. However, this method of proof easily extends to a general finite character $*$-operation.

THEOREM 7. Let D be an integral domain and let $*$ be a finite character $*$-operation on D. Then $(\cap A_\alpha)^* = \cap A_\alpha^*$ for every collection $\{A_\alpha\} \subseteq f(D)$ with $\cap A_\alpha \neq 0$ if and only if every element of $F(D)$ is a $*$-ideal (i.e., $*$ is the identity $*$-operation).

Proof. The implication ($\Leftarrow$) is obvious.

($\Rightarrow$). By Corollary 4.1 it suffices to show that every maximal ideal M of D is a $*$-ideal. Suppose that M is a maximal ideal of D that is not a $*$-ideal. Then $M^* = D$. Since $*$ has finite character, there is a finitely generated ideal $I \subseteq M$ with $I^* = D$. Let $S = \{I \subseteq M | I$ is a finitely generated ideal with $I^* = D\}$. Then $S \neq \phi$ and S is closed under finite products. Let $0 \neq x \in M$. Shrink M to a prime ideal P minimal over (x). It follows from [5, Théorème 9] that P is a $*$-ideal. Now $J = \cap\{(x)+I | I \in S\} \neq 0$, so $J^* = \cap\{((x)+I)^* | I \in S\} = \cap D = D$. Hence there is a finitely generated ideal $J_0 \subseteq J \subseteq M$ with $J_0^* = D$. Now $J_0 \subseteq J \subseteq (x) + J_0^2 \subseteq P + J_0^2$. Hence, in the integral domain $\overline{D} = D/P$, $\overline{J}_0 = \overline{J}_0^2$. Since $\overline{J}_0$ is finitely generated, $\overline{J}_0 = \overline{0}$, so $J_0 \subseteq P$. Hence $D = J_0^* \subseteq P^*$, a contradiction.

COROLLARY 7.1. Let D be a Mori domain. Then D is a crescent domain if and only if D is a one-dimensional Gorenstein domain.

We remark that Theorem 7 is not true without the hypothesis that $*$ has finite character. For it is easily proved that for any collection $\{A_\alpha\} \subseteq F(V)$ with $\cap A_\alpha \neq 0$ where V is a valuation domain, we have $\cap A_{\alpha v} = (\cap A_\alpha)_v$.

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