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THE $D + XD_S[X]$ CONSTRUCTION FROM GCD-DOMAINS

Muhammad ZAFRULLAH*

Department of Mathematics, University College London, London WC1E 6BT, United Kingdom

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Let D be an integral domain, S a multiplicative set in D and X an indeterminate over D. We show that if D is a GCD-domain, then the behaviour of $D^{(S)} = \{a_0 + \sum a_iX^i \mid a_0 \in D,\ a_i \in D_S\} = D + XD_S[X]$ depends upon the relationship between S and the prime ideals P of D such that D_P is a valuation domain. We use this study to construct locally GCD-domains which are not GCD-domains. These domains have, each, at least one prime t-ideal P such that PD_P is not a t-ideal. We also give an example of an ideal A with $A \subsetneq A_t \subsetneq A_v \subsetneq D$. The $D^{(S)}$ construction is also used to construct, from lattice ordered groups, Riesz groups which are not lattice ordered.

Introduction

Given that S is a multiplicative set in a commutative integral domain D and that X is an indeterminate over D, we can construct an integral domain $D^{(S)} = \{a_0 + \sum a_iX^i \mid a_0 \in D \text{ and } a_i \in D_S\} = D + XD_S[X]$. In this note we show that if D is a GCD-domain, then the behaviour of $D^{(S)}$ depends on the relationship between S and the essential primes of D. (A prime ideal P is *essential* in D if D_P is a valuation domain.) We use this dependence to produce examples of locally GCD-domains which are not Prüfer v-multiplication domains and are Schreier domains. Each of these integral domains D has a prime t-ideal P such that PD_P is not a prime t-ideal of D_P. From these integral domains we can also construct examples of integral domains D with integral ideals A such that $A \subsetneq A_t \subsetneq A_v \subsetneq D$.

Throughout hence the symbols D, K, X and $D^{(S)}$ will denote, respectively, a commutative integral domain, its field of fractions, an indeterminate over D and the $D + XD_S[X]$ construction from D.

To give a clear idea of our results we need to recall some notions from multiplicative ideal theory. Our main references in this direction are [6] and [10].

Let $F(D)$ denote the set of non-zero fractional ideals of D. Defined on $F(D)$ are operations like $A \mapsto (A^{-1})^{-1} = A_v$ called the *v-operation* and $A \mapsto \bigcup B_v = A_t$ (where B ranges over finitely generated D-submodules of A) called the *t-*

* Current address: Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, U.S.A.

operation. For details the reader may refer to [6, Sections 32 and 34]. Yet, for our purposes we note the following:

(1) $D_v = D$ and if $A \in F(D)$, then $A \subseteq A_t \subseteq A_v$, $(A_v)_v = A_v$ and $(A_t)_t = A_t$.

(2) If $A, B \in F(D)$, then $A \subseteq B$ implies $A_v \subseteq B_v$, and if A is principal, then $(AB)_v = AB_v$.

So in general, if A is an integral ideal of D, then $A \subseteq A_t \subseteq A_v \subseteq D$. As a consequence of our studies of $D^{(S)}$, we construct an example of an integral domain T with an ideal A such that $A \subsetneq A_t \subsetneq A_v \subsetneq T$.

An ideal A in $F(D)$ is called a *v-ideal* if $A = A_v$ and a *v-ideal of finite type* if $A = B_v$ for some finitely generated $B \in F(D)$. An integral domain D is called a *Prüfer v-multiplication domain* (*PVMD*) if the set $H(D)$ of v-ideals of finite type is a group under the *v-multiplication*: $(AB)_v = (A_vB)_v = (A_vB_v)_v$. Further an ideal $A \in F(D)$ is called a *t-ideal* if $A = A_t$ and a *maximal t-ideal* if A is integral and maximal w.r.t. being an integral t-ideal. A maximal t-ideal is a prime ideal (cf. [7]). If $A \in F(D)$ is finitely generated and if S is a multiplicative set of D, it can be shown by using [6, Example 20, p. 432] that $(AD_S)_v = (A_vD_S)_v$ (see [15] for an alternative treatment). Using this result it can be shown that if P is a prime ideal of D such that PD_P is a t-ideal of D_P, then P is a t-ideal of D. This may lead one to think that if P is a prime t-ideal of D, then PD_P is a t-ideal of D_P. Using the $D + XD_S[X]$ construction, we give an example of an integral domain T with a prime t-ideal P such that PT_P is not a t-ideal of T_P. In the following we indicate how this can be achieved:

If P is an essential prime in D, then PD_P is a t-ideal of D_P because every finitely generated ideal of a valuation domain is principal and a prime ideal is the union of principal ideals contained in it. So every essential prime ideal is a t-ideal. A prime ideal minimal over an ideal of the type $0 \neq (a):(b) \neq D$ is called an *associated prime* (cf. [1]) and according to [15], if P is an associated prime of D, then PD_P is a t-ideal of D_P. Thus an associated prime is also a t-ideal. According to [12] D is a *P-domain* if every associated prime of D is essential. Further, according to [7] D is a PVMD if and only if every maximal t-ideal of D is essential. Using the facts that every t-ideal is contained in a maximal t-ideal and that if $P \subseteq Q$ are prime ideals with Q essential, then P is essential, we conclude that a PVMD is a P-domain and that every prime t-ideal is essential in a PVMD. Now if we can construct a P-domain which is locally a PVMD (or a locally GCD-domain) but not a PVMD, then we have our example. For if in a locally PVMD D, for every maximal t-ideal P, PD_P is a t-ideal in D_P, then D being locally a PVMD, D_P is a PVMD with its maximal ideal a t-ideal which makes D_P a valuation domain. But then, as we already know, D is a PVMD. We now discuss the means by which we propose to get to P-domains which are locally PVMD's but not PVMD's. According to [2] an element $x \in D$ is *primal* if $x \mid ab$ in D implies that $x = a_1b_1$ where $a_1 \mid a$ and $b_1 \mid b$. An integrally closed integral domain D is called *Schreier* if every element of D is primal. It is known that a PVMD is a GCD-domain if and only if it is a Schreier domain (cf. [6, Example 7, p. 430].

These considerations give the following picture:

$$\text{GCD-domain} \Rightarrow \text{PVMD} \Rightarrow P\text{-domain}.$$

Now according to [4] if D is a GCD-domain, then $D^{(S)}$ is Schreier for every multiplicative set S. So to show that $D^{(S)}$ is a GCD-domain we only have to show that it is a PVMD. On the other hand, if we construct $D^{(S)}$ from a GCD-domain D such that $D^{(S)}$ is locally GCD but not GCD, then we have an example of a P-domain which is not a PVMD. This follows from the fact that D is a P-domain if an only if it is locally a P-domain [12].

Now to show how we get to a $D^{(S)}$ with the required property we need to introduce some more terminology. A non-zero prime ideal P in a GCD-domain is called a *PF-prime* if for all $x, y \in P$, $\text{GCD}(x, y) \in P$. Using the fact that for every finitely generated ideal A in a GCD-domain, A_v is principal, we can confirm that in a GCD-domain a prime t-ideal is a *PF*-prime. According to [14] a *PF*-prime is essential and conversely. So, in a GCD-domain the notions of prime t-ideal, essential prime and *PF*-prime coincide. Indeed a *maximal PF-prime* can be defined in an obvious manner. We shall say that a prime ideal P *intersects a multiplicative set S in detail* if for all non-zero prime ideals $Q \subseteq P$, $Q \cap S \neq \emptyset$. In Section 1, we prove the following theorem:

Theorem 1. *Let D be a GCD-domain and let S be a multiplicative set in D. Then $D^{(S)}$ is a GCD-domain if and only if for each PF-prime P of D with $P \cap S = \emptyset$ there exists $d \in P$ such that d is not divisible by any non-unit of S.*

As a consequence of the proof of this theorem we conclude that if $D^{(S)}$ is a GCD-domain and if P is a *PF*-prime of D with $P \cap S \neq \emptyset$, then P intersects S in detail.

In Section 2, we show that if D is a GCD-domain and if S is a multiplicative set of D with the property that every *PF*-prime P which intersects S, intersects it in detail, then $D^{(S)}$ is a P-domain which may not be a GCD-domain and hence may not be a PVMD. We establish a criterion for $D^{(S)}$ to be a non-PVMD, P-domain and show that if E is the ring of entire functions and S the multiplicative set generated by the principal primes of E, then $E^{(S)}$ is a P-domain which is not a PVMD. In fact $E^{(S)}$ turns out to be a locally GCD-domain.

The study of the $D + XD_S[X]$ construction could be of interest in another area. We know that if D is a GCD-domain, then for any multiplicative set S, $D^{(S)}$ is a Schreier domain [4]. If T is a Schreier domain its group of divisibility $G(T)$ is a Riesz group (cf. [2]). Now the divisibility group of a GCD-domain is a lattice ordered (l.o.) group and it is well known (cf. e.g. [6, Theorem 18.6]) that given any abelian l.o. group A we can construct a GCD-domain, which is Bezout and whose group of divisibility is isomorphic to A. Thus the $D + XD_S[X]$ construction provides a rather mechanical method of constructing Riesz groups from l.o. groups. The procedure would be as follows: We take up an abelian l.o. group A and from it construct a GCD-domain D such that $G(D)=A$. Now for any multiplicative set in D, $D^{(S)}$ will be a Schreier domain. It only remains to find out for which S, $D^{(S)}$ will be Schreier but not GCD. The divisibility group of such a non-GCD, $D^{(S)}$, will be a Riesz group which is not l.o. It is interesting to note that there do exist l.o. groups from which we can construct only l.o. groups via the $D+XD_S[X]$ construction. These l.o. groups must be eliminated if we wish to make a general statement about the construction of Riesz groups which are not l.o. groups. In Section 3 we show that these l.o. groups are isomorphic to small cardinal sums of subgroups of real numbers.

In Section 4 we discuss some applications of the results of earlier sections. For example, we construct locally GCD-domains from locally GCD-domains via the $D+XD_S[X]$ construction. This section also includes the two examples already discussed in the introduction.

1. Proof of Theorem 1

The $D+XD_S[X]$ constructions were studied in [4] where it was shown that if D is a GCD-domain, then $D^{(S)}$ is a GCD-domain if and only if for all $d \in D$, GCD(d, X) exists. Theorem 1 and its proof connect the event of $D^{(S)}$ being a GCD-domain with the intrastructural relationship between the PF-primes of D and the multiplicative set S. The proof of Theorem 1 is based on the following string of lemmas:

Lemma 1.1. *Let V be a non-trivial valuation domain and let V' be an* overring (*ring between V and its quotient field*) *of V with $V' \supsetneq V$. Then the following are equivalent*:

(1) $V+XV'[X]=\{a_0+\Sigma_i a_iX^i \mid a_0 \in V,\ a_i \in V'\}$ *is a GCD-domain*;
(2) $V'=K$, *the quotient field of V*;
(3) $V+XV'[X]$ *is a Bezout domain*.

Proof. (1)$\Rightarrow$(2). Suppose on the contrary that $V' \neq K$. Since V is a valuation domain and V' is an overring of V, $V'=V_P$ where P is a prime ideal of V. Thus $V+XV'[X]=V+XV_S[X]$ where $S=V-P$. Now let $0 \neq a \in P$. Then, since V is a valuation domain, every element of S divides a. Similarly, by the construction, every element of S divides X. Now let $d=\text{GCD}(a, X)$. Then since $d \mid X$ and $d \mid a$, $d \in S$ and so for all n, $d^n \mid X$ and $d^n \mid a$. This contradicts GCD$(a/d, X/d)=1$ if d is a non-unit, and contradicts $V' \neq V$ if d is a unit. From this it follows that if $V+XV_P[X]$ is a GCD-domain, with P non-maximal, then $P=0$ and hence $V'=K$.

(2)$\Rightarrow$(3). Follows from [4].

(3)$\Rightarrow$(1). Obvious, because a Bezout domain is a GCD-domain. □

Lemma 1.2. *Let S be a multiplicative set of D, let $D^{(S)} = D + XD_S[X]$ and let $\bar{P} = P + XD_S[X]$ be a prime ideal in $D^{(S)}$. Then $\bar{P}$ is essential if and only if*
(i) *P is essential and*
(ii) *P intersects S in detail.*

Proof. We note that if S is a multiplicative set in D, then $D_S = K$ if and only if each non-zero prime ideal of D meets S. We also note that if S and T are two multiplicative sets of D, then $D_{ST} = (D_S)_{T^*} = (D_S)_T$ where T^* is the saturation of T in D_S and $ST = \{st \mid s \in S \text{ and } t \in T\}$. Now let $\bar{P}$ be essential, then $(D^{(S)})_{\bar{P}}$ is a valuation overring of $D[X]$. Hence $(D^{(S)})_{\bar{P}} \cap K$ is the valuation domain D_P. Note that $D^{(S)} - \bar{P} = M = \{f(X) \in D^{(S)} \mid f(0) \not\in P\}$. So $(D^{(S)})_{\bar{P}} = (D^{(S)})_M$. But as $(D^{(S)})_{\bar{P}} \supsetneq (D^{(S)})_{D-P} = D_P + XD_{S(D-P)}[X]$, we have $(D^{(S)})_{\bar{P}} = (D_P + XD_{S(D-P)}[X])_{M^*}$ where $M^* = \{f(X) \in (D^{(S)})_{(D-P)} \mid f(0) \text{ is a unit in } D_P\}$. Now since D_P is a valuation domain and $D_{S(D-P)}$ is an overring of D_P, $D_{S(D-P)}$ is at least a valuation domain. This renders $D_P + XD_{S(D-P)}[X] = T'$, a Schreier domain (cf. [4]).

We note that each $f(X) \in M^*$ is a primitive polynomial of T' and hence is a product of irreducible elements of T'. Each of these irreducible elements is a prime because T' is Schreier and an irreducible element in a Schreier domain is a prime (cf. [2]). Further, since for each prime $p(X) \in M^*$ $p(X)T' \cap D_P = (0)$, $p(X)T'$ is a rank one prime. Now according to [11], $(D^{(S)})_{\bar{P}} = (D_P + XD_{S(D-P)}[X])_{M^*}$ is a GCD-domain if and only if $D_P + XD_{S(D-P)}[X]$ is a GCD-domain. But, as $\bar{P}$ is essential, $(D^{(S)})_{\bar{P}}$ is a GCD-domain and we conclude that $D_P + XD_{S(D-P)}[X]$ is a GCD-domain. Now by Lemma 1.1, $D_P + XD_{S(D-P)}[X]$ is a GCD-domain if and only if $D_{S(D-P)}$ is the field of fractions of D_P and hence of D. This is possible if and only if for each non-zero prime ideal $Q \subseteq P$, $Q \cap S \neq \emptyset$.

Conversely, let $\bar{P} = P + XD_S[X]$ with P essential and intersecting S in detail. Then $(D^{(S)})_{\bar{P}} \supseteq (D^{(S)})_{(D-P)} = D_P + (XD_S[X])_{(D-P)} = D_P + XD_{S(D-P)}[X]$. Now let $Q \neq 0$ be any prime ideal of D. If $Q \subseteq P$, then, as P intersects S in detail, $Q \cap S \neq \emptyset$ and hence $Q \cap S(D-P) \neq \emptyset$ and if $Q \not\subseteq P$, then $Q \cap (D-P) \neq \emptyset$ and hence $Q \cap S(D-P) \neq \emptyset$. Consequently every non-zero prime ideal of D intersects $S(D-P)$ and $D_{S(D-P)}$ is the quotient field K of D. Now $(D^{(S)})_{\bar{P}}$ is a quasi local overring of the Bezout domain $D_P + XK[X]$ (cf. [4]) and hence is a valuation domain. □

Lemma 1.3. *Let P be a* t*-ideal in a Schreier domain D. Then no two elements of P are coprime.*

Proof. Suppose that $a,b \in P$ are coprime. Then according to [2], $aD \cap bD = abD$. Now $(a, b)^{-1} = (1/a) \cap (1/b) = (aD \cap bD)/ab = D$ and so $(a, b)_v = ((a, b)^{-1})^{-1} = D^{-1} = D \subseteq P$, a contradiction. □

Lemma 1.4. *Let $\bar{P}$ be a prime* t*-ideal in $D^{(S)} = D + XD_S[X]$ such that $\bar{P} \cap D = P \neq (0)$ and let D be a GCD-domain. Then P is a PF-prime ideal of D.*

Proof. We note that D is a GCD-domain and that, according to [4], $D^{(S)}$ is a Schreier domain. Now by Lemma 1.3, no two elements of $\bar{P}$ are coprime. Consequently no two elements of P are coprime. Now let $a, b \in P$. Then since $\mathrm{GCD}(a, b) = d$ exists, we can write $a = a_1 d$ and $b = b_1 d$ where a_1 and b_1 are coprime. Because no two elements of P are coprime, at least one of a_1, b_1 is not in P and hence $d = \mathrm{GCD}(a, b) \in P$. $\square$

Proof of Theorem 1. The proof is based on the fact that a Schreier domain is a GCD-domain if and only if it is a PVMD. We note that D is a GCD-domain and so $D^{(S)}$ is a Schreier domain. So to show that $D^{(S)}$ is a GCD-domain it is sufficient to show that each maximal t-ideal of $D^{(S)}$ is essential. For this we split the set T of maximal t-ideals of $D^{(S)}$ into the following three sets:

$$L = \{\bar{P} \in T \mid \bar{P} \cap D = (0)\},$$

$$M = \{\bar{P} \in T \mid \bar{P} \cap D \neq (0) \text{ and } \bar{P} \cap S = \emptyset\},$$

$$N = \{\bar{P} \in T \mid \bar{P} \cap S \neq \emptyset\}.$$

(i) If $\bar{P} \in L$, then $(D^{(S)})_{\bar{P}} = (K[X])_{\bar{P}K[X]}$ which is a valuation domain.

(ii) If $\bar{P} \in M$, then by Lemma 1.4, $P = \bar{P} \cap D$ is a *PF*-prime of D such that $P \cap S = \emptyset$. Hence by the hypothesis there is an element $d \in P$ such that for any non-unit $s \in S$, $s \nmid d$. Since X is divisible by elements of S only, we have $\mathrm{GCD}(d, X) = 1$. Hence $X \not\in \bar{P} = \{a_0 + \sum (a_i/b_i)X^i \mid b_i \in S \text{ and } a_0, a_i \in P\}$. Let $H = D^{(S)} - \bar{P}$. Then $H = \{a_0 + \sum (a_i/b_i)X^i \mid b_i \in S \text{ and at least one } a_i \not\in P\}$. Consequently $(D^{(S)})_{\bar{P}} \supseteq (D^{(S)})_{(D-P)} = D_P[X]$, because $P \cap S = \emptyset$. Thus $(D^{(S)})_{\bar{P}} = (D_P[X])_{H^*}$ where H^* is the saturation of H in $D_P[X]$. But as, by Lemma 1.4, D_P is a valuation domain, we have $H^* = \{f(X) \in D_P[X] \mid A_f = D_P\}$ where A_f is the ideal generated by the coefficients of $f(X)$. Consequently $(D^{(S)})_{\bar{P}} = D_P(X)$, the trivial extension of D_P to $K(X)$.

(iii) If $\bar{P} \in N$, then $\bar{P} = P + XD_S[X]$ where, by Lemma 1.4, P is a *PF*-prime of D. We show that P intersects S in detail. For this, let $Q \subseteq P$ be a non-zero prime ideal such that $Q \cap S = \emptyset$. Then by the hypothesis there must be $d \in P$ such that no non-unit of S divides d. But then for any $s \in S \cap P$, $\mathrm{GCD}(s, d) = 1$. This contradicts the fact that $\bar{P}$ is a maximal t-ideal (cf. Lemma 1.4). Now by Lemma 1.2, $(D^{(S)})_{\bar{P}}$ is a valuation domain. Thus having shown that every maximal t-ideal of $D^{(S)}$ is essential, we conclude that $D^{(S)}$ is a PVMD and hence a GCD-domain.

Conversely, if $D^{(S)}$ is a GCD-domain, then for each $d \in D$, $\mathrm{GCD}(d, X)$ exists. Now if P is such that $P \cap S = \emptyset$, let $d \in P$ and let $d' = \mathrm{GCD}(d, X)$. Then $d_1 = d/d' \in P$ such that no non-unit $s \in S$ divides d_1. $\square$

Corollary 1.5. *Let D be a GCD-domain and let S be a multiplicative set in D. Then the following are equivalent:*

(1) $D^{(S)}$ *is a GCD-domain*;
(2) *For each element* $d \in D$, $d = d_1 s$ *where* $s \in S$ *and* d_1 *is coprime to all* $s \in S$;
(3) *For each prime* t-*ideal* P *of* D *with* $P \cap S = \emptyset$, *there is at least one* $d \in P$ *such that* d is coprime to each element of S. □

Corollary (to proof) 1.6. *Let* $D^{(S)} = D + XD_S[X]$ *be a GCD-domain. If* $\bar{P}$ *is a prime* t-*ideal of* $D^{(S)}$, *then the following hold*:
(1) If $\bar{P} = P + XD_S[X]$, *then* P *intersects* S *in detail*;
(2) If $\bar{P} \cap S = \emptyset$, *then* $\bar{P} \cap D = (0)$ *or*

$$\bar{P} = P[X]D_S[X] \cap D^{(S)} \quad \text{where } P = \bar{P} \cap D .$$

Proof. (1) is obvious from Lemma 1.2, and (2) is immediate from the description of $\bar{P}$ in case (ii) of the proof of Theorem 1. □

2. Construction of *P*-domains

An integral domain D is called *essential* if it has a family $\{P_i\}_{i \in I}$ of essential primes such that $D = \bigcap D_{P_i}$. Griffin [7] proved that a PVMD is essential. In addition he conjectured that there should be an essential domain which is not a PVMD. This conjecture was affirmed by Heinzer and Ohm [9]. Now according to [1], for any integral domain D, we have $D = \bigcap D_P$ where P ranges over the associated primes (of principal ideals) of D. So a P-domain is essential. It was pointed out in [12] that the example of the non-PVMD essential domain provided in [9] was in fact a P-domain. Recently, Heinzer [8] has constructed an example of an essential domain which is not a P-domain. This gives us the following picture: GCD-domain ⇒ PVMD ⇒ P-domain ⇒ essential domain and in general no arrows can be reversed. Now once it is established that the P-domains are a distinct class of rings, it would be in order to have a ring-theoretic method of constructing them. The following result is a clue to this method:

Theorem 2.1. *Let* D *be a* P-*domain and let* S *be a multiplicative set in* D *such that each associated prime of* D *that intersects* S *intersects it in detail. Then* $D^{(S)}$ *is a* P-*domain.*

The proof is based on Lemma 1.2 and the following two lemmas:

Lemma 2.2. *Let* S *be a multiplicative set of* D *such that for each associated prime* P *of* D *with* $P \cap S \neq \emptyset$, D_P *is a valuation domain. If* D_S *is a* P-*domain, then so is* D. □

The proof of this lemma is immediate once we note that $D_S = \bigcap D_P$ where P ranges over associated primes with $P \cap S = \emptyset$ [1].

Lemma 2.3. *Let S be a multiplicative set in D. If $\bar{P}$ is an associated prime of $D^{(S)}$ with $\bar{P} \cap S \neq \emptyset$, then $\bar{P} \cap D$ is an associated prime of D.*

Proof. If $\bar{P}$ is an associated prime of $D^{(S)}$, then, according to [1], for each $a \in \bar{P}$ there is $b \in D^{(S)}$ such that $\bar{P}$ is minimal over (a):(b). Now since $\bar{P} \cap S \neq \emptyset$, for each $s \in \bar{P} \cap S$ there is $h(X) \in D^{(S)}$ such that $\bar{P}$ is minimal over $(s):(h(X))$. We note that $h(X) = h_0 + \sum h_i X^i$ where $h_0 \in D$ and $h_i \in D_S$. Since each of $h_i X^i$ $(i \geq 1)$ is divisible by each of $s \in S$, we have $rh(X) \in (s)$ if and only if $rh_0 \in (s)$. Thus $(s):(h(X)) = A + XD_S[X]$ where $A = (s):(h_0)$ in D. Now as $\bar{P} \cap S \neq \emptyset$, $\bar{P} = P + XD_S[X]$ and as $\bar{P}$ is minimal over $(s):(h_0) + XD_S[X]$, P is minimal over $(s):(h_0)$. Whence $P = \bar{P} \cap D$ is an associated prime of D. □

Proof of Theorem 2.1. If D is a P-domain, then, according to [12], so are D_S and $D_S[X]$. So by Lemma 2.2 it is sufficient to show that for each associated prime $\bar{P}$ of $D^{(S)}$ with $\bar{P} \cap S \neq \emptyset$, $\bar{P}$ is essential.

Now by Lemma 2.3, $\bar{P} = P + XD_S[X]$ where P is an associated prime of D. Since D is a P-domain, P is essential. We have $P \cap S \neq \emptyset$ and so by the hypothesis, P intersects S in detail. But then Lemma 1.2 applies and $\bar{P} = P + XD_S[X]$ is an essential prime. □

We note that Theorem 2.1 gives us a method of constructing a P-domain but it does not distinguish between P-domains and PVMD's. To find P-domains which are not PVMD's we use an indirect method via Theorem 1. For this we note the following points:

(i) If D is a GCD-domain and S is a multiplicative set in D, then $D^{(S)}$ is a Schreier domain;

(ii) A PVMD which is also a Schreier domain is a GCD-domain;

(iii) If D is a GCD-domain and S is a multiplicative set in D, then, according to Theorem 1, the existence of a PF-prime with $P \cap S = \emptyset$ such that every element of P is divisible by at least one non-unit from S implies that $D^{(S)}$ is not a GCD-domain.

Directly from points (i)–(iii) follows the theorem below.

Theorem 2.4. *Let D be a GCD-domain. Suppose that S is a multiplicative set in D such that each maximal PF-prime which intersects S intersects it in detail. If there exists a PF-prime P disjoint from S such that every element of P is divisible by at least one non-unit from S, then $D^{(S)}$ is a P-domain which is not a PVMD.* □

To work out simple and practical examples we need the notion of a *rigid element*. A non-zero non-unit $r \in D$ is called rigid if for all $x, y \mid r$, $x \mid y$ or $y \mid x$ (cf. [3, p. 129]). If D is a Schreier domain and r is a rigid element, then the set $\{x \in D \mid x \text{ is non-coprime to } r\}$ is a prime ideal which may be called the *prime ideal associated with* r and may be denoted by $P(r)$.

Corollary 2.5. *Let D be a GCD-domain and let P be a PF-prime of D. If $D + XD_P[X]$ is a GCD-domain, then P is a maximal PF-prime such that every PF-prime contained in P contains a rigid element.*

Proof. Let P be a *PF*-prime and let $D + XD_P[X]$ be a GCD-domain. Then, as $P \cap (D - P) = \emptyset$, by Theorem 1, there exists $d \in P$ such that d is not divisible by any non-unit element of $D - P$. So if $d_1, d_2 \mid d$ where d_1, d_2 are non-units, then $d_1, d_2 \in P$. Now P being a *PF*-prime, $\text{GCD}(d_1, d_2) \in P$. Thus every two non-unit factors of d are non-coprime and this, in a GCD-domain, is equivalent to saying that d is a rigid element.

That P is a maximal *PF*-prime follows from the fact that any ideal A properly containing P intersects $D - P$ and so cannot be a *PF*-prime, because then it would contain d and an element $s \in D - P$ which are coprime. So P is a maximal *PF*-prime associated with the rigid element d. Now let Q be a non-zero prime ideal contained in P. Then as Q is a *PF*-prime and $Q \cap (D - P) = \emptyset$, by Theorem 1, there exists $h \in Q$ such that h is not divisible by any non-unit of $D - P$. Then again any two non-unit factors of h lie in P and hence are non-coprime and from this we conclude as before that h is a rigid element. □

Corollary 2.5 gives us a clue to the type of GCD-domains to be selected to construct non-PVMD P-domains. For instance, we could construct an example from a GCD-domain which has a rank one maximal *PF*-prime which is not associated with a rigid element. In this case $D + XD_P[X]$ is a Schreier P-domain which is not a PVMD. A large number of GCD-domains with this property exist. Of these we take the following ring as a simple object for consideration:

Let K be an algebraically closed field of characteristic zero and let $S = \{x^a \mid a \text{ is a rational} \geq 0\}$. Then it is easy to establish that the ring $K[S] = \bigcup_{n=1}^{\infty} K[x^{1/n}]$ is a one-dimensional Bezout domain. This Bezout domain has only one type of rigid elements, namely x^a where $a > 0$ and hence only one prime ideal associated with a rigid element. That is $P = \{y \in K[S] \mid \text{GCD}(y, x) \neq 1\}$. So we select a non-zero prime $\bar{P} \neq P$ and conclude that $D + XD_{\bar{P}}[X]$ is a P-domain which is not a PVMD.

We conclude this section with a rather interesting and unusual example of a P-domain which is not a PVMD.

Example 2.6. Let E be the ring of entire functions. It is well known that E is a Bezout domain and that every non-zero non-unit $x \in E$ is uniquely expressible as a countable product $x = \prod p_i^{a_i}$ where a_i are natural numbers. The uniqueness of this expression indicates that each principal prime ideal of E is a maximal ideal of rank one. Now let S be generated by the principal primes of E. Then $E^{(S)} = E + XE_S[X]$ is a P-domain which is not a PVMD. The reasons for this conclusion are given below.

(i) Every prime ideal which intersects S is principal, hence of rank one and hence it intersects S in detail. (This follows from the fact that if a prime ideal P

intersects S, it contains a finite product of primes and hence contains a principal prime . . . which is maximal.)

(ii) Every non-zero non-unit of E is divisible by principal primes and hence for each prime P with $P \cap S = \emptyset$ it is true that each $x \in P$ is divisible by an infinite number of non-units from S.

Clearly by (i), $E^{(S)}$ is a P-domain and by (ii), $E^{(S)}$ is not a GCD-domain. Now E being Bezout, $E^{(S)}$ is Schreier and Schreier non-GCD means non-PVMD.

3. Construction of integral domains whose groups of divisibility are Riesz groups

A directed partially ordered group G is called a *Riesz group* if for all a_1, a_2, b_1, $b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$ there exists some $c \in G$ such that $a_i \leq c \leq b_j$ for all $i, j = 1, 2$. These groups are a generalization of lattice ordered groups and they arise from the work of F. Riesz on linear operators. In view of their importance any construction which gives rise to these groups is worth looking into. The $D + XD_S[X]$ construction from a GCD-domain gives rise to integral domains whose groups of divisibility are Riesz groups and thus it proves itself to be a candidate for study.

In continuation with our remarks in the introduction, we shall be mainly interested in eliminating the l.o. groups whose $D + XD_S[X]$ construction gives rise to l.o. groups alone. For this it would be sufficient to find out the GCD-domains D for which $D^{(S)}$ is a GCD-domain for all multiplicative sets S. To give our intended result a suitable statement we define a *generalized UFD* (*GUFD*) as a GCD-domain each of whose non-zero non-units can be expressed as a product of finitely many rigid elements r with the following property:

(f) For each non-unit $h \mid r$ there is an n such that $r \mid h^n$.

Since the powers of a rank one principal prime have property (f), a UFD is a GUFD.

Theorem 3.1. *Let D be a GCD-domain. Then for $D^{(S)}$ to be a GCD-domain for all S, it is necessary and sufficient that D is a GUFD.*

Proof. By the hypothesis and by Corollary 2.5 every PF-prime of D is a maximal PF-prime and it contains a rigid element. From this it follows that every PF-prime of D is of rank one. Further, if r is a rigid element, then since $P(r) = \{x \in D \mid \mathrm{GCD}(x, r) \neq 1\}$ is a PF-prime of D, which must be of rank one, we conclude that for all non-units $h \mid r$ there should exist an n such that $r \mid h^n$. So every rigid element of D has property (f). Further, since, according to [14], every minimal prime of a principal ideal of a GCD-domain is a PF-prime, we conclude that every prime ideal contains a rigid element. Now using the GCD-property it is easy to show that the multiplicative set T generated by all the rigid elements with property (f) is saturated. If $T \neq D - \{0\}$, then there must be a non-zero ideal disjoint with T. But then this prime ideal must contain a rigid element and this gives the contradiction needed to ensure that every non-zero non-unit of D is expressible as a product of finitely many rigid elements. Hence according to our definition, D is a GUFD.

Conversely, if D is a GUFD, then it is easy to show that each *PF*-prime of D is associated with a rigid element. Now a direct application of Theorem 1 shows that if D is a GUFD, then $D^{(S)}$ is a GCD-domain for all multiplicative sets S. □

Remark 3.2. The generalized UFD's were studied in the author's doctoral thesis submitted to the University of London in 1974. In the thesis the idea of GUFD's was developed parallel to that of UFD's.

Now it is easy to verify that the group of divisibility of a GUFD is isomorphic to a direct sum of additive subgroups of the set of reals (in fact, the divisibility group of a GUFD is isomorphic to a small cardinal sum of these additive subgroups of the set of reals). The general statement we had in mind can now be made as follows:

Corollary 3.3. *Let D be a GCD-domain such that it is not a generalized UFD. Then for some multiplicative set S of D, $G(D^{(S)})$ is a Riesz group which is not a l.o. group.* □

4. Applications

4.1. Constructing a locally GCD-domain from a locally GCD-domain

An integral domain D is called a *locally GCD-domain* if for each maximal ideal M of D, D_M is a GCD-domain. It is easy to see that if D is a locally GCD-domain, then so is D_S for every multiplicative set S of D.

Proposition 4.1. *Let D be a locally GCD-domain and let S be a multiplicative set of D such that for every maximal ideal M of D with $M \cap S \neq \emptyset$, M intersects S in detail. Then $D^{(S)}$ is a locally GCD-domain.*

Proof. Let M be a maximal ideal of $D^{(S)}$. Then the following cases arise:

(1) $M \cap D = (0)$;
(2) $M \cap D \neq (0)$ but $M \cap S = \emptyset$;
(3) $M \cap S \neq \emptyset$.

(1) If $M \cap D = (0)$, then

$$(D^{(S)})M = ((D^{(S)})_{D-\{0\}})_{M(D^{(S)})_{D-\{0\}}} = (K[X])_{MK[X]} .$$

(2) If $M \cap D \neq (0)$ but $M \cap S = \emptyset$, then

$$(D^{(S)})_M = ((D^{(S)})_S)_{M(D^{(S)})_S} = (D_S[X])_{MD_S[X]} .$$

Now let $M \cap D = P$. Then, as $M \cap S = \emptyset$, we have $M = MD_S[X] \cap D^{(S)}$ and so $P = MD_S[X] \cap D$. From this it is easy to conclude that $MD_S[X] \cap D_S = PD_S$. But then

$$(D^{(S)})_M = (D_S[X])_{MD_S[X]} \supseteq (D_S[X])_{D_S - PD_S} \supseteq (D_S)_{PD_S}[X] = D_P[X]$$

which is a GCD-domain containing $D_S[X]$. Now $(D^{(S)})_M$, being a quotient ring of a GCD-domain is a GCD-domain.

(3) Finally, if $M \cap S \neq \emptyset$, then $M = P + XD_S[X]$ where $P = M \cap D$ and $(D^{(S)})_M \supseteq (D^{(S)})_{D-P} = D_P + XD_{S(D-P)}[X]$. Now, reasoning in the way we did in the proof of Theorem 1, $D_{S(D-P)} = K$. So $D_P + XD_{S(D-P)}[X]$ is a GCD-domain. Now, $(D^{(S)})_M$, being a localization of the GCD-domain $D_P + XD_{S(D-P)}[X]$, is a GCD-domain. □

Corollary 4.2. *If D is a locally GCD-domain, then so is $D[X]$.* □

Proposition 4.1 is a rather difficult result to work with. Yet it simplifies matters a great deal if we restrict out attention to locally GCD-domains whose prime spectra form trees under inclusion (that is, Prüfer domains). By doing so we constructed P-domains from Bezout domains in Section 2. We note as a corollary to Proposition 4.1 that the ring $E^{(S)}$ of Example 2.6 is a locally GCD-domain.

4.2. The existence of a maximal t*-ideal P with PD_P not a* t*-ideal*

Having indicated the area where non-PVMD, locally GCD-domains can be found, we turn our attention to their applications. (But first some introduction.)

Locally GCD-domains arise as a natural concept in a number of ways, viz. as locally factorial domains and as Prüfer and Bezout domains. We recall that a GCD-domain is a PVMD and hence a P-domain. We also recall that, according to [12], D is a P-domain if and only if it is locally a P-domain. So a locally GCD-domain is a P-domain. Consequently, a locally GCD-domain which is not a PVMD can be put forward as an example of an essential domain which is not a PVMD. But this we have already done. So we put locally GCD-domains to a different use. We use them to derive, indirectly, the following result:

Proposition 4.3. *Let P be a maximal* t*-ideal in a commutative integral domain D. Then it is not necessary that PD_P should be a* t*-ideal of D_P.*

Proof. Suppose that the proposition is not true and let D be a locally GCD-domain which is not a PVMD. Further, let P be a maximal t-ideal of D. Then by our assumption PD_P is a t-ideal. But since D_P is a GCD-domain, PD_P must be a PF-prime and hence D_P is a valuation domain. But then if the above is true for all the maximal t-ideals, then, according to [7], D is a PVMD – a contradiction. Hence if D is a locally GCD-domain which is not a PVMD, then D has a maximal t-ideal P such that PD_P is not a t-ideal of D_P. □

The need for Proposition 4.3 arose from a rather misleading (yet correct) result of [15], which says that if A is a finitely generated ideal of D and S a multiplicative set of D, then $(AD_S)_v = (A_vD_S)_v$. Now if $S = D - P$ and if $A \subseteq P$, one could easily be misled into thinking that if $A_vD_P \subseteq PD_P$, then $(A_vD_P)_v \subseteq PD_P$. We note that the contradiction required for the proof of Proposition 4.3 can also be provided by taking D to be locally PVMD. So we have the following corollary:

Corollary 4.4. *The following are equivalent for D:*
(1) *D is a PVMD;*
(2) *D is locally a PVMD and for every prime* t*-ideal P of D, PD_P is a* t*-ideal.* □

4.3. Example of an ideal I with $I \subsetneq I_t \subsetneq I_v \subsetneq D$

The v- and t-operations are special cases of the so called $*$-operations. For details on $*$-operations the reader is referred to [6]. For the purposes of this example we note that for every fractionary ideal A of D, $A \subseteq A_* \subseteq A_v$ (and hence $A_t \subseteq A_v$). Moreover, if $\{D_i\}_{i\in I}$ is a family of overrings of D with $D = \bigcap_i D_i$, then the operation defined by $A \to \bigcap_{i\in I} AD_i$ on $F(D)$, the set of fractional ideals of D, is a $*$-operation (cf. [6, 32.5]).

Lemma 4.5. *Let D be an essential domain. Then for all $a, b \in D - \{0\}$, $((a, b)(a) \cap (b))_v = (ab)$.*

Proof. Let $\{P_i\}_{i\in I}$ be a family of essential primes of D such that $D = \bigcap D_{P_i}$ and define $*$ on $F(D)$ by $A \mapsto A_* = \bigcap AD_{P_i}$. Then for any $a, b \in D - \{0\}$ and for any $P \in \{P_i\}$

$$\left(\frac{(a, b)((a) \cap (b))}{ab}\right) D_P = (a, b)D_P \frac{aD_P \cap bD_P}{ab}$$
$$= (a, b)D_P((a, b)D_P)^{-1} = D_P ,$$

because D_P is a valuation domain. Thus

$$\left((a, b)\left(\frac{(a) \cap (b)}{ab}\right)\right)_* = \bigcap_i \left((a, b)\frac{((a) \cap (b))}{ab}\right) D_{P_i}$$
$$= \bigcap D_{P_i} = D .$$

Now, as $A_* \subseteq A_v$ for all $A \in F(D)$, we have for all $a, b \in D - \{0\}$

$$\left((a, b)\frac{(a) \cap (b)}{ab}\right)_v = D$$

or $((a, b)((a) \cap (b)))_v = abD$. □

Now, according to [7], if A is a finitely generated fractional ideal, then $(AB)_t = D$ if and only if $B_v = X_v$ where X is finitely generated. Further, according to [15] an essential domain D is a PVMD if and only if for all $a, b \in D - \{0\}$, $(a) \cap (b)$ is of finite type. From these observations we derive the following result:

Proposition 4.6. *An essential domain D is a non-PVMD if and only if there is a pair of non-zero elements $a, b \in D$ such that $((a, b)((a) \cap (b))/ab)_t \neq D$.* □

Corollary 4.7. *In a non-PVMD essential domain D there is at least one pair of non-zero elements such that*

$$(a, b)((a) \cap (b)) \subseteq ((a, b)((a) \cap (b)))_t \subsetneq ((a, b)((a) \cap (b)))_v \subsetneq D. \quad \square$$

Now to ensure the remaining inequality we proceed as follows: Let X be an indeterminate over an integral domain D. Then for any $A \in F(D)$, $A^{-1}[X] = (A[X])^{-1}$ [13]. Using the same equation again we get $A_v[X] = (A[X])_v$. Now to avoid mentioning too many references we choose D to be a P-domain which is not a PVMD. Then $D[X]$ is a P-domain which is not a PVMD (cf. [12]). Then as $(1/ab)((a) \cap (b)) = (a, b)^{-1}$ we have for $a, b \in D - \{0\}$ with properties as in Corollary 4.7,

$$\left((a, b)\,\frac{(a) \cap (b)}{ab}\right)[X] = (aT + bT)\left(\frac{aT \cap bT}{ab}\right)$$

where $T = D[X]$, and so

$$\left((aT + bT)\,\frac{aT \cap bT}{ab}\right)_v = \left((a, b)\,\frac{(a) \cap (b)}{ab}\right)_v [X] = D[X] = T$$

and, as in Corollary 4.7, we have $A = (aT + bT)(aT \cap bT)$ such that $A \subseteq A_t \subsetneq A_v \subsetneq T$.

Now we note that for any non-zero non-unit $c \in D$, $cT \cap XT = cXT$ which gives $(cT + XT)_v = T$. So we define $I = (cT + XT)A$ where A is as defined above. Then, noting that if F is a finitely generated fractional ideal, then $F_v = F_t$, we conclude that

$$I_t = ((cT + XT)A)_t = ((cT + XT)_t A)_t = ((cT + XT)_v A)_t = A_t\,,$$
$$I_v = ((cT + XT)A)_v = ((cT + XT)_v A)_v = A_v\,.$$

This gives $I \subsetneq I_t \subsetneq I_v \subsetneq T$.

Remark 4.8. This example was constructed to answer a question raised, in a personal communication, by Alain Bouvier about the existence of the ideal I of the above example.

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References

[1] J. Brewer and W. Heinzer, Associated primes of principal ideals, Duke Math. J. 41 (1974) 1–7.
[2] P. Cohn, Bezout rings and their subrings, Math. Proc. Cambridge Philos. Soc. 64 (1968) 251–264.
[3] P. Cohn, Free Rings and their Relations (Academic Press, New York, 1971).
[4] D. Costa, J. Mott and M. Zafrullah, The construction $D + XD_S[X]$, J. Algebra 53 (1978) 423–439.
[5] L. Fuchs, Riesz groups, Ann. Scuola Norm. Sup. Pisa 19 (1965) 1–34.
[6] R. Gilmer, Multiplicative Ideal Theory, (Dekker, New York, 1972).
[7] M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19 (1967) 710–722.
[8] W. Heinzer, An essential integral domain with a non essential localization, Canad. J. Math. 33 (2) (1981) 400–403.
[9] W. Heinzer and J. Ohm, An essential ring which is not a v-multiplication ring, Canad. J. Math. 25 (1973) 856–861.
[10] P. Jaffard, Les Systèmes d'Ideaux (Dunod, Paris, 1960).
[11] J. Mott and M. Schexnayder, Exact sequences of semi-value groups, J. Reine Angew. Math. 283/284 (1976) 388–401.
[12] J. Mott and M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35 (1981) 1–26.
[13] T. Nishimura, On the v-ideals of an integral domain, Bull. Kyoto Univ. Ed. Ser. B 17 (1961) 47–50.
[14] P. Sheldon, Prime ideals in GCD-domains, Canad. J. Math. 26 (1974) 98–107.
[15] M. Zafrullah, On finite conductor domains, Manuscripta Math. 24 (1978) 191–203.