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The Construction $D + XD_S[X]$

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If D is a commutative integral domain and S is a multiplicative system in D, then $T^{(S)} = D + XD_S[X]$ is the subring of the polynomial ring $D_S[X]$ consisting of those polynomials with constant term in D. In the special case where $S = D^* = D\backslash\{0\}$, we omit the superscript and let T denote the ring $D + XK[X]$, where K is the quotient field of D.

Since $T^{(S)}$ is the direct limit of the rings $D[X/s]$, where $s \in S$, we can conclude that many properties hold in $T^{(S)}$ because these properties are preserved by taking polynomial ring extensions and direct limits. Moreover, the ring $T^{(S)}$ is the symmetric algebra $S_D(D_S)$ of D_S considered as a D-module. In addition, $D_S[X]$ is a quotient ring of $T^{(S)}$ with respect to S; in fact, in the terminology of [10], $T^{(S)}$ is the composite of D and $D_S[X]$ over the ideal $XD_S[X]$. (The most familiar of the composite constructions is the so-called $D + M$ construction [1], where generally M is the maximal ideal of a valuation ring.)

The ring $T^{(S)}$, therefore, provides a test case for many questions about direct limits, symmetric algebras, and composites.

The state of our knowledge of T is considerably more advanced than that of $T^{(S)}$; generally speaking, we often show that a property holds in T if and only if it holds in D. In other cases we show that $T^{(S)}$ does not have a given property if $D_S \neq K$. For example, if $T^{(S)}$ is a Prüfer domain, then $D_S[X]$ is a Prüfer domain and D_S is therefore equal to K. We show that T is Prüfer (Bezout) if and only if D is Prüfer (Bezout). Yet $T^{(S)}$ is a GCD-domain if D is a GCD-domain and the greatest common divisor of d and X exists in $T^{(S)}$ for each

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$d \in D^*$. Thus, we have a method for constructing GCD-domains that are not Bezout domains. This observation led to a counterexample to a conjecture of Sheldon [11].

Coherence generalizes the notions of both Prüfer domain and Noetherian domain. Clearly, the ring $T^{(S)}$ is Noetherian only in the case that D is Noetherian and $D_S = D$. We show that $T^{(S)}$ is coherent if D is Noetherian and, moreover, that T is coherent if and only if D is coherent. Therefore, this construction can be used to add to the known list of examples of non-Noetherian non-Prüfer coherent domains.

An elementary divisor domain (EDD) is an integral domain with the property that every matrix is equivalent to a diagonal matrix. In [8], Kaplansky showed that an EDD is necessarily a Bezout domain (that is, that finitely generated ideals are principal) and that a Bezout domain for which 2×2 triangular matrices are equivalent to diagonal matrices is EDD. It is an open question whether or not every Bezout domain is EDD; however, in [4], Butts and Dulin showed that, at that time, all known methods of constructing Bezout domains gave rise to elementary divisor domains. We have mentioned that T is a Bezout domain if D is; thus it is not surprising that we ask if this construction also produces an EDD. We show in Section 4.6 that T is EDD if and only if D is EDD.

1. $D + DX_S[X]$ and GCD-Domains

Recall that a GCD-domain is an integral domain in which each pair of nonzero elements have a greatest common divisor. Cohn [2] designates such rings as HCF-rings (for highest common factor) and in that paper, he discusses the relationship between GCD-domains and the so-called Schreier domains. We recall the definition of a Schreier domain.

An element X of an integral domain R is said to be *primal* if $X \mid ab$ implies that $X = X_1X_2$, where $X_1 \mid a$ and $X_2 \mid b$. The element X is *completely primal* if every factor of X is primal. An integrally closed domain R is a *Schreier domain* if every nonzero element is primal. An integrally closed domain is Schreier if and only if its group of divisibility satisfies the Riesz interpolation property [3], and since a GCD-domain has a lattice-ordered group of divisibility, a GCD-domain is clearly a Schreier domain.

Since each of the properties, integral closure and Schreier, is preserved under polynomial ring extensions and direct limits we see that $T^{(S)}$ is integrally closed or Schreier if D is integrally closed or Schreier.

THEOREM 1.1. *Suppose that S is a multiplicative set in D. Then $T^{(S)} = D + XD_S[X]$ is a GCD-domain if and only if D is a GCD-domain and GCD(d, X) exists in $T^{(S)}$ for each $d \in D^*$.*

Proof. We may assume without loss of generality that S is saturated. If $D + XD_S[X]$ is a GCD-domain, then GCD(d, X) exists for each $d \in D^*$. Conversely, if D is a GCD-domain, D is Schreier and so is $T^{(S)}$. Suppose that the given condition holds and consider an arbitrary nonzero nonunit $a \in T^{(S)}$. We can write $a = u + \sum_{i=1} u_i X^i$, $u \in D$, $u_i \in D_S$; $a = u + (b/s) X^r(u' + \sum_{i=1} u_i' X^i)$, $b \in D$ and $s \in S$.

Corresponding to $u = 0$ or $u \neq 0$, a can be one of two possible types:

(i) $a = (b/s) X^r\{u' + \sum_{i=1} u_i' X^i\}$, where the expression in braces belongs to $T^{(S)}$ and is a primitive polynomial in $D_S[X]$.

(ii) $a = u_1\{u_2 + (b'/s) X^t(u' + \sum u_i' X^i)\}$, where $u_1 = \text{GCD}(u, (b/s) X^t)$ and the expression in braces is a primitive polynomial in $D_S[X]$.

Now in both cases the factorizations of the expressions in braces depend upon their degrees in X, which are finite and so these expressions are products of *irreducible elements* or *atoms*.

It is easy to see that an atom in a Schreier domain is a *prime*, and in view of this we conclude that the expressions in braces in the above two cases are products of primes.

Now consider two arbitrary nonzero nonunits $a, b \in T^{(S)}$. Here $a = u_1 + \sum_{i=1} a_i X^i$; $b = u_2 + \sum_{i=1} b_i X^i$, $u_i \in D$, a_i, $b_i \in D_S$. The following three cases arise:

(1) u_1 and u_2 are both nonzero.

(2) $u_1 = 0$; $u_2 \neq 0$ (or $u_1 \neq 0$; $u_2 = 0$).

(3) $u_1 = 0 = u_2$.

In the first case, $a = up_1 \cdots p_r$, $b = vq_1 \cdots q_s$, where $u, v \in D$ and, for each i, p_i and q_i are primes not in D. It can be verified that GCD$(a, b) = d(\text{GCD}(u, v))$, where $d = \text{GCD}(p_1 \cdots p_r, q_1 \cdots q_s)$.

In case (2), $a = up_1 \cdots p_r$ and $b = (m/s_1) X^t q_1 \cdots q_s$. If we show that $h = \text{GCD}(u, (m/s_1))$ exists, then it will follow that GCD$(a, b) = hd$, where $d = \text{GCD}(p_1 \cdots p_r, q_1 \cdots q_s)$. Since $k = \text{GCD}(u, X)$, we have $u = u_1 k$, $X = x_1 k$, where GCD$(u_1, x_1) = 1$ and $k \in S$. Thus, $k^n \mid X$ for each integer n, and this, in turn, implies that $k \mid x_1$, that is, that GCD$(u_1, X) = 1$ and GCD$(u, X^t/s_1) = k_1$, an associate of k. Moreover, if $c \mid u_1$ and $c \mid mX^t/s_1$, then $c \mid m$ by the Schreier property. But since D is a GCD-domain, GCD(u_1, m) exists and $h = k(\text{GCD}(u_1, m)) = \text{GCD}(u, (m/s_1) X^t)$ exists.

Finally, in case (3), $a = (m/s_1) X^{t_1} p_1 \cdots p_r$; $b = (m_2/s_2) X^{t_2} q_1 \cdots q_s$. If, say, $t_1 < t_2$, then GCD$(a, b) = \text{GCD}(m_1, m_2)(X^{t_1}/s_1)$, where $d = \text{GCD}(p_1 \cdots p_r, q_1 \cdots q_s)$. If, on the other hand, $t_1 = t_2 = t$, then GCD$(a, b) = \{\text{GCD}(m_1, m_2)/\text{LCM}(s_1, s_2)\} X^t d$, where $d = \text{GCD}(p_1 \cdots p_r, q_1 \cdots q_s)$ and LCM(s_1, s_2) is the least common multiple of s_1 and s_2 in D. Therefore, any two elements of $T^{(S)}$ have a GCD.

COROLLARY 1.2. *If D is a UFD and S is a multiplicative set in D, then $D + XD_S[X]$ is a GCD-domain.*

Proof. We may assume that S is saturated. If $d \in D^*$, then we can write $d = d_1 s_1$ such that d_1 is not divisible by any nonunit of S and $s_1 \in S$. Then obviously $\text{GCD}(d_1, X) = s_1$ and the result follows.

COROLLARY 1.3. *If D is a GCD-domain and K is its field of fractions, then $T = D + XK[X]$ is GCD-domain.*

To see that there exist GCD-domains D with multiplicative sets S such that $T^{(S)}$ is not a GCD-domain, we consider the following:

EXAMPLE 1.4. Let D be a discrete valuation domain of rank 2 with maximal ideal pD. Let $S = \{p^i\}_{i=1}^{\infty}$ and consider $D + XD_S[X]$. If we pick an element q in the nonzero minimal prime ideal of D, and consider the common factors of q and X, we find that $p^i \mid q$ and $p^i \mid X$ for all i. But X and q have no greatest common divisor and this implies that $T^{(S)}$ is not a GCD-domain.

2. PRIME IDEAL STRUCTURE OF $D + XD_S[X]$

THEOREM 2.1. *Let* $L = \{P_L \in \text{Spec } T^{(S)} \mid P_L \cap S \neq \varnothing\}$ *and* $M = \{P \in \text{Spec } T^{(S)} \mid P \cap S = \varnothing\}$. *Then* $\text{Spec } T^{(S)} = L \cup M$. *Moreover, L and M are isomorphic as partially ordered sets under containment to* $\{P \in \text{Spec } D \mid P \cap S \neq \varnothing\}$ *and* $\text{Spec}(D_S[X])$, *respectively. Furthermore, each $P_L \in L$ is of the form* $(P_L \cap D) + XD_S[X]$.

Proof. Obviously, $D_S[X] = (T^{(S)})_S$ and $M \simeq \text{Spec } D_S[X]$. To show that L is isomorphic to $\{P \in \text{Spec } D \mid P \cap S \neq \varnothing\}$ we need only observe that any any $P_L \in L$ contains the kernel $XD_S[X]$ of the natural homomorphism from $T^{(S)}$ onto D. But this is immediate, since there is an $s \in P_L \cap S$ and s divides each element of $XD_S[X]$ in $T^{(S)}$.

We now proceed to indicate the relationship of chains of prime ideals of $T^{(S)}$ with those of $D[X]$. The observation that $D[X]_S = (T^{(S)})_S$ is crucial and leads to the following definitions.

DEFINITION 2.2. Suppose that R and R_1 are integral domains with $R \subseteq R_1$. We say that R and R_1 are in *accord* at the multiplicative system $S \subseteq R$ if $R_S = (R_1)_S$ and $U(R) = U(R_1)$. We denote by $\text{Spec}_S R$ the set of all those prime ideals of R that are disjoint from S. The set M of Theorem 2.1 is $\text{Spec}_S T^{(S)}$.

It is well known that there is a one-to-one order preserving correspondence between $\text{Spec}_S R$ and $\text{Spec } R_S$. Moreover, it follows that if $R \subseteq R_1$ and R and R_1 are in accord at S, then there is a one-to-one order preserving correspondence between $\mathrm{Spec}_S\, R$ and $\mathrm{Spec}_S\, R_1$. In particular, since $D[X]$ and $T^{(S)}$ are in accord at S, the maps $\sigma\colon \mathrm{Spec}_S\, D[X] \to \mathrm{Spec}_S\, T^{(S)}$ and $\tau\colon \mathrm{Spec}_S\, T^{(S)} \to \mathrm{Spec}_S\, D[X]$ defined by $\sigma(P) = PD_S[X] \cap T^{(S)}$ and $\tau(Q) = Q \cap D[X]$ are one-to-one and inverse of each other.

LEMMA 2.3. *Let P be a prime ideal of $D[X]$. Define*

$$\begin{aligned} P^e = PT^{(S)} &= P \cap D + XD_S[X] && \text{if } P \cap S \neq \varnothing \\ &= PD_S[X] \cap T^{(S)} && \text{if } P \cap S = \varnothing. \end{aligned}$$

Then P^e the unique prime ideal of $T^{(S)}$ such that $(P^e \cap D[X])^e = P^e$. Moreover, if Q is a prime ideal of $T^{(S)}$ and $P = Q \cap D[X]$, then $Q = P^e$.

Proof. We can apply Theorem 2.1 and properties of quotient rings to conclude P^e is a prime ideal of $T^{(S)}$. Moreover, it is clear that

$$\begin{aligned} P^e \cap D[X] &= P \cap D + XD[X] && \text{if } P \cap S \neq \varnothing \\ &= P && \text{if } P \cap S = \varnothing. \end{aligned}$$

This observation leads immediately the conclusion that $(P^e \cap D[X])^e = P^e$.

Finally, suppose that $P = Q \cap D[X]$, where Q is a prime ideal of $T^{(S)}$. If $Q \cap S \neq \varnothing$, then $Q = Q \cap D + XD_S[X]$, $P = Q \cap D + XD[X] = P \cap D + XD[X]$, and $P^e = Q$. On the other hand, if $Q \cap S = \varnothing$, then $Q = P^e$ because $D[X]$ and $T^{(S)}$ are in accord at S.

DEFINITION 2.4. The prime ideal P^e defined in Lemma 2.3 will be called the elevation of P and we will say that P elevates to P^e. Similarly if C is a collection of prime ideals of $D[X]$, the set of primes $C^e = \{P^e \mid P \in C\}$ will be called the elevation of C.

Remark 2.5. We see from Lemma 2.3 that any chain of prime ideals of $T^{(S)}$ is the elevation of some chain of prime ideals of $D[X]$. Thus, the Krull dimension of $T^{(S)}$ (denoted $\dim T^{(S)}$) is less than or equal to $\dim D[X]$. However, not every chain of prime ideals of $D[X]$ elevates to a *chain* in $T^{(S)}$. For example, let $T = Z + XQ[X]$, where Z and Q denote the integers and rational numbers, respectively. Let $P_1 = (p + X)\, Z[X]$ and $P_2 = (p, X) = pZ + XZ[X]$. Then $P_1 \subseteq P_2$, $1 + X/p \in P_1{}^e$, but $1 + X/p \notin P_2{}^e = pZ + XQ[X]$. Note that for this example $S = Z^*$ and $P_1 \cap S = \varnothing$, while $P_2 \cap S \neq \varnothing$. One additional observation: even if a chain of prime ideals in $D[X]$ elevates to a chain in $T^{(S)}$, the length of the elevated chain may be smaller than that of the original chain. For this example, let $P_3 = (p)\, Z[X]$ and let P_2 be the same ideal as above. Then $P_3 \subset P_2$, but $P_3{}^e = P_2{}^e = pZ + XQ[X]$. Here the reason is that $P_3 \cap S = P_2 \cap S$. In general, if P is any prime ideal of $D[X]$, and if $P \cap S \neq \varnothing$, then $P^e = (Q[X])^e$, where $Q = P \cap D$.

With this interpretation, we have the following theorem.

THEOREM 2.6. $\dim D_S[X] \leqslant \dim T^{(S)} \leqslant \dim D[X]$. *Moreover*, $\dim T^{(S)} = \sup\{$*lengths of chains of prime ideals of the form* C^e, *where* C *is a chain of prime ideals of* $D[X]\}$.

Since Theorem 2.1 implies that $\dim T^{(S)} \geqslant 1 + \dim D$, we have the following corollary to Theorem 2.6.

COROLLARY 2.7. *Suppose that* D *is such that* $\dim D[X] = 1 + \dim D$ (*for example, if* D *is Noetherian or if* D *is Prüfer*) *or suppose that the multiplicative system is such that* $\dim D_S[X] = \dim D[X]$. *Then, in these cases*, $\dim T^{(S)} = \dim D[X]$.

Theorem 2.6 has led to some conclusions, but a further question arises immediately. Which chains of prime ideals in $D[X]$ elevate to chains in $T^{(S)}$? Two answers are readily obtained. Suppose that C is the chain of primes $P = P_1 \supset P_2 \supset \cdots \supset P_r \supset \cdots \supset P_n$. If $P_1 \cap S = \varnothing$, then since $D[X]$ and $T^{(S)}$ are in accord at S, C^e is a chain in $T^{(S)}$ of the same length. On the other hand, if $P_n \cap S \neq \varnothing$, then for each i, $P_i^e = P_i \cap D + XD_S[X]$, and C elevates to a chain in $T^{(S)}$. Unfortunately in this latter case, the length of C^e may be smaller than that of C.

But when we consider arbitrary chains in $D[X]$, a further difficulty arises. For if $P_r \cap S \neq \varnothing$ and $P_{r+1} \cap S = \varnothing$, then each of the chains $P_1 \supset P_2 \supset \cdots \supset P_r$ and $P_{r+1} \supset \cdots \supset P_n$ elevates to chains in $T^{(S)}$, but we cannot say that $P_{r+1}^e \subseteq P_r^e$.

DEFINITION 2.8. We call a chain $P = P_1 \supset P_2 \supset \cdots \supset P_n$ of prime ideals of D an *S-chain* if $P_n \cap S \neq 0$. We also say that the above chain has length n. The *S-height* of P is the number of prime ideals in the longest S-chain descending from P, while the supremum of $\{S\text{-height } P \mid P \in \operatorname{Spec} D\}$ is the *S-dimension of* D, denoted by S-$\dim D$. (Obviously, S-$\dim D \leqslant \dim D$, and if $S = D^*$, then S-$\dim D = \dim D$.)

If P is a prime ideal of $D[X]$ and $P \cap S = \varnothing$, where S is a multiplicative system in D, then height P = height P^e = height $PD_S[X]$. But if $P \cap S \neq \varnothing$, then $P^e = Q + XD_S[X]$, where $Q = P \cap D$, and Theorem 2.1 shows that S-height Q = S-height P^e = S-height P. If fact, all we know is that S-height $P \leqslant 1 + 2$ (S-height Q). Equality may hold, for example, if $S = D^*$ and if $\dim D[X] = 1 + 2 \dim D$.

We see, therefore, that S-$\dim D$ contributes more to determining $\dim T^{(S)}$ than does S-$\dim D[X]$. In fact, we have the following corollaries to Theorem 2.1 and 2.6.

COROLLARY 2.9. *Suppose that* $\dim D < \infty$. *Then* $1 + S\text{-}\dim D \leqslant 1 + \dim D \leqslant \max\{1 + \dim D, \dim D_S[X]\} \leqslant \dim T^{(S)} \leqslant S\text{-}\dim D + \dim D_S[X]$.

COROLLARY 2.10. *The dimension of* $T = D + XK[X]$ *is equal to* $1 + \dim D$.

Proof. This is immediate from Corollary 2.9, since, in this case, $S\text{-}\dim D = \dim D$ and $\dim D_S[X] = \dim K[X] = 1$.

Corollary 2.10 shows that for some multiplicative systems S, $1 + S\text{-}\dim D = \dim T^{(S)} = S\text{-}\dim D + \dim D_S[X]$. Thus, the inequalities in Corollary 2.9 are, in some sense, the best possible estimates.

We ask: Is $\dim T^{(S)} = S\text{-}\dim D + \dim D_S[X]$, for all multiplicative systems S? The following example shows that the answer is, in general, no. This example will be useful later as a counterexample to a conjecture of Sheldon.

EXAMPLE 2.11. Let D be a PID and S a multiplicative system of D such that $D_S \neq K$. We show that $\dim T^{(S)} = 2$, and that $S\text{-}\dim D + \dim D_S[X] = 3$. Clearly $\dim T^{(S)} \geqslant 2$. On the other hand, if $P_L \in L$, then $P_L = P + XD_S[X]$ and P_L does not properly contain another prime in L since D is one-dimensional. Moreover, $D_S[X]$ is two-dimensional. Thus any prime ideal $P_M \in M$ has height $\leqslant 2$. We show that $\text{ht}(P_L) \leqslant 2$ by observing that P_L contains no prime $P_M \in M$ of height exactly 2. Any such prime P_M is such that $Q = P_M \cap D \neq (0)$. But Q and P are relatively prime ideals in D since D is one-dimensional. Thus, $P_M \not\subseteq P_L$ and $\text{ht}(P_L) = 2$.

There is at least one other sitaution where the answer is yes to the above question.

PROPOSITION 2.12. *If D is a valuation ring of finite Krull dimension, then for any multiplicative system S in D,* $1 + \dim D = \dim T^{(S)} = S\text{-}\dim D + \dim D_S[X]$.

Proof. Without loss of generality we may assume that S is saturated and, therefore, that S is the complement of a prime ideal of D. Then $S\text{-}\dim D =$ depth of $P = \dim D - \text{ht}(P)$, and $\dim D_S[X] = 1 + \text{ht}(P)$.

One might conjecture that $\dim T^{(S)}$ is equal to $m = \max\{1 + S\text{-}\dim D, \dim D_S[X]\}$. But this conjecture is false, for there are examples where $m < 1 + \dim D$. For instance, consider the following example.

EXAMPLE 2.13. Let D be a rank 2 valuation ring with nonzero prime ideals $P_1 \supset P_2$. Let $S = D \backslash P_2$. Then D_S is a rank one valuation ring, and $\dim D_S[X] = 2$. Clearly $1 + S\text{-}\dim D = 2$ and $\dim T^{(S)} = 3$.

Unfortunately, we have not been able to determine the dimension of $T^{(S)}$ exactly; what we desire is some function of $S\text{-}\dim D$, $\dim D$, $\dim D_S[X]$, and $\dim D[X]$ that precisely describes $\dim T^{(S)}$ for all domains D and all multiplicative systems S.

3. Sheldon's Conjecture

For a GCD-domain R, there exist prime ideals P of R such that R_P is a valuation ring. Moreover, the supremum of $\dim R_P$ for all such prime ideals of R is called the *prime filter* dimension of R and is denoted by PF-dim R. In [11], Sheldon conjectured that a GCD-domain R for which $\dim R =$ PF-dim $R < \infty$ is necessarily a Bezout domain. We give a counterexample to this conjecture.

Example 3.1. Let D be a PID and let S be a multiplicative system for which $D_S \neq K$. We have already observed that $\dim T^{(S)} = 2$. Moreover, $T^{(S)}$ is a GCD-domain, but not Bezout since $D_S \neq K$ (for then $D_S[X]$ is not Bezout). We need only observe that PF-dim $T^{(S)} = 2$. By Proposition 4.2 of [11], $1 \leqslant$ PF-dim $T^{(S)} \leqslant \dim T^{(S)} = 2$. If we are able to show that $T^{(S)}$ has at least one prime ideal P for which $T_P^{(S)}$ is a rank 2 valuation ring, the result will follow.

But this holds for any prime $P_L \in L$, for $P_L = P + XD_S[X]$ and $T_{P_L}^{(S)}$ is a GCD-domain (being a localization of a GCD-domain) in which no two nonzero nonunit elements are relatively prime. Therefore $T_{P_L}^{(S)}$ is a valuation ring necessarily of rank 2.

We recall that if P is a prime ideal of a GCD-domain D such that $\mathrm{GCD}(x, y) \in P$ for each pair of elements x and y of P, then D_P is a valuation ring [11, p. 99]. Moreover, if $S = D^*$, then for a pair of elements a, b in the prime ideal $P_L = P + XK[X]$ of T, $\mathrm{GCD}(a, b) \in P_L$. This observation establishes the following result.

Proposition 3.2. *If D is a GCD-domain and $k =$ PF-dim $D < \infty$, then* PF-dim $T = k + 1$.

In view of this proposition, we state the following:

Theorem 3.3. *For each positive integer $n \geqslant 2$, there exists a non-Bezout GCD-domain R such that* PF-dim $R = \dim R = n$.

Proof. We have an example R_2 for $n = 2$. For $n = 3$, we take $R_2 + XK_2[X]$, where K_2 is the quotient field of R_2. Similarly, the result follows for all n by induction.

4. The $D + XK[X]$ Construction

Up to this point, we have allowed the multiplicative system S to be quite arbitrary for the most part. But now we turn our attention to the special case where $S = D^*$. In this case recall that we are using the notation $T = D +$

$XK[X]$. We are able to obtain in some cases more specific information about T than for $T^{(S)} = D + XD_S[X]$.

4.1. *The Ideals of T*

First we give a description of certain of the ideals of T.

LEMMA 4.11. *Let I be an ideal of $T = D + XK[X]$. The following are equivalent.*

(1) $I \cap D \neq 0$.

(2) $I \supset XK[X]$.

(3) $IK[X] = K[X]$.

If any of these hold, then $I = (I \cap D) + XK[X] = (I \cap D)\,T$.

Proof. (1) $\rightarrow$ (2). Let $a \in I \cap D$, $a \neq 0$. Then $XK[X] = aXK[X] \subseteq I$ and hence $I = (I \cap D) + XK[X]$.

(2) $\rightarrow$ (1). If $I \supset XK[X]$, then $I = I \cap D + XK[X]$, and therefore $I \cap D \neq \{0\}$.

(1) $\rightarrow$ (3) is clear. The last assertion follows from (1) as in (1) $\rightarrow$ (2).

PROPOSITION 4.12. *Each ideal of T is of the form $f(X)FT = f(X) \cdot (F + XK[X])$, where F is a nonzero D-submodule of K such that $f(0)F \subseteq D$, and $f(X) \in K[X]$.*

The finitely generated ideals of T are of the form $f(X)\,JT$, where J is a finitely generated ideal of D and $f(X) \in T$.

Proof. First observe that any subset of T of the form $f(X)FT$ is in fact an ideal of T.

Next let I be an ideal of T. If $IK[X] = K[X]$, then $I \cap D \neq 0$ and $I = (I \cap D) + XK[X] = (I \cap D)\,T$.

If $IK[X] \neq K[X]$, then $IK[X] = f(X)\,K[X]$ for some nonconstant $f(X) \in K[X]$. Then there is a nonzero element $\alpha \in K$ such that $\alpha f(X) \in I$. Let $F = \{\alpha \in K \mid \alpha f(X) \in I\}$. Then F is a D-submodule of K.

Since $F \neq 0$ and $f(X)F \subseteq I$, $I \supseteq f(X)T = f(X)(F + XK[X])$. But if $h(X) \in I$, then $h(X) = f(X)(\alpha_0 + \cdots + \alpha_n X^n)$, where $\alpha_0, \ldots, \alpha_n \in K$; whence $h(X) = \alpha_0 f(X) + h'(X)$, where $h'(X) \in f(X)\,XK[X] \subseteq I$. Hence $\alpha_0 \in F$ and $h(X) \in f(X)(F + XK[X])$. Thus $I = f(X)(F + XK[X]) = f(X)FT$, from which it also follows that $f(0)F \subseteq D$.

To prove the second assertion, let I be a finitely generated ideal of T and write it according to the first statement as $I = f(X)FT$. Then F is a finitely generated D-module and there is an element $d \in D^*$ such that $dF \subseteq D$.

If $f(0) = 0$, then $f(X)/d \in T$, dF is a finitely generated ideal of D, and $I = (f(X)/d)(dF)\,T$.

$* I \supseteq f(X) F T$

If $f(0) = \alpha \neq 0$, then $f(X)/\alpha \in T$, αF is finitely generated ideal of D, and $I = (f(X)/\alpha)(\alpha F)\, T$.

COROLLARY 4.13. *$T = D + XK[X]$ is a Bezout domain if and only if D is.*

A ring is said to have the *n*-generator property if every finitely generated ideal has a bais s of *n* elements.

COROLLARY 4.14. *$T = D + XK[X]$ has the n-generator property if and only if D does.*

Proof. The sufficiency follows from Proposition 4.12. For the necessity, let J be a finitely generated ideal of D. Then $J = JT \cap D \simeq JT/XK[X]$. Since JT has a basis of n elements, so does J.

COROLLARY 4.15. *$T = D + XK[X]$ is a Prüfer domain if and only if D is.*

Proof. Suppose D is a Prüfer domain. Let I be a finitely generated ideal of T and write $I = f(X)\, JT$, where J is a finitely generated ideal of D. Since J is invertible, $JJ^{-1} = D$. As sets, $J \subseteq JT$ and $J^{-1} \subseteq (JT)^{-1}$ and therefore $1 \in JJ^{-1} \subseteq (JT)(JT)^{-1}$. Thus $(JT)(JT)^{-1} = T$ and $I = f(X)\, JT$ is invertible.

By the class group $C(D)$ of a Prüfer domain D we mean the group of equivalence classes of invertible fractional ideals modulo the group of principal fractional ideals.

If D is Prüfer, then by Corollary 4.15, T is also, and furthermore $J \to JT$ is a homomorphism of $C(D)$ into $C(T)$. Since for integral ideals J of D, $J \simeq JT/XK[X]$, J and JT are simultaneously principal. This shows that the homomorphism $C(D) \to C(T)$ is injective. Furthermore Proposition 4.12 shows that the map is surjective. Hence $C(D) \simeq C(T)$.

We summarize these observations in the following:

COROLLARY 4.16. *If D is a Prüfer domain, then the class group of T is isomorphic to the class group of D.*

Recall that the valuative dimension, $\dim_v(R)$, of a domain R is the supremum of $\dim V$ for all valuation overrings V of R. For a Prüfer domain R, $\dim_v(R) = \dim R$.

COROLLARY 4.17. *The valuative dimension of T is $1 + \dim_v D$.*

Proof. If V is a valuation overring of T, then $W = V \cap K$ is a valuation overring of D, and $W + XK[X]$ is a Prüfer domain of dimension equal to $\dim W + 1 \leqslant \dim_v D + 1$. Thus, $\dim V \leqslant \dim_v D + 1$. On the other hand, if W is a valuation overring of D such that $\dim W = \dim_v D$, then

$W + XK[X]_{(X)}$ is a valuation overring of T with dimension equal to $\dim_v D + 1$. Hence $\dim_v T = \dim_v D + 1$.

An integral domain D is said to have the QR-property if every overring (that is, every ring between D and K) of D is a quotient ring of D. Such a domain is necessarily a Prüfer domain. But, more than that a Prüfer domain D has the QR-property if and only if for each finitely generated ideal I of D there exists $d \in I$ and a positive integer n such that $I^n \subseteq dD$ [5, p. 337].

This characterization of the QR-property, Proposition 4.12 and Corollary 4.15 readily yield the following result.

Corollary 4.18. *$T = D + XK[X]$ has the QR-property if and only if D does.*

4.2. *The Prime Spectrum of T*

Our knowledge of the prime ideals of T considerably exceeds the information contained in Theorem 2.1.

Theorem 4.21. *The nonzero prime ideals of $T = D + XK[X]$ are the ideals $Q + XK[X]$, where Q is a prime ideal of D, and the principal ideals $f(X)\, T$, where $f(X)$ is irreducible in $K[X]$ and $f(0) = 1$. The height one primes of T are $XK[X]$ and the principal prime ideals $f(X)\, T$. The maximal ideals of T are those of the form $M + XK[X]$, where M is a maximal ideal of D, and the principal primes $f(X)\, T$ described above.*

Proof. Let P be a nonzero prime ideal of T. If $X \in P$, then for every $d \in D$, $(X/d)^2 = (X/d^2)\, X \in P$, and hence $X/d \in P$. Thus $XK[X] \subseteq P$. Let $Q = P \cap D$. Then $P = Q + XK[X]$.

Suppose $X \notin P$. Then by Lemma 4.11, $P \cap D = (0)$ and $P = PK[X] \cap T$, where $PK[X]$ is a proper prime ideal of $K[X]$. Moreover, $PK[X] = f(X)\, K[X]$, where $f(X) \in P$ is an irreducible element of $K[X]$ such that $f(0) = 1$. But then $P = PK[X] \cap T = f(X)\, K[X] \cap T = f(X)\, T$, as desired. The last equality is a consequence of the fact that $f(0) = 1$.

That the primes $f(X)\, T$ and $XK[X]$ are height one is seen by localizing at $D\backslash\{0\}$. That they are the only height one primes follows from the first assertion.

The only assertion which still requires proof is that of the maximality of the ideals $f(X)\, T$, where $f(X)$ is irreducible in $K[X]$ and $f(0) = 1$. No such prime ideal is contained in any other or in $XK[X]$ since they are all of height one. Furthermore, the fact that $f(0) = 1$ excludes $f(X)$ from every prime ideal of the form $Q + XK[X]$. Hence $f(X)\, T$ is maximal.

Now let us discuss the maximal spectrum of T. Recall that an ideal is a j-ideal if it is an intersection of maximal ideals. We say that a prime j-ideal is a j-prime. Moreover, the j-dimension of a ring R is the supremum of the lengths of chains of j-primes. Max-Spec(R) is the subspace of Spec(R) consisting of the maximal ideals of R; j-Spec(R) is the subspace of Spec(R) consisting of the j-primes of R.

THEOREM 4.22. $j\text{-dim}(T) = 1 + j\text{-dim}(D)$.

Proof. First observe that T has an infinite number of height one maximal ideals $f(X)\,T$, where $f(X)$ is irreducible in $K[X]$ and $f(0) = 1$. Moreover, the zero ideal of T is the intersection of all these maximal ideals, that is, (0) is a j-ideal of T. Or, in other words, the Jacobson radical of T is (0). Now if P is a j-prime of D, then $P = \bigcap M_\alpha$, where each M_α is a maximal ideal of D. Consequently, $P + XK[X] = \bigcap (M_\alpha + XK[X])$ and $P + XK[X]$ is a j-ideal of T. Thus, $j\text{-dim}\ T \geqslant j\text{-dim}\ D + 1$.

On the other hand, suppose that $0 \subset Q_1 \subset \cdots \subset Q_n$ is a chain of j-prime ideals of T. If $Q_n = P_n + XK[X]$, where P_n is a prime ideal of D, then each of $Q_1, \ldots, Q_n$ has the same form since all of the other kind of prime ideals of T are maximal. Thus, $Q_i = P_i + XK[X]$, and $P_1 \subset P_2 \subset \cdots \subset P_n$ is a chain of j-primes of D. Therefore, $n \leqslant 1 + j\text{-dim}\ D$.

If Q_n does not have this form, then Q_n is a height one maximal ideal of T and $n = 1$. In either case, $n \leqslant 1 + j\text{-dim}\ D$, and the theorem is proved.

Recall that Spec(R) is Noetherian if and only if R satisfies the ascending chain condition on radical ideals. Max-Spec(R) and j-Spec(R) are simultaneously Noetherian and this occurs if and only if R satisfies the ascending chain condition on j-ideals.

THEOREM 4.23. Spec(T) (respectively max-Spec(T)) is Noetherian if and only if Spec(D) (respectively, max-Spec(D)) is Noetherian.

Proof. Spec(D) (max-Spec(D)) is homeomorphic to a subspace of Spec(T) (max-Spec(T)). Therefore, Spec(D) (max-Spec(D)) is Noetherian if Spec(T) (max-Spec(T)) is Noetherian.

Conversely, suppose that Spec(D) (max-Spec(D)) is Noetherian and that $0 = I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of radical ideals (j-ideals) of T. Observe that there are only finitely many height one maximal ideals of the form $f(X)\,T$ containing a given I_K. For each k, let A_k denote the index set for the set of all prime (maximal) ideals of the form $P_\alpha + XK[X]$ that contain I_k. Let B_k denote the finite index set of maximal ideals of the form $f(X)\,T$ that contain I_k. Clearly then $I_k = \bigcap_{\alpha \in A_k} (P_\alpha + XK[X]) \bigcap_{i \in B_k} (f_i(X)\,T$ and $A_1 \subseteq A_2 \subseteq \cdots$ and $B_1 \subseteq B_2 \subseteq \cdots$. Clearly there is a k_0 such that $B_k = B_{k_0}$ for all $k \geqslant k_0$. Then for $k \geqslant k_0$, $I_k = I_{k+1}$ if and only if $A_k = A_{k+1}$.

Now $\bigcap_{\alpha \in A_{k_0}} P_\alpha \subseteq \bigcap_{\alpha \in A_{k_0+1}} P_\alpha \subseteq \cdots$ is an ascending chain of radical ideals (j-ideals) of D. Since D satisfies the appropriate chain condition, there is an integer $N \geqslant k_0$ such that for $k \geqslant N$, $A_k = A_N$. Thus Spec(T) (max-Spec(T)) is Noetherian.

4.3. *Coherence*

Recall that a ring R is coherent if every finitely generated ideal of R is finitely presented.

THEOREM 4.31. *$T = D + XK[X]$ is coherent if and only if D is coherent.*

Proof. First note that T is a faithfully flat D-module (one way to observe this is to note that $T = S_D(K)$ and use a result of Lazard [9, *p.* 84]). Therefore $JT \simeq J \otimes_D T$ for any finitely generated ideal J of D.

If D is coherent and I is a finitely generated ideal of T, then $I = f(X)\,JT$, where J is a finitely generated ideal of D. Hence $I \simeq JT$ as T-modules. Now $JT \simeq J \otimes_D T$ and since J is a finitely presented D-module, JT is a finitely presented T-module. Hence, I is a finitely presented T-module, and T is coherent.

Conversely, if J is a finitely generated ideal of D, then $JT = J \otimes_D T$ is a finitely generated ideal of T and, hence, is finitely presented. Since T is a faithfully flat D-module, J is finitely presented. Thus, D is coherent.

Under certain conditions we can prove that $T^{(S)}$ is coherent.

THEOREM 4.32. *If D is a Noetherian domain and S is a multiplicative system in D, then $T^{(S)} = D + XD_S[X]$ is coherent. In fact, $T^{(S)}[\{X_\lambda\}]$ is coherent for any family $\{X_\lambda\}$ of indeterminates.*

Proof. The proof is a direct application of a result of Greenberg and Vasconcelos. For if Y is a finite family of indeterminates and if D is Noetherian, then the following diagram satisfies the hypothesis of Proposition 4.1 of [6]:

$$\begin{array}{ccc} T^{(S)}[Y] = D[Y] + XD_S[X, Y] & \longrightarrow & D_S[X, Y] \\ \downarrow & & \downarrow \\ D[Y] & \longrightarrow & D_S. \end{array}$$

Hence $T^{(S)}[Y]$ is coherent.

If Y is an infinite family of indeterminates, observe that $T^{(S)}[Y]$ is the "flat direct limit" of the coherent rings $T^{(S)}[Y]$, where Y_0 is a finite subset of Y. Therefore, in this case, $T^{(S)}[Y]$ is coherent.

4.4. *Divisorial Ideals of T*

Suppose that I is a fractional ideal of an integral domain R. The intersection I_v of all principal fractional ideals of R that contain I is called the *v-ideal* or *divisorial ideal associated with I*. If $I = I_v$, we say that I is a *v-ideal* or a *divisorial* ideal. A v-ideal I is a v-ideal of finite type if $I = F_v$ for some finitely generated fractional ideal F of R. The map $I \to I_v$ is called the v-operation of R. A basic development of the v-operation and divisorial ideals can be found in Section 34 of [5]. A domain R is a v-domain if the v-operation satisfies the property that for any finitely generated ideals A, B, and C of R, $(AB)_v \subseteq (AC)_v$ implies that $B_v \subseteq C_v$.

In this section, we examine the v-operation on $T = D + XK[X]$. We wish to prove that T is a v-domain if and only if D is a v-domain. The proof requires knowledge of the structure of the finitely generated ideals of T and the following lemma.

LEMMA 4.41. *If I is a nonzero ideal of D, then $(IT)_v = (I + XK[X])_v = I_v + XK[X]$.*

Proof. Suppose $\alpha T \supset IT$, $\alpha \in K(X)$. Write $\alpha = f(X)/g(X)$, where $f(X)$ and $g(X)$ are relatively prime elements in $K[X]$. Let $d \in I\backslash\{0\}$. Then $dg(X) = f(X)\,h(X)$, where $h(X) \in T$, and, therefore, $f_0 = f(X) \in K$. We have then that $\alpha = f_0/g(X)$. If $X \mid g(X)$, then $g(X)\,T \subseteq f_0(D + XK[X])$, and $IT \subseteq T \subseteq (f_0/g(X))\,T = \alpha T$. If $X \nmid g(X)$, then $g_0 = g(0) \neq 0$. Moreover, $g'(X) = g(X)/g_0 \in T$, so that $T/g'(X) \supseteq T$. Note that $I \subseteq (f_0/g_0)\,D$. Hence, $IT = I + XK[X] \subseteq (f_0/g_0)\,D + XK[X] = (f_0/g_0)\,T \subseteq T(f_0/g_0)\,1/g'(X) = \alpha T$. In either case, there is an element $\beta \in K$ such that $IT \subseteq \beta T \subseteq \alpha T$. Therefore,

$$\begin{aligned}(IT)_v &= \bigcap \{\alpha T \mid \alpha T \supseteq IT,\ \alpha \in K(X)\}\\ &= \bigcap \{\beta T \mid \beta T \supseteq IT,\ \beta \in K\}\\ &= \bigcap \{\beta D + XK[X] \mid \beta D \supseteq I,\ \beta \in K\}\\ &= I_v + XK[X].\end{aligned}$$

THEOREM 4.42. *$T = D + XK[X]$ is a v-domain if and only if D is a v-domain.*

Proof. We show that if $(AB)_v \subseteq (AC)_v$, then $B_v \subseteq C_v$ for finitely generated ideals of T. Let $A = f(X)\,A'T$, $B = g(X)\,B'T$ and $C = h(X)\,C'T$, where A', B', and C' are finitely generated ideals of D and $f(X), g(X), h(X) \in T$. Then $(AB)_v = f(X)\,g(X)(A'B'T)_v = f(X)\,g(X)(A'B' + XK[X])_v = f(X)\,g(X)\cdot((A'B')_v + XK[X])$, and similarly $(AC)_v = f(X)\,h(X)((A'C')_v + XK[X])$. Now $(AB)_v\,K[X] = f(X)\,g(X)\,K[X]$ and $(AC)_v\,K[X] = f(X)\,h(X)\,K[X]$ since $(A'B')_v$ and $(A'C')_v$ are nonzero ideals of D. Therefore, $f(X)\,g(X)\,K[X] \subseteq f(X)\,h(X)\,K[X]$ and $g(X) = h(X)\,k(X)$, where $k(X) \in K[X]$. If $X \mid k(X)$, then $B \subseteq h(X)\,XK[X] \subseteq C$ and $B_v \subseteq C_v$. If $X \nmid k(X)$, then $k(0) = \alpha \in K\backslash\{0\}$, and $\alpha(A'B')_v \subseteq (A'C')_v$. Thus, $\alpha B'_v \subseteq C'_v$ since D is a v-domain. But then $B_v = (g(X)\,B'T)_v = g(X)(B'T)_v = g(X)(B'_v + XK[X]) = h(X)\,k(X)(B'_v + XK[X]) \subseteq h(X)(\alpha B'_v + XK[X]) \subseteq h(X)(C'_v + XK[X]) = h(X)(C'T)_v = C_v$.

Conversely, suppose that T is a v-domain. Suppose that A, B, and C are finitely generated ideals of D such that $(AB)_v \subseteq (AC)_v$. Then $((AB)\,T)_v \subseteq$

$((AC)T)_v$, and therefore $(BT)_v \subseteq (CT)_v$. In other words, $B_v + XK[X] \subseteq C_v + XK[X]$. From this it is immediate that $B_v \subseteq C_v$.

The v-operation on a domain R determines an equivalence relation on the set of ideals of R. The equivalence class determined by an ideal A is denoted by div(A) and is called the divisor class of A. The set $D(R)$ of all divisor classes of R is a semigroup under div A + div B = div(AB). $D(R)$ is a group if and only if R is completely integrally closed. Since T and $K[X]$ have a common ideal and $K[X]$ is completely integrally closed, the complete integral closure of T is $K[X]$. Thus, if $D \neq K$, T is not completely integrally closed, and therefore div(T) is not a group. It is natural, nevertheless, to inquire if the subsemigroup H of divisor classes div(A), where A is finitely generated, forms a group. In other words, is it possible for T to be a Prüfer v-multiplicative ring?

If T is a Prüfer v-multiplicative ring, then Lemma 4.41 readily shows that D is. To see the converse we use the fact that an integral domain D is a Prüfer v-multiplicative ring if and only if D_P is a valuation ring for each maximal t-ideal P of D. A t-ideal is a union of finite v-ideals [7, p. 17]. If P is a prime ideal of T such that $P \cap D = 0$, then $T_P \supseteq K[X]$ and T_P is a valuation ring. Therefore, we show that if P is a maximal t-ideal of T such that $P \cap D \neq 0$, then $P = Q + XK[X]$ such that Q is a maximal t-ideal of D. This fact and the assumption that D is a Prüfer v-multiplicative ring yield the conclusion that T_P is a local overring of the Bezout domain $D_Q + XK[X]$, and is therefore a valuation ring.

Suppose, then, that $P = Q + XK[X]$ is a maximal t-ideal of T, where $Q \neq 0$. Lemma 4.41 shows that Q is a t-ideal of D. If Q is not a maximal t-ideal of D, then there is a t-ideal of D such that $Q' \supset Q$. But then $P' = Q' + XK[X]$ is a t-ideal of T such that $P' \supset P$. In summary we have the following result.

THEOREM 4.43. *$T = D + XK[X]$ is a Prüfer v-multiplication ring if and only if D is.*

4.5 *The Group of Divisibility of T*

If R is an integral domain with quotient field L, then the group of divisibility $V_L(R)$ of R is the group $L^*/U(R)$, partially ordered by $R^*/U(R)$, where $R^* = R\backslash\{0\}$ and $U(R)$ is the group of units of R. If G and H are partially ordered groups, let $G \oplus_L H$ and $G \oplus_C H$ denote the lexicographic and cardinal sum of G and H, respectively. In like manner, let $\sum^* G_\alpha$ denote the cardinal sum of the family of partially ordered groups G_α.

Let S be the subset of T consisting of all polynomials $f(X) \in R$ such that $f(0) \in U(D)$. Then S is a multiplicative system and $T_S = D + XK[X]_{(X)}$. Moreover, S is generated by the prime elements $f(X)$, where $f(X)$ is irreducible in $K[X]$ and $f(0) \in U(D)$. By results of [10], the group divisibility of T_S is the lexicogrphic sum of the group of divisibility of D and the group of divisibility of $K[X]_{(X)}$. That is, $V_{K(X)}(T_S) = V_K(D) \oplus_L Z$. Moreover, the prime elements satisfy the UF-property described in [10]. Therefore, the group of divisibility of T is the cardinal sum of the subgroup generated by the prime elements of S and $V_{K(X)}(T_S)$. Let us summarize:

THEOREM 4.51. *The group of divisibility of T is $\sum^* Z_f \otimes_C (V_K(D) \oplus_L Z)$, where Z_f denotes a copy of Z for each irreducible $f(X)$ in S.*

It follows then that T inherits many properties from D that are characterized by the group of divisibiltity.

In particular, we could have concluded, without Theorem 1.1, that T is a GCD-domain if and only if D is a GCD-domain.

4.6 *Elementary Divisor Domains*

We gave the definition of EDD in the Introduction and we shall not repeat it here, but one well-known fact that we use in this section should be mentioned. A Bezout domain D is EDD if and only if for each triplet $a, b, c \in D$ such $(a, b, c)\, D = D$, there exist $p, q \in D$ such that $(pa, pb + qc)\, D = D$.

THEOREM 4.61. *$T = D + XK[X]$ is an EDD if and only if D is EDD.*

Proof. It is easily shown that T an EDD implies that D is EDD. The converse is much more difficult.

Suppose D is EDD. Then D is Bezout and, therefore, T is Bezout. Consequently, it is sufficient to show that any matrix $\begin{bmatrix} a(X) & b(X) \\ 0 & d(X) \end{bmatrix}$, where $(a(X), b(X), d(X))\, T = T$, is equivalent (over T) to a diagonal matrix. First let us simplify the problem by establishing the following lemma.

LEMMA 4.62. *If $A = \begin{bmatrix} a(X) & b(X) \\ c(X) & d(X) \end{bmatrix}$ is a 2×2-matrix over T, where* GCD$(a(X), b(X), c(X), d(X)) = 1$, *then A is equivalent to a matrix $\begin{bmatrix} a & b'(X) \\ c'(X) & d'(X) \end{bmatrix}$, where $a \in D$.*

Two things to keep in mind are the following basic reductions:

(1) In a Bezout domain $[a\ b]\begin{bmatrix} p & s \\ q & r \end{bmatrix} = [d\ 0]$ where $d =$ GCD(a, b), $d = pa + qb$, $a = dr$, and $b = -ds$. Similarly, $\begin{bmatrix} p & q \\ s & r \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$. Since $pr - qs = 1$, $[a, b]$ is equivalent to $[d\ 0]$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ is equivalent to $\begin{bmatrix} d \\ 0 \end{bmatrix}$.

(2) A matrix $\begin{bmatrix} a(X) & b(X) \\ 0 & d(X) \end{bmatrix}$ where $a(X) \neq 0$, $b(X) \neq 0$, $0 \neq d(X) \in T$ is equivalent to $\begin{bmatrix} a(X) & b'(X) \\ 0 & d(X) \end{bmatrix}$ where degree $b'(X) \leqslant \min\{$degree $a(X)$, degree $d(X)\}$. For if $b(X) = b_0 + \cdots b_n X^n$, $a(X) = a_0 + \cdots a_m X^m$ where $n > m$, then $\begin{bmatrix} a(X) & b(X) \\ 0 & d(X) \end{bmatrix}\begin{bmatrix} 1 & -b_n/a_m X^{n-m} \\ 0 & \end{bmatrix} = \begin{bmatrix} a(X) & b'(X) \\ 0 & d(X) \end{bmatrix}$, where degree $b'(X) \leqslant n - 1$, Repeating, we can get degree $b'(X) \leqslant$ degree $a(X)$. Carrying out a similar procedure, we can get degree $b'(X) \leqslant$ degree $d(X)$.

Now we proceed to prove the lemma by induction on the degree of $a(X)$. Suppose the lemma is true for all such matrices where the degree of the entry in the first row and first column has degree less than that of $a(X)$.

By (1) above we could replace $a(X)$ by the GCD of $a(X)$ and $c(X)$, the GCD of $a(X)$ and $b(X)$, or by the GCD of $b(X)$ and $d(X)$ (after interchanging columns). If any of these have smaller degree, we use the inductive hypothesis. Thus, we may assume that A is equivalent to $\left[\begin{smallmatrix} a(X) & b(X) \\ 0 & d(X) \end{smallmatrix}\right]$, where GCD($a(X)$, $b(X)$) and $d^*(X) = \text{GCD}(b(X), d(X))$), and both have the same degree as $a(X)$. By (2) above, we may assume that degree $b(X) =$ degree $a^*(X) =$ degree $a(X) =$ degree $d^*(X)$. Therefore, $a(X) = aa^*(X)$, $b(X) = ba^*(X) = b'd^*(X)$, where $a, b, b' \in D$. But since $a(X)$, $b(X)$, and $d(X)$ are relatively prime in T, $a^*(X)$ and $d^*(X)$ are relatively prime in T and hence in $K[X]$. But since $ba^*(X) = b'd^*(X)$, this can be true only if $a^*(X)$ and $d^*(X)$ have degree zero. Therefore, the lemma is proved.

We return to the proof of the theorem.

By applying the lemma, and (1) and (2) above, we see that $A = \left[\begin{smallmatrix} a(X) & b(X) \\ 0 & d(X) \end{smallmatrix}\right]$ is equivalent to a matrix $B = \left[\begin{smallmatrix} a & b \\ 0 & g(X) \end{smallmatrix}\right]$ where $a, b \in D$. We show that B can be diagonalized using the fact that the matrix $\left[\begin{smallmatrix} a & b \\ 0 & g_0 \end{smallmatrix}\right]$, where g_0 is the constant term of $g(X)$, can be diagonalized over D. Since $(a, b, g(X))\, T = T$, $(a, b, g_0)\, D = D$ and there are elements $p, q \in D$ such that $(pa, pb + qg_0)\, D = D$. We consider the ideal $(pa, pb + qg(X))\, T$. Since T is a Bezout domain this ideal is principal and is generated by $h(X)$. There are elements $r, s \in D$ such that $1 = par + (pb + qg_0)\, s$, hence,

$$\begin{aligned} par + (pb + qg(X))\, s &= par + (pb + qg_0)\, s + qs(g(X) - g_0) \\ &= 1 + qs(g(X) - g_0) = h(X)\, t(X), \end{aligned}$$

where $t(X) \in T$. Thus, the constant term h_0 of $h(X)$ is a unit of D. But since $h(X)$ divides $pa \in D$, the degree of $h(X)$ is zero. Hence $(h(X))\, T = T$ and the theorem is proved.

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