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1980
Annali di Matematica pura ed applicata
(IV), Vol. CXXIII, pp. 331-355

Topologically Defined Classes of Commutative Rings (*) (**).

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Sunto. – *In questo lavoro viene studiata l'operazione di somma amalgamata* [13] *di spazi spettrali* [24] *e vengono esaminate in dettaglio alcune proprietà algebriche degli anelli che intervengono in tale operazione. Dei risultati ottenuti vengono poi fornite numerose applicazioni alla teoria dei « $D + \mathfrak{m}$ » domini di* GILMER [19], *a quella della seminormalizzazione di* TRAVERSO [38] *e a quella delle CPI-estensioni nel senso di* BOISEN-SHELDON [5].

0. – Introduction.

Recently, several Authors have investigated problems relevant to commutative unitary rings, dealing with topological methods and motivations (cf. for instance [5], [8], [9], [18], [24], [27], [28], [29], [31], [32], [35]). In the present paper, we demonstrate various general results concerning the operation of « attaching of spectral spaces » and the algebraic structure of the rings intervening in such an operation, taking into consideration several different applications, principally to the theory of « $D + \mathfrak{m}$ » domains introduced by GILMER [19], to that of the CPI-extensions in the sense of BOISEN-SHELDON [5] and to that of TRAVERSO's seminormalization [38].

More precisely, in the first section we demonstrate practical results of comparison between the closure of a subset of the prime spectrum of a ring in Zariski's topology and its closure in the constructible (or patch) topology, and we apply the said results to the study of the amalgamated sum of two spectral spaces. FERRAND [17] (and, marginally, ANANTHARAMAN [1]) has also dealt with this study, but with different motivations and applications to the problem of finite non-flat descent of schemes. In Section 2, we apply the techniques and results of the preceeding section to build up spectral spaces « attaching » (cf. [13]) a spectral space which has only one minimal point to another spectral space which has only one maximal (= closed) point, over such a closed point. This construction, expounded from an algebraic point of view, generalizes that concerning the « $D + \mathfrak{m}$ » domains [19] [21] and permits the putting in evidence and, hence, the elimination of the hypotheses and the unnatural restrictions submitted in the algebraic case. Some results stated in this section, relevant essentially to the topological and ordering properties of these spaces, are easily deduced

(*) Entrato in Redazione il 15 novembre 1978.

(**) Work performed under the auspices of C.N.R. (Gruppo Nazionale per le Strutture Algebriche, Geometriche e loro Applicazioni).

from those of Section 1 and they generalize considerably analogous results, proven by algebraic means, concerning the « $D + \mathfrak{m}$ » domains (GILMER [19], PAPICK [32], KIKUCHI [26]). Further results of this paragraph, relevant to the transfer and noetherianity properties, permit, in particular, a widening of knowledge of such domains.

In Section 3, we start with the construction of a spectral space by attaching a spectral space, having only a finite number of maximal points, with a spectral space having a unique minimal point, by amalgamating these points. Such a construction, from an algebraic point of view, is closely linked to that of « $S = K + \mathfrak{j}$ » rings (cf. NAGATA [30, E 2.1 p. 204]), K being a local ring, R a semi-local ring dominating K and $\mathfrak{j}$ Jacobson's radical of R. An algebraic-topological treatment of the rings arising from this topological-spectral construction is developed on the basis of the results of Section 1. Furthermore, we give several examples in order to illustrate, by iterating the « glueing & amalgamating » process, how to build up spectral spaces with pre-determinated topological and ordering peculiarities. Therefore, this type of construction comes in handy when in pursuit of examples and counter-examples relevant to the problem, stated by LEWIS [27] and founded on previous results proven by KAPLANSKY [25] and HOCHSTER [24], of characterizing the partially ordered sets isomorphically equivalent (as partially ordered sets) to a prime spectrum of a ring, endowed with the partial ordering associated with Zariski's topology (or, equivalently, determined by $\subseteq$) (cf. also [8], [28]). Making use in the « local case » of the construction examined in this paragraph, we show some notable algebraic-topological applications of it to the process of « glueing over pre-fixed points » and to the seminormality (cf. TRAVERSO [38]) and we outline the connections with the problem of « glueing prime ideals » (cf. PEDRINI [33]).

In the last section, we show how the problem, stated by BOISEN and SHELDON [5], of finding an overring of a given domain D having Pospec (= prime spectrum endowed only with the partial-ordering structure defined by $\subseteq$) order-isomorphic to the subset of Pospec (D) consisting of all prime ideals of D comparable to a fixed ideal, can be easily studied and solved, making use of the techniques introduced in the present paper. After having preliminarily recovered the principal results relevant to the CPI-extensions [5], we supply further results for this theory, expecially with regard to the problem of characterizing, making use only of the relation $\subseteq$, the Pospec of the CPI-extension of a domain with respect to a non-prime ideal. Finally, this paragraph concludes with several results concerning the transfer of properties resorting essentially to topological and ordering properties, and referring particularly to some classes of G-domains [18] [29] [36] and GD-domains [32].

1. – A topologically defined ring-theoretic operation.

Let A be a ring, we denote with X^{Zar}[resp. X^{Const}] the prime spectrum of the ring A, $X = \mathrm{Spec}\,(A)$, endowed with the Zariski topology [resp. with the constructible (1) topology]. For every subset Y of X, we denote with $\mathrm{Ad}_{\mathrm{Zar}}(Y)$ [resp. $\mathrm{Ad}_{\mathrm{Const}}(Y)$] the closure of Y in X^{Zar} [resp. X^{Const}], and we set

$$^{\mathrm{gen}}Y = \{x \in X \mid x \text{ is a generization } (^2) \text{ in } X^{\mathrm{Zar}} \text{ of a point of } Y\}$$
$$^{\mathrm{sp}}Y = \{x \in X \mid x \text{ is a specialization } (^2) \text{ in } X^{\mathrm{Zar}} \text{ of a point of } Y\}\,.$$

It is well known that the set X of all prime ideals of A can also be viewed simply as a *poset* (i.e. partially ordered set) with respect to the set-theoretic inclusion. Afterwards, we shall use the locution *the Pospec of A*, or just Pospec (A), to refer to this partially ordered set.

It is easily seen that the identity map $\mathrm{id}_X\colon X^{\mathrm{Const}} \to X^{\mathrm{Zar}}$ is a continuous map, therefore, for every $Y \subseteq X$, $\mathrm{Ad}_{\mathrm{Const}}(Y) \subseteq \mathrm{Ad}_{\mathrm{Zar}}(Y)$. Furthermore, every closed of X^{Zar} being stable for specializations, for every $Y \subseteq X$ it happens that $^{\mathrm{sp}}Y \subseteq \mathrm{Ad}_{\mathrm{Zar}}(Y)$.

(1.1) LEMMA. – *Let A be a ring and let $X = \mathrm{Spec}\,(A)$. For every subset $Y \subseteq X$ it happens that:*

$$\mathrm{Ad}_{\mathrm{Zar}}(Y) = {}^{\mathrm{sp}}(\mathrm{Ad}_{\mathrm{Const}}(Y))\,.$$

PROOF. – From the remarks of the beginning of the present section, we deduce that $^{\mathrm{sp}}(\mathrm{Ad}_{\mathrm{Const}}(Y)) \subseteq \mathrm{Ad}_{\mathrm{Zar}}(Y)$. On the other hand, if $x \in \mathrm{Ad}_{\mathrm{Zar}}\big({}^{\mathrm{sp}}(\mathrm{Ad}_{\mathrm{Const}}(Y))\big)$, then, for every fundamental open set $D(f)$ of X^{Zar}, $D(f) \ni x$, we have $D(f) \cap {}^{\mathrm{sp}}(\mathrm{Ad}_{\mathrm{Const}}(Y)) \neq \emptyset$, hence $D(f) \cap \mathrm{Ad}_{\mathrm{Const}}(Y) \neq \emptyset$, because every open set of X^{Zar} is stable for generizations. In the compact space X^{Const} [23] [31], $\mathrm{Ad}_{\mathrm{Const}}(Y)$ and $D(f)$ are closed sets, therefore:

$$\emptyset \neq \bigcap_{\substack{f \in A \\ D(f) \ni x}} (D(f) \cap \mathrm{Ad}_{\mathrm{Const}}(Y)) = \Big(\bigcap_{\substack{f \in A \\ D(f) \ni x}} D(f)\Big) \cap \mathrm{Ad}_{\mathrm{Const}}(Y)\,.$$

Being $\bigcap_{\substack{f \in A \\ D(f) \ni x}} D(f) = {}^{\mathrm{gen}}\{x\}$, we conclude that $x \in {}^{\mathrm{sp}}(\mathrm{Ad}_{\mathrm{Const}}(Y))$.

(1.2) REMARK. – In general, for every subset Y of X, the following inclusion holds:

$$\mathrm{Ad}_{\mathrm{Const}}({}^{\mathrm{sp}}Y) \subseteq {}^{\mathrm{sp}}(\mathrm{Ad}_{\mathrm{Const}}(Y))\,.$$

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(1) The *constructible topology* [23] [3, p. 48] or *patch topology* [24] on X is that topology having, as subbase of closed sets, all the closed sets of X^{Zar} and all the quasi-compact open sets of X^{Zar}. Hence, a subbase of closed sets of X^{Const} is given by $\{V(f) \mid f \in A\} \cup \{D(f) \mid f \in A\}$. It is easily seen that a subset Y of X is closed in X^{Const} if, and only if, there exists a ring-homomorphism $\varphi\colon A \to B$ such that $Y = {}^a\varphi(\mathrm{Spec}\,(B))$.

(2) We say that $x \in X$ is a *generization* [resp. *specialisation*] of a point $y \in X$ if $y \in \mathrm{Ad}_{\mathrm{Zar}}(x)$ [resp. $x \in \mathrm{Ad}_{\mathrm{Zar}}(y)$]; cf. [23, 0.2.1.2].

In fact, it is straightforward that $\mathrm{Ad}_{\mathrm{Zar}}(Y) = \mathrm{Ad}_{\mathrm{Zar}}({}^{sp}Y) = {}^{sp}\big(\mathrm{Ad}_{\mathrm{Const}}({}^{sp}(Y))\big)$, from which the above inclusion follows. We note that the strict inclusion can occur, as the following example shows.

(1.3) EXAMPLE. – Let $X = \mathrm{Spec}\,(\boldsymbol{Z})$, let Y be a non-finite proper subset of $\mathrm{Max}\,(\boldsymbol{Z})$ and let ξ be the generic point of X. We know that X^{Const} is the Alexandroff compactification of the discrete space $\mathrm{Max}\,(\boldsymbol{Z})$ [18, Par. 2 Lemme et Eq. 1], hence $\mathrm{Ad}_{\mathrm{Const}}(Y) = Y \cup \{\xi\}$ and therefore ${}^{sp}(\mathrm{Ad}_{\mathrm{Const}}(Y)) = X$. Whereas, ${}^{sp}Y = Y$ and then $\mathrm{Ad}_{\mathrm{Const}}({}^{sp}Y) = Y \cup \{\xi\}$.

Let $u: A \to C$ be a ring-homomorphism and $v: B \to C$ a surjective ring-homomorphism. We denote by D the pull-back of A and B over C (i.e. $D = A \times_C B = \{(a, b) \in A \times B \mid u(a) = v(b)\}$), and by $u': D \to B$, $v': D \to A$ the restrictions to D of the canonical projections. Let $X = \mathrm{Spec}\,(A)$, $Y = \mathrm{Spec}\,(B)$, $Z = \mathrm{Spec}\,(C)$, $W = \mathrm{Spec}\,(D)$, $\alpha = {}^a u: Z \to X$, $\beta = {}^a v: Z \to Y$, $\alpha' = {}^a u': Y \to W$, $\beta' = {}^a v': X \to W$. We get the following commutative diagrams:

$$\begin{tikzcd} D \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B \arrow[r, "v"'] & C \end{tikzcd} \qquad \begin{tikzcd} W & X \arrow[l, "\beta'"'] \\ Y \arrow[u, "\alpha'"] & Z \arrow[l, hook', "\beta"] \arrow[u, "\alpha"'] \end{tikzcd}$$

The map $\beta: Z \hookrightarrow Y$ being a closed embedding, we identify Z with its image in Y, in order to simplify the notations.

(1.4) THEOREM. – *With the foregoing notations and hypotheses, let $X \cup_\alpha Y$ be the topological space obtained by attaching X to Y, over the closed set Z, by the continuous map α* (3). *Then, $X \cup_\alpha Y$ is a spectral space* [24] *homeomorphic to* $\mathrm{Spec}\,(D)$ (4).

PROOF. – From the definition of D itself, we deduce immediately that:

(*a*) *v' is a surjective homomorphism (and, therefore, β' is a closed embedding*; we identify for greater convenience X with its image in W under β').

(*b*) *Let $\mathfrak{b} = \mathrm{Ker}\,(v)$ and $\mathfrak{d} = \mathrm{Ker}\,(v')$, then $u'|_{\mathfrak{d}}: \mathfrak{d} \to \mathfrak{b}$ is an isomorphism of modules (subordinate to $u': D \to B$). Therefore, the conductor* (5) *of u' contains $\mathfrak{d}$ and, hence, it is easily seen that, for every $h \in \mathfrak{d}$, the canonical homomorphism $D_h \to B_{u'(h)}$ is an isomorphism* (cf. also [6, Ch. 5 Par. 1 Ex. 16]).

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(3) $X \cup_\alpha Y$ is the quotient space of the disjoint union of X and Y, modulo the equivalence relation generated by: $\alpha(z) \sim z$, for each $z \in Z$ [13; Ch. 6, 6.1].

(4) We notice that a similar statement is contained in the unpublished paper [17]. We give here a simplified proof of it, making use of Lemma (1.1).

(5) The *conductor of a ring-homomorphism* $f: A \to B$ is, by definition, the ideal $\mathfrak{f} = \mathrm{Ann}_A(\mathrm{Coker}\,(f)) = \mathrm{Ann}_A(B/\mathrm{Im}\,(f))$.

From the last assertion of the statement (b) we deduce, in particular, the other following assertions:

(c) *For every prime ideal* $\mathfrak{p}$ *of* D, $\mathfrak{p} \not\supseteq \mathfrak{d}$, *if* $\mathfrak{q}$ *is the unique prime ideal of* B *such that* $u'^{-1}(\mathfrak{q}) = \mathfrak{p}$, *then* $\mathfrak{q} \not\supseteq \mathfrak{b}$ *and* $B_{\mathfrak{q}} \cong D_{\mathfrak{p}}$.

(d) *The map* $\alpha' \colon Y \to W$ *restricted to* $Y \setminus Z = \alpha'^{-1}(W \setminus X)$ *establishes a scheme-isomorphism* (and hence, in particular, an homeomorphism between topological spaces and an order-isomorphism between partially ordered sets) *with* $W \setminus X$ (we notice that $X \cong V(\mathfrak{d})$ and $\alpha'^{-1}(X) \cong V(\mathfrak{b}) \cong Z$).

The equality $\beta' \circ \alpha = \alpha' \circ \beta$ allows us to affirm that:

(e) *There exists a unique continuous map* $\sigma \colon X \cup_\alpha Y \to W$ *which commutes the following diagram*:

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, hook, "\beta'"] & \\
X \cup_\alpha Y \arrow[r, "\sigma"] & W \\
Y \arrow[u] \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

From the statements (a) and (d) it follows that:

(f) $\sigma \colon X \cup_\alpha Y \to W$ *is a bijective map*; *therefore, in particular* $W = X \cup \alpha'(Y)$.

To conclude, that is, to show that σ is an homeomorphism, it is sufficient to prove that, if F is a subset of W such that $\alpha'^{-1}(F)$ is a closed set of Y and $\beta'^{-1}(F)$ is closed set of X, then F is a closed set of W. By applying Lemma (1.1) we obtain that F is a closed set if, and only if, $F = {}^{sp}(\mathrm{Ad}_{\mathrm{Const}}(F))$. We remark that $\mathrm{Ad}_{\mathrm{Const}}(F) = F$ (see Note ([1])), in fact, if $\alpha'^{-1}(F) = V(\mathfrak{s})$ and if $\beta'^{-1}(F) = V(\mathfrak{r})$ then F is the image, under ${}^a h$, of the spectrum of the D-algebra $h \colon D \to A/\mathfrak{r} \times B/\mathfrak{s}$, $x \mapsto (v'(x) + \mathfrak{r}, u'(x) + \mathfrak{s})$. Now, the conclusion follows immediately, because F is stable for specializations, so being $F \cap X$ and $F \cap (W \setminus X)$.

(1.5) COROLLARY. – *We preserve the notations and hypotheses of the beginning of this section and of the preceding theorem* (1.4).

(1) *The map* $\mathfrak{a} \mapsto v'^{-1}(\mathfrak{a})$ *establishes an isomorphism between the lattice of all the ideals of* A *and that of all the ideals of* D *containing* $\mathfrak{d}$. *This map defines, by restriction, an isomorphism between* Pospec (A) *and the partially ordered subset of* Pospec (D) *which consists of all the prime ideals of* D *containing* $\mathfrak{d}$ (*this isomorphism, obviously, coincides with the one which can be deduced from the closed embedding* $\beta' \colon$ Spec (A) $\to$ $\to$ Spec (D)).

(2) *For every prime ideal* $\mathfrak{q}$ *of* B, $\mathfrak{q} \not\supseteq \mathfrak{b}$, *the map* $\mathfrak{h} \mapsto u'^{-1}(\mathfrak{h})$ *establishes a bijection, which preserves the inclusion, between the set of all the ideals of* B *which are primary for* $\mathfrak{q}$ *and the set of all the ideals of* D *which are primary for* $\mathfrak{p} = u'^{-1}(\mathfrak{q})(\not\supseteq \mathfrak{b})$.

(3) *The map defined in the statement* (2), *by restriction to the prime ideals, determines the isomorphism* $\mathrm{Pospec}\,(B)\diagdown V(\mathfrak{b}) \xrightarrow{\sim} \mathrm{Pospec}\,(D)\diagdown V(\mathfrak{b})$ *described formerly* (1.4(*d*)).

(4) *If* $u: A \to C$ *is injective* [resp. *surjective, of finite type, integral, finite*], *then* $u': D \to B$ *is injective* [resp. *surjective, of finite type, integral, finite*].

(5) *If* A' *is the integral closure of* A *in* C, *then* $D' = A' \times_C B$ *is the integral closure of* D *in* B.

(6) *If* u *is an injective homomorphism and if* $\mathfrak{b}$ *is a regular ideal of* B, *then* $\mathrm{Tot}\,(B) \simeq \mathrm{Tot}\,(D)$ (*where* $\mathrm{Tot}\,(-)$ *denote the total ring of fractions of the ring* $-$).

(7) *If* u *is an injective homomorphism and if* B *is an integral domain, then* D *is an integral domain with the same field of quotients as* B, *and in this field* B *and* D *have the same complete integral closure.*

(8) *If* C *is the field of quotients of a normal domain* A *and if* B *is a normal domain hen* D *is a normal domain.*

Proof. – For (1), (2), (3), (4) apply (1.4). (5) follows from (4) and [6, Ch. 5 p. 15]. The verifications of (6), (7) and (8) are straightforward (cf. also [19, 22.5]).

(1.6) Corollary. – *We preserve the notations and hypotheses of the beginning of this section.* W *and* Z *are noetherian spaces, if and only if,* X *and* Y *are noetherian spaces.*

Proof. – Apply (1.4) and [6; Ch. 2 Par. 4, N. 2].

(1.7) Remark. – It is not true, in general, that if A and B are noetherian rings then $A \times_C B$ is a noetherian ring. For instance, if $k \subsetneq K$ are two algebraically closed fields, if $A = k$, $B = K[T]$, $C = K$, if $v: K[T] \to K$ is the canonical surjective homomorphism $T \mapsto 0$ and, finally, if u is the inclusion $k \subsetneq K$, then the ring $k \times_K K[T]$ is not a noetherian ring. In fact, the ideal $(\{(0, aT) | a \in K\})$ is not finitely generated.

(1.8) Proposition. – *We preserve the notations and hypotheses of the beginning of this section.* $A \times_C B$ *and* C *are noetherian rings,* u' *is a finite homomorphism if, and only if,* A *and* B *are noetherian rings and* u *is a finite homomorphism.*

Proof. – We suppose that $A \times_C B$ is a noetherian ring and that u' is a finite homomorphism. Then, A is noetherian, as a quotient ring of $A \times_C B$ and B is also noetherian, because u' is of finite type [3, Cor. 7.7]. Furthermore, $u: A \times_C B/\mathfrak{b} \to B/\mathfrak{b}$ is necessarily finite. Conversely, keeping in mind the statement (1.5(4)), it suffices to show that $A\times_C B$ is noetherian and, to this end, it is enough to verify that $\mathfrak{b} = \operatorname{Ker}(v')$ is an ideal of finite type. But this follows easily, since $\mathfrak{b}$ is an ideal of finite type of B and u' is a finite homomorphism (cf. (1.4(b)) and [23; **0**. 6.4.8]).

(1.9) PROPOSITION. – *We preserve the notations and hypotheses of the beginning of the present section. If S is a multiplicatively closed set in the ring D, then indicating $S_A = v'(S)$, $S_B = u'(S)$, $S_C = u\circ v'(S) = v\circ u'(S)$, we obtain that*

$$S^{-1}D \cong S_A^{-1}A\times_{S_C^{-1}C} S_B^{-1}B\,.$$

Conversely, if S_A is a multiplicatively closed set of A and if S_B is a multiplicatively closed set of B and if $u(S_A) = v(S_B) = S_C$, then $S_A^{-1}A\times_{S_C^{-1}C} S_B^{-1}B \cong (S_A\times_{S_C} S_B)^{-1}D$.

PROOF. – The verifications are straightforward.

2. – Application to the « $D+\mathfrak{m}$ » constructions and to the composition of valuation rings.

As we have already mentioned in the introduction, in this section we apply the techniques and results of the preceding section to build up spectral spaces by « amalgamating » a spectral space having a unique minimal point, with a spectral space having only a unique maximal (= closed) point, over this closed point. The principal applications of such a construction concern the $D+\mathfrak{m}$ domains [19], [21]. Particular attention is devoted to the examination of the transfer properties, expecially with regard to some classes of G-domains [18], [25], [29], [36] and GD-domains [11], [12], [32]. On this subject, after having explicitly shown many different possibilities of construction of spectral spaces, we prove that the Artin-Tate theorem, concerning the noetherian G-domains [10] [25], may in no wise be extended to the case of G-domains with noetherian spectrum. We remark, among other things, that the costruction of a valuation ring by composition given by Nagata [30, p. 35] is included, as a very particular case, in the one examined in this section.

Let $(V, \mathfrak{m}, k(V))$ be a local ring, let D be a subdomain of $k(V)$ and let K be the quotient field of D. We now consider the following diagram:

$$\begin{array}{ccc} D_1 = D\times_{k(V)} V & \xrightarrow{v'} & D \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ V & \xrightarrow{v} & k(V) \end{array}$$

We quote $X = \operatorname{Spec}(D)$, $Y = \operatorname{Spec}(V)$, $P = \operatorname{Spec}(k(V))$, $X_1 = \operatorname{Spec}(D_1)$ and we denote by y the closed point of Y, image of P under $\beta = {}^a v: P \to Y$, by x the generic point of X, image of P under $\alpha = {}^a u: P \to X$, and by z the point of X_1 image of P under $\gamma = {}^a(v\circ u') = {}^a(v'\circ u): P \to X_1$. Let $\alpha' = {}^a u': Y \to X_1$ and $\beta' = {}^a v': X \to X_1$.

(2.1) PROPOSITION. – *With the notations and hypotheses of the beginning of this section, we have:*

(1) *X_1 is a topological space homeomorphic to $X \cup_z Y$; $X_1 \setminus X$ is a scheme isomorphic to $Y \setminus \{y\}$ under α' restricted to $Y \setminus \{y\}$.*

(2) *The closed embedding $\beta' : X = \mathrm{Spec}\,(D) \hookrightarrow X_1 = \mathrm{Spec}\,(D_1)$ has the image equal to ${}^{sp}\{z\}$ and it establishes an order-monomorphism of* Pospec (D) *into* Pospec (D_1).

(3) *The continuous map $\alpha' : Y = \mathrm{Spec}\,(V) \to X_1 = \mathrm{Spec}\,(D_1)$ is injective, has the image equal to ${}^{gen}\{z\}$ and establishes an order-monomorphism of* Pospec (V) *into* Pospec (D_1).

(4) *Every prime ideal of X_1 is comparable with z, more precisely $X_1 = {}^{sp}\{z\} \cup {}^{gen}\{z\}$.*

(5) $\dim\,(D_1) = \dim\,(D) + \dim\,(V)$.

(6) Pospec (D_1) *is a tree if, and only if,* Pospec (D) *is a tree and* Pospec (V) *is a totally ordered set.*

(7) Spec (D_1) *is a noetherian space if, and only if,* Spec (D) *and* Spec (V) *are noetherian spaces.*

In the particular case in which $D = K$ is a field, we have the following commutative diagram:

$$\begin{tikzcd} \tilde{V} = K \times_{k(V)} V \arrow[r, "\tilde{v}"] \arrow[d, hook, "\tilde{j}"'] & K \arrow[d, hook, "j"] \\ V \arrow[r, "v"] & k(V) \end{tikzcd}$$

therefore:

(8) $\iota = {}^a\tilde{j} : Y = \mathrm{Spec}\,(V) \to \tilde{Y} = \mathrm{Spec}\,(\tilde{V})$ *is a homeomorphism and, hence, it establishes an isomorphism between* Pospec (V) *and* Pospec $(\tilde{V})$.

(9) *$\tilde{V}$ is local ring.*

(10) *If $\tilde{y}$ is the unique closed point of $\tilde{Y}$, then $\iota|_{Y \setminus \{y\}} : Y \setminus \{y\} \to \tilde{Y} \setminus \{\tilde{y}\}$ is a scheme-theoretic isomorphism.*

PROOF. – Apply (1.4), (1.5) and (1.6).

(2.2) PROPOSITION. – *We preserve the notations and hypotheses of the beginning of this section and of Proposition* (2.1).

(1) *For every prime ideal $\mathfrak{p} \supseteq \mathfrak{p}_z$ of the ring D_1, denoting by $\mathfrak{q}$ the unique prime ideal of D which corresponds to $\mathfrak{p}$* (2.1(2)), *we have: $D_1/\mathfrak{p} \cong D/\mathfrak{q}$. In particular if $\mathfrak{p} = \mathfrak{p}_z$, then $D_1/\mathfrak{p}_z \cong D$.*

(2) *The map* $\mathfrak{b} \mapsto v'^{-1}(\mathfrak{b})$ *establishes an isomorphism between the lattice of all the ideals of* D *and that of all the ideals of* D_1 *containing* $\mathfrak{p}_z$.

(3) *We suppose now that* $K = k(V)$. *For every prime ideal* $\mathfrak{p} \subseteq \mathfrak{p}_z$ *of* D_1, *denoting by* $\mathfrak{q}$ *the unique prime ideal of* V *which corresponds to* $\mathfrak{p}$, *we have*: $(D_1)_\mathfrak{p} \simeq V_\mathfrak{q}$. *In particular*, $(D_1)_{\mathfrak{p}_z} \simeq V$.

(4) *We suppose that* $D = K \subsetneq k(V)$. *For every prime ideal* $\mathfrak{q}$ *of* V, *we denote by* $\tilde{\mathfrak{q}}$ *the prime ideal* $\mathfrak{q} \cap \tilde{V}$. *If* $\mathfrak{q} \neq \mathfrak{m}$, *then* $V_\mathfrak{q} \cong \tilde{V}_{\tilde{\mathfrak{q}}}$.

(5) *In the general case*, $D \subseteq K \subseteq k(V)$, *for every prime ideal* $\mathfrak{q}$ *of* V, *if* $\tilde{\mathfrak{q}}$ *is the unique prime ideal of* $\tilde{V}$ *associated with the prime ideal* $\mathfrak{q}$ *of* V, *corresponding to the prime ideal* $\mathfrak{p} \subsetneq \mathfrak{p}_z$ *of* D_1, *then we have*: $(D_1)_\mathfrak{p} \simeq \tilde{V}_{\tilde{\mathfrak{q}}} \simeq V_\mathfrak{q}$. *When* $\mathfrak{p} = \mathfrak{p}_z$, $(D_1)_{\mathfrak{p}_z} \cong \tilde{V} = K \times_{k(V)} V$.

(6) *If* $\mathfrak{p}$ *is a prime ideal of* D_1 *containing the ideal* $\mathfrak{p}_z$ *and if* $\mathfrak{q} = \mathfrak{p}/\mathfrak{p}_z$ *is the unique prime ideal of* D *corresponding to* $\mathfrak{p}$, *then* $(D_1)_\mathfrak{p} \simeq D_\mathfrak{q} \times_{k(V)} V$.

(7) *The map* $\mathfrak{a} \mapsto u'^{-1}(\mathfrak{a})$ *establishes a bijection, which preserves the ordering given by* $\subseteq$, *between the set of all the* $\mathfrak{q}$-*primary ideals of* V, $\mathfrak{q}$ *being an arbitrary prime ideal of* V, $\mathfrak{q} \neq \mathfrak{m}$, *and the set of all the* $\mathfrak{p}$-*primary ideals of* D_1, *being* $\mathfrak{p} = \mathfrak{q} \cap D_1 \subsetneq \mathfrak{p}_z$. *When* $K = k(V)$ *such a bijection holds also in the case* $\mathfrak{q} = \mathfrak{m}$.

(8) *If* $D \subsetneq k(V)$, *then the conductor* $\mathfrak{f}$ *of* $u' : D_1 \hookrightarrow V$ *coincides with* $\mathfrak{p}_z$.

(9) D_1 *is an integral domain if, and only if,* V *is an integral domain. In this case,* D_1 *and* V *have the same complete integral closure in their common field of quotients.*

(10) *If* D' *is the integral closure of* D *in* $k(V)$, *then* $D_1' = D' \times_{k(V)} V$ *is the integral closure of* D_1 *in* V. *If* V *is a valuation ring, then* D_1' *is the integral closure of* D_1 *in its field of quotients.*

PROOF. – For (1) and (2) cf. (1.4(*a*)) and (1.5(1)). (3) and (4) follow from (1.4(*c*)), (1.9) and (2.1(1), (10)). (5) ensues from (3), (4) and (1.4(*c*)). (6) is a particular case of (1.9); in fact, if $S = D_1 \setminus \mathfrak{p}$, with $\mathfrak{p} \supseteq \mathfrak{p}_z$, then $u'(S) \subseteq u'(D_1 \setminus \mathfrak{p}_z) \subseteq V \setminus \mathfrak{m}$. For (7) cf. (1.5(2)). (8): we already know that $\mathfrak{p}_z \subseteq \mathfrak{f}$ (1.4(*b*)). On the other hand, V being a local ring, every element of $V \setminus \mathfrak{m}$ is a unit of V, therefore if $x \in D_1 \setminus \mathfrak{p}_z$, then $x \notin \mathfrak{f}$, because otherwise, $\mathfrak{f}$ would be equal to the ideal (1) and hence, V would be equal to D_1, that is $D = K = k(V)$. The statement (9) is a particular case of the statements (1.5(6), (7)). The first part of the statement (10) follows from (1.5(5)); the second part can be proven using an argument quite similar to the one used to establish the assertion (1.5(8)).

(2.3) THEOREM. – *We preserve the notations of the beginning of this section.* D_1 *is a noetherian domain if, and only if,* V *is a noetherian domain,* $D = K$ *and* $[k(V) : K] < \infty$.

PROOF. – From what is already known (cf. (1.8)), it suffices to show that the homomorphism $u' : D_1 \to V$ is finite in order to conclude, passing to the quotient rings, that u is finite, $D = K$ is a field and $[k(V):K] < \infty$. Now, if $\mathfrak{p}_2$ is a D_1-module of finite type, then $\mathfrak{m} = \mathfrak{p}_2 V$ is a D_1-module of finite type (1.4(b)). Furthermore, D_1 and V have the same field of quotients (2.2(9)) and the conductor of $D_1 \hookrightarrow V$ is the ideal $\mathfrak{p}_2$, which is a non-zero ideal, hence V is a D_1-module of finite type (cf. also [25, Ch. 1 Ex. 41 (a) p. 46]).

(2.4) THEOREM. – *We preserve the notations of the beginning of this section. We suppose that $k(V)$ is the field of quotients of D.*

(1) *D_1 is a valuation ring if, and only if, V and D are valuation rings.*

(2) *D_1 is a discrete valuation ring if, and only if, V is a discrete valuation ring and $D = k(V)$* (i.e. $D_1 = V$).

(3) *D_1 is a Prüfer domain if, and only if, D and V are Prüfer domains.*

(4) *D_1 is a S-domain* ([1]) *if, and only if, D and V are S-domains.*

PROOF. – (1) It is clear that if D_1 is a valuation ring, then $V = (D_1)_{\mathfrak{p}_2}$ and $D = D_1/\mathfrak{p}_2$ are also valuation rings. For the converse the verification is straightforward [30, p. 35]. (2) follows from (1) and from (2.3). (3) ensues from (1) and (2.2(6)), bearing in mind that a Prüfer domain is characterized by having all its localizations at prime ideals equal to valuation rings. (4) is a simple consequence of (2.2(2) and (7)).

(2.5) REMARK. – If the field of quotients K of D is isomorphic to a proper subfield of $k(V)$, by imposing on V and D the properties of the type enounced above (cf. (2.4)), we cannot conclude, in general, that the same property holds for D_1, as the following example (concerning the statements (1)-(3)) shows.

(2.6) EXAMPLE. – If $V = K[\![T]\!]$, $D = k$, k being a proper subfield of K, and if $v: K[\![T]\!] \to K$, $T \mapsto 0$, then $D_1 = k \times_K K[\![T]\!]$ is isomorphic to the subring $k + TK[\![T]\!]$ of $K[\![T]\!]$, which is not a valuation ring in its field of quotients $K(\!(T)\!)$.

As regards the S-domains, the difficulties mentioned in (2.5) rise since, in general, we cannot describe the behaviour of the $\mathfrak{m}$-primary ideals when we pass from V to D_1. On this subject, we recall that a prime ideal $\mathfrak{p}$ of a commutative ring is called a *branched prime*, when there exists at least one $\mathfrak{p}$-primary ideal $\mathfrak{h}$, $\mathfrak{h} \neq \mathfrak{p}$; otherwise, $\mathfrak{p}$ is called an *unbranched prime*. If $\mathfrak{m}$ is an unbranched prime ideal, we can strengthen the statement (2.4(4)) in the following manner; *if D and V are S-domains and $\mathfrak{m}$ is unbranched in V then D_1 is an S-domain and $\mathfrak{p}_2$ is unbranched* (cf. 2.2(7)). Therefore, as a simple and direct application of the topological techniques of Section 1, we have reobtained, as a particular case, the principal results concerning the S-domains proven in [26] using algebraic methods.

([1]) Cf. [20] and [26].

(2.7) THEOREM. – *We preserve the hypotheses and notations of the beginning of the present section.*

(*a*) *D_1 is a G-domain* ([2]) $\Leftrightarrow$ *V is a G-domain.*

(*b*) *D_1 is a Goldman ring* ([3]) $\Leftrightarrow$ *V and D are Goldman rings.*

(*c*) *D_1 is a g-ring* [resp. *a locally pqr domain*] ([4]) $\Leftrightarrow$ *V and D are g-rings* [resp. *locally pqr domains*].

If we suppose that $k(V)$ is the field of quotients of D, then:

(*d*) *D_1 is a strong G-domain* ([5]) $\Leftrightarrow$ *V and D are strong G-domains.*

(*e*) *D_1 is an i-domain* ([6]) $\Rightarrow$ *V and D are i-domains.*

(*f*) *D_1 is an open domain* [resp. *a propen domain*] ([6]) $\Rightarrow$ *V and D are open domains* [resp. *D is an open domain and V is a propen domain*].

(*g*) *D_1 is a GD-domain* ([7]) $\Rightarrow$ *V and D are GD-domains.*

(*h*) *If V is a valuation ring the statements* (*e*), (*f*) *and* (*g*) *can be inverted.*

PROOF. – (*a*). It suffices to remark that the generic point of X_1 is open if, and only if, the generic point of Y is open. (*b*). It is easily seen that X_1 is a T_D topological space ([8]) if, and only if, X and Y are T_D topological spaces. (*c*). It is straightforward that X_1 is an Alexandroff-discrete topological space ([9]) if, and only if, X and Y are Alexan-

---

([2]) A *G-domain* D is an integral domain such that its field of quotients is a finitely generated algebra over D [25, 1.3]. Topologically, a reduced ring is a G-domain if, and only if, ts prime spectrum is an irriducible space and the generic point is open [25, Th. 18] [18, Lemma 2].

([3]) A *Goldman ring* A is a ring such that every prime ideal $\mathfrak{p}$ is a *G-ideal* (i.e. $A/\mathfrak{p}$ is a G-domain) [18] [35]. A reduced ring is a Golman ring if, and only if, its prime spectrum is a T_D topological space [18, Prop. 1]; cf. also the following Note ([8]).

([4]) A *g-ring* A [35, Prop. 6] [18, Par. 3] is a ring such that for every prime ideal $\mathfrak{p}$ of A there exists $f \in A \setminus \mathfrak{p}$ in such a way that $A_{\mathfrak{p}} \cong A_f$. A reduced ring A is a g-ring if, and only if, Spec (A) is an Alexandroff-discrete topological space; cf. also the following Note ([9]). An integral g-ring is called in [36] a *locally pqr domain*.

([5]) A *strong G-domain* A [36] is an integral domain such that every overring of A is equal to A_f for some $f \in A$.

([6]) An *i-domain* [resp. *open domain*] A, with K as a field of quotients, is an integral domain such that for every overring B, $A \subseteq B \subseteq K$, the map Spec $(B) \to$ Spec (A) is injective [resp. open]. A *propen domain* A is an integral domain, such that the map Spec $(B) \to$ Spec (A) is open for every overring B, $A \subseteq B \subsetneq K$; [32].

([7]) A *GD-domain* A is an integral domain such that the «going-down» property holds for every overring B of A [11], [12], [32].

([8]) A *T_D-space* X (cf. [4, 3.1], [9, Par. 6], [18]) is a topological space such that for every point $x \in X$, the derived set $\{x\}'$ is a closed set.

([9]) An *Alexandroff-discrete topological space* X is a T_0-space such that, for every subspace $Y \subseteq X$, the following property holds:

$$Ad_X(Y) = \bigcup_{y \in Y} Ad_X(y)$$

(cf. [4, Par. 5], [9, Par. 6], [18, Par. 1]).

droff-discrete topological spaces. (d). It is well known that a strong G-domain is simply a locally pqr Prüfer domain [36, Th. 3.5]. The conclusion follows from (c) and (2.4(3)). The statements (e), (f) and (g) follow easily from (2.1(2), (3), (4)) and (2.2(1), (3)). (h). The statement (e) reverses, because D_1 is an i-domain if, and only if $D_1 \subset D_1'$ is an i-extension, D_1' being the integral closure of D_1 in its fields of quotients, and D_1' is a Prüfer domain [32, 2.13]. In fact, in the present situation $D_1' = D' \times_{k(V)} V$ where D' is the integral closure of D' in $k(V)$ (2.2(10)). The statement (g) can be inverted: it is well-known that an integral domain A is a GD-domain, if and only if, $A \subset A[u]$ satisfies GD for every u in the field of quotients of A [12]. Therefore, if $u \notin V$, then $u^{-1} \in \mathfrak{m}$, hence $D_1[u] = D_1[(u^{-1})^{-1}] = (D_1)_{u^{-1}}$. The reversibility of ($f$) now follows from [32, 3.2 and 5.1].

(2.8) PROPOSITION. – *If $(V, \mathfrak{m}, k(V))$ is a valuation ring containing, as subring, a field k isomorphic to $k(V)$, then there exists an isomorphism $\sigma: D + \mathfrak{m} \xrightarrow{\sim} D_1$ which commutes the following diagram:*

$$\begin{tikzcd} D + \mathfrak{m} \arrow[r, "\sigma", "\sim"'] \arrow[d, hook] & D_1 \arrow[d, hook] \\ k + \mathfrak{m} \arrow[r, phantom, "="] & V \end{tikzcd}$$

PROOF. – For the universality property of D_1, there exists a unique homomorphism σ which commutes the following diagram:

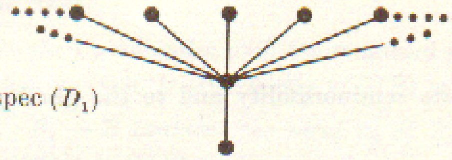
$$\begin{tikzcd} & D \\ D + \mathfrak{m} \arrow[ur, "v''"] \arrow[r, dashed, "\sigma"] \arrow[d, hook, "u''"] & D_1 \arrow[u, "v'"'] \arrow[d, hook, "u'"] \\ k + \mathfrak{m} \arrow[r, phantom, "="] & V \end{tikzcd}$$

where $u''(x + y) = u(x) + y$, $v''(x + y) = x$ for all $x \in D$, $y \in \mathfrak{m}$. The verification that σ a bijective map is straightforward.

The Proposition (2.8) shows how the results of the present section, deduced in a natural and direct way from the main statements of the previous section, generalize some of those which GILMER [19, Th. A p. 560], [21] and PAPICK [32; 3.27, 3.28, 5.26] have proven for the $D + \mathfrak{m}$ domains, further widening the knowledge about this theory.

(2.9) EXAMPLE. – Let k be an arbitrary non-finite field. Set $K = k(X)$, $D = k[X]$, $V = K[Y]_{(Y)}$. Let $u: D = k[X] \hookrightarrow k(X)$, $v: V \twoheadrightarrow k(V) \cong K$ be the canonical homomorphisms. The Pospec of the domain $D_1 = D \times_K V \cong k[X] + Yk(X)[Y]_{(Y)} \subsetneq k(X, Y)$ can be represented by the following diagram:

Pospec (D_1)



(2.10) EXAMPLE. – Let k, K, V be as in (2.9). Set $D = k[X]_{(X)}$ and let $u: D \hookrightarrow K$, $v: V \twoheadrightarrow k(V) \cong K$ be the canonical homomorphisms. The Pospec of the domain $D_2 = D \times_K V \cong k[X]_{(X)} + Yk(X)[Y]_{(Y)} \subsetneq k(X, Y)$ can be represented by the following diagram:

Pospec (D_2)

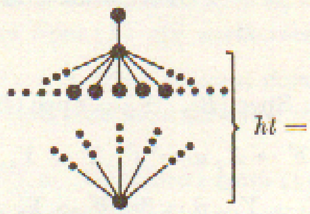
We note that D_1 and D_2 are non-noetherian rings, even if Spec (D_1) and Spec (D_2) are noetherian topological spaces (2.1(7)). Furthermore, D_1 and D_2 are G-domains (2.7(a)) and D_2 is a valuation ring. These examples show that none of the conditions of the Artin-Tate Theorem (i.e. semi-locality and dim≤ 1), which characterize the noetherian G-domains [10, Th. 2.5], holds in the case of G-domains with noetherian spectra.

(2.11) EXAMPLE. – Let k be an arbitrary field. Set

$$K = k(X)\,, \quad D = k[X]_{(X)}\,, \quad V = K[Y_1, Y_2, \ldots, Y_n]_{(Y_1, Y_2, \ldots, Y_n)}\,.$$

Let $u: D \hookrightarrow K$ $v: V \twoheadrightarrow K$ be the canonical homomorphisms. The Pospec of the domain $D_3 = D \times_K V \subsetneq k(X)(Y_1, Y_2, \ldots, Y_n)$ can be represented by a diagram of the following type:

Pospec (D_3)



In general, we can take $K = k(X_i | i \in I)$, $D = S^{-1}k[X_i | i \in I]$, $V = K[Y_j | j \in J]_{\mathfrak{m}}$, where I and J are arbitrary sets of indexes, S is a multiplicative set of $k[X_i | i \in I]$

and $\mathfrak{m}$ is a maximal ideal of $K[Y_j|j \in J]$ such that $k(\mathfrak{m}) \simeq K$. The properties of the Pospec of the domain $D_4 = D \times_K V$ are easily deduced from those of Pospec (D) and Pospec (V).

3. – Applications to the seminormality and to the glueing of prime ideals.

In this section, we intend to build up a spectral space by « glueing & amalgamating » a spectral space having a finite number of maximal (= closed) points with another having only one minimal point and we develop an algebraic-topological study of the rings intervening in such a construction. Then, we apply this study to recover, in a natural and direct way, the main properties of the seminormalized of a ring [38] and to enlarge the knowledge of this theory, especially in connection with the glueing process over points of a spectral space. Furthermore, we explicitly construct several examples to show how the « glueing & amalgamating » process comes in handy in studying spectral sets [27], [28].

First at all, we show that the « $A + \mathfrak{J}(B)$ » rings (similar to the « $D + \mathfrak{m}$ » domains, cf. § 2) introduced by NAGATA [30, E.2.1 p. 204] and revided later by LEWIS [27] can be easily obtained and studied with the methods introduced in Par. 1.

Let $(A, \mathfrak{m}, k)$ be a local ring and B a semi-local ring such that, for every maximal ideal $\mathfrak{m}_i$, the field $K_i = k(\mathfrak{m}_i) = B_{\mathfrak{m}_i}/\mathfrak{m}_i B_{\mathfrak{m}_i}$ is an extension of the field k, $i = 1, 2, \ldots, r$. Let $j: k \hookrightarrow \prod_{i=1}^{r} K_i$ be the canonical monomorphism. In the following commutative diagram all the homomorphisms are canonical:

$$\begin{tikzcd}
B_2 = A \times_{\prod K_i} B \arrow[r, "v''"] \arrow[d, "h'"'] \arrow[dd, bend right=60, "u'"'] & A \arrow[d, "h"] \arrow[dd, bend left=60, "u"] \\
B_1 = k \times_{\prod K_i} B \arrow[r, "v'"] \arrow[d, hook, "j'"'] & k \arrow[d, hook, "j"] \\
B \arrow[r, "v"] & \prod_{i=1}^{r} K_i
\end{tikzcd}$$

Set $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $Y_i = \operatorname{Spec}(B_i)$, $i = 1, 2$, $P = \operatorname{Spec}(k)$, $F = \operatorname{Spec}\left(\prod_{i=1}^{r} K_i\right)$, $\alpha = {}^a u: F \to X$, $\alpha' = {}^a u': Y \to Y_2$, $\gamma = {}^a h: P \to X$, $\gamma' = {}^a h': Y_1 \to Y_2$, $\varepsilon = {}^a j: F \to P$, $\varepsilon' = {}^a j': Y \to Y_1$, $\beta = {}^a v: F \to Y$, $\beta' = {}^a v': P \to Y_1$, $\beta'' = {}^a v'': X \to Y_2$. We denote by y_1 [resp. y_2, x] the image of P into Y_1 [resp. Y_2, X]. We identify $F = \{z_1, z_2, \ldots, z_r\}$ [resp. X] with its image into Y [resp. Y_2] under the closed embedding β [resp. β''].

(3.1) PROPOSITION. – *We preserve the notations introduced above.*

(1) $Y_1^2 \simeq P \cup_e Y$; $Y_2 \simeq X \cup_a Y \simeq X \cup_v Y_1$.

(2) B_1 *and* B_2 *are local rings.*

(3) *The conductor of* B *into* B_1 *coincides with the maximal ideal* $\mathfrak{n}_1$ *of* B_1 *(kernel of the homomorphism* v'*), which, in turn, is isomorphic to* $\mathfrak{J}(B)$ *under* j'.

(4) *The conductor of* $u': B_2 \to B$ *contains the ideal* $\mathfrak{n}_2$ *of* B_2 *(kernel of the homomorphism* v''*), which is isomorphic to* $\mathfrak{J}(B)$ *under* u'.

(5) $Y_1 \setminus \{y_1\}$ *is scheme-theoretically isomorphic to* $Y \setminus F$.

(6) *For every prime ideal* $\mathfrak{q}$ *of* B, $\mathfrak{q} \neq \mathfrak{m}_i$ *for all* $i = 1, 2, \ldots, r$, *the map* $\mathfrak{h} \mapsto \mathfrak{h} \cap B_1$ *establishes a bijection, which preserves the inclusion* $\subseteq$, *between the set of all the* $\mathfrak{q}$*-primary ideals of* B *and the set of all* $\mathfrak{p}$*-primary ideals of* B_1, *where* $\mathfrak{p} = \mathfrak{q} \cap B \subsetneqq \mathfrak{n}_1$. *Hence, in particular, by restriction to the set of all the prime ideals, this map establishes an isomorphism between* Pospec $(B) \setminus F$ *and* Pospec $(B_1) \setminus \{y_1\}$.

(7) $Y_2 \setminus X$ *is scheme-theoretically isomorphic to* $Y \setminus F$.

(8) *For every prime ideal* $\mathfrak{q}$ *of* B, $\mathfrak{q} \neq \mathfrak{m}_i$ *for all* $i = 1, 2, \ldots, r$, *the map* $\mathfrak{h} \mapsto u'^{-1}(\mathfrak{h})$ *establishes a bijection, which preserves the inclusion* $\subseteq$, *between the set of all the* $\mathfrak{q}$*-primary ideals of* B, *and the set of all* $\mathfrak{p}$*-primary of* B_2, *where* $\mathfrak{p} = u'^{-1}(\mathfrak{q})$, $\mathfrak{p} \not\supseteq \mathfrak{n}_2$. *In particular, by restriction to the set of all the prime ideals, this map defines an isomorphism between* Pospec $(B) \setminus F$ *and* Pospec $(B_2) \setminus X$.

(9) *If* B *is a* k*-algebra* (¹), *then* B_1 *is a* k*-algebra; more exactly,* $B_1 \simeq k + \mathfrak{J}(B)$.

(10) *If* B *dominates* (²) A, *then* B_1 *and* B_2 *dominate* A; *more exactly,* $B_1 = A + \mathfrak{J}(B)$ *and* B_2 *is a* A*-ring trivially augmented onto* A, *with* $\mathfrak{J}(B)$ *as augmentation ideal* [23, **0**.16.1].

(11) B_1 [resp. B_2] *is a noetherian ring and* B *is a* B_1*-algebra* [resp. B_2*-algebra*] *of finite type if, and only if,* B *is a noetherian ring* [resp. A *and* B *are noetherian rings*] *and* $[K_i : k] < \infty$ *when* $i = 1, 2, \ldots, r$.

(12) B *is a finite* B_1*-algebra (and also a finite* B_2*-algebra) if, and only if,* $[K_i : k] < \infty$, $i = 1, 2, \ldots, r$.

(13) Spec (B_1) [resp. Spec (B_2)] *is a noetherian space if, and only if,* Spec (B) *is a noetherian space* [resp. Spec (A) *and* Spec (B) *are noetherian spaces*].

(14) *If* B *is an integral domain then* B_1 *is an integral domain and* B *and* B_1 *have the same field of quotients, and in this field they have the same complete integral closure.*

PROOF. – (1) is a particular case of (1.4). (2) follows from (1). (3) and (4) are particular cases of (1.4(b)). (5) and (7) are particular cases of the statement (1.4(d)),

(¹) We mean, obviously, that $k \hookrightarrow B \xrightarrow{v} \prod K_i$ coincides with j.
(²) We mean that $A \subseteq B \xrightarrow{v} \prod K_i$ coincides with u.

whereas (6) and (8) are particular cases of the statements (1.5(2), (3)). (9) and (10): for the universality property of B_1 [resp. B_2], there exists a unique homomorphism $k \hookrightarrow B_1$ [resp. $A \hookrightarrow B_2$] which commutes the following diagrams:

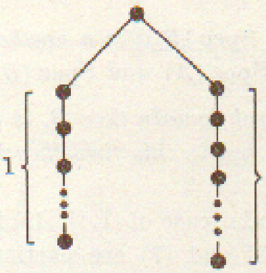
$$
\begin{tikzcd}
 & k \arrow[dr, hook, "j"] & \\
k \arrow[ur, equal] \arrow[r, hook, dashed] \arrow[dr, hook] & B_1 \arrow[u, "v'"] \arrow[d, "j'"] & \prod K_i \\
 & B \arrow[ur, "v"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[r, "h"] & k \arrow[dr, hook, "j"] & \\
{[A} \arrow[ur, equal] \arrow[r, hook, dashed] \arrow[ddr, hook] & B_2 \arrow[u, "v''"] \arrow[d, "h'"] & & {\prod K_i]} \\
 & B_1 \arrow[d, hook, "j'"] & & \\
 & B \arrow[uurr] & &
\end{tikzcd}
$$

If we identify k [resp. A] with its image into B and B_1 [resp. into B_1 and B_2] and $\mathfrak{n}_1$ [resp. $\mathfrak{n}_1$ and $\mathfrak{n}_2$] with $\mathfrak{F}(B)$, then we have: $B_1 \supseteq k + \mathfrak{F}(B)$ [resp. $B_1 \supseteq A + \mathfrak{F}(B)$ and $B_2 \supseteq A + \mathfrak{F}(B)$]. Furthermore, if $x \in B_1 \setminus \mathfrak{F}(B)$ then $v'(x) \in k \subset B_1$ [resp. if $x \in B_1 \setminus \mathfrak{F}(B)$, then $v'(x) = a + \mathfrak{m}$ with $a \in A$; if $y \in B_2 \setminus \mathfrak{F}(B)$, then $v''(y) \in A$] hence $x - v'(x) \in \mathfrak{n}_1 \cong \mathfrak{F}(B)$ [resp. $x - a \in \mathfrak{n}_1 \cong \mathfrak{F}(B)$; $y - v''(y) \in \mathfrak{n}_2 \cong \mathfrak{F}(B)$]. The statements (11) and (12) follow from (1.8) and (1.5(4)), bearing in mind that, in this case, the following affirmations are equivalent: (i) j is a homomorphism of finite type; (ii) j is a finite homomorphism; (iii) $[K_i : k] < \infty$, for all $i = 1, 2, \ldots, r$. (13) follows from (1.6) and (14) from (1.5(7)).

Proposition (3.1) provides a technique easy to visualize in order to build up new spectral spaces and spectral sets (³) by the « glueing & amalgamating » process.

(3.2) EXAMPLE. – If V is a valuation ring of rank n and W is a valuation ring of rank m and if $k(V) = k(W) = k$, then $V \times_k W = A$ is a ring, having a noetherian spectrum, and the following picture describes its Pospec:

Pospec (A) $\quad ht = n - 1$

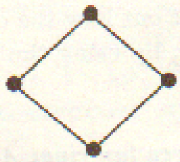


$ht = m - 1$

(³) A partially ordered set is termed *spectral set* if it is isomorphic to the Pospec of some ring [27], [28].

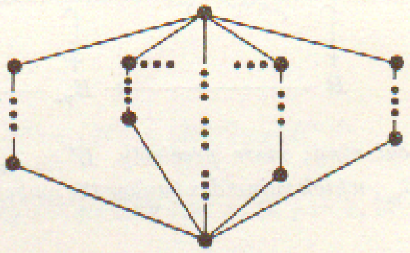
Explicitly, when $n = m = 1$, we can take as V and W two valuation rings of dimension 1 [resp. two discrete valuation rings] having the same residue field k. The ring $B = V \times_k W$ has a noetherian spectrum [resp. is noetherian]. When $n = m = 2$, we can take as V the ring associated with the valuation of the field $K = k(X, Y)$ (k being an arbitrary field) with $\boldsymbol{Z} \oplus \boldsymbol{Z}$ (lexicographically ordered) as value group, defined by the map $f_v \colon X \mapsto (1, 0)$, $Y \mapsto (0, 1)$, and as W the ring associated with the valuation (of the same field and with the same value group) defined by the map $f_W \colon X \mapsto (0, 1)$, $Y \mapsto (1, 0)$ (cf. [19, Par. 15], [22, Ex. 4.3], [26, Ex. 1]). It is easy to show that $k(V) = k(W) = k$. Set $C = V \times_k W$. If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the maximal ideals of the semilocal domain $D = V \cap W$, it is easy to show that $\boldsymbol{k}(\mathfrak{m}_1) = \boldsymbol{k}(\mathfrak{m}_2) = k$, then the domain $E = k \times_{k(\mathfrak{m}_1) \times (\mathfrak{m}_2)} D \cong k + \mathfrak{J}(D)$ has a Pospec of the following type:

Pospec (E)



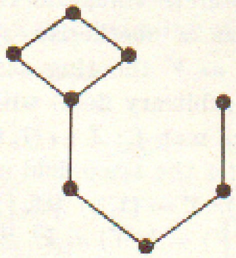
More generally, let $\{V_i \mid i = 1, 2, \ldots, r\}$ be a family of valuation rings of the same field, not two of which are comparable. If we suppose that every residue field $k(V_i)$ is an extension of one fixed field k ($i = 1, 2, \ldots, r$), then the domain $F = k \times_{\prod_i k(V_i)} (\bigcap_i V_i)$ has a Pospec which can be represented by a diagram of the following type:

Pospec (F)



Obviously if we fix the field of quotients K of the domain E (or D, or V, or W), then we can find some rings A', B', C', D', E' having the Pospec isomorphic-respectively-to that of A, B, C, D, E, and coefficient field $k' = K$ (the choice of the coefficient field being quite arbitrary). Therefore, we can go on glueing & amalgamating. For instance, if D' and E are integral domains with the Pospec described above and if $k' = \boldsymbol{k}(\mathfrak{m}_1) = \boldsymbol{k}(\mathfrak{m}_2) = K$, then the Pospec of $G = E \times_{k(\mathfrak{m}_1)} D'$ has the following form:

Pospec (G)



The preceding examples give an idea of how to use the glueing & amalgamating process and suggest how to build up many other spectral sets with pre-established properties.

We purpose now to point out how the concept of *seminormality* [38] can be studied in a natural and direct way, by using the topological-algebraic techniques introduced above.

(3.3) PROPOSITION. – *Given two rings $A \subseteq B$, we suppose that A is a noetherian ring, B is A-finite (hence, noetherian). For every $x \in X = \operatorname{Spec}(A)$, we denote by B'^x the subring of B obtained by B glueing over* (4) *x. Let $\Phi_x = \{y_1, y_2, \ldots, y_n\}$ be the fiber over x of the canonical map $Y = \operatorname{Spec}(B) \to X = \operatorname{Spec}(A)$. Set $F_x = \prod_{i=1}^{n} \boldsymbol{k}(y_i)$, $B_{\mathfrak{p}_x} = A_{\mathfrak{p}_x} \otimes_A B$, $A_{\mathfrak{p}_x}$ being the localization at the prime $x = \mathfrak{p}_x$, and let $u_x \colon \boldsymbol{k}(x) \hookrightarrow F_x$ be the canonical monomorphism. We have a commutative diagram of the following type:*

$$\begin{tikzcd} B^x = L^x \times_{B_{\mathfrak{p}_x}} B \arrow[r] \arrow[d, hook] & L^x = \boldsymbol{k}(x) \times_{F_x} B_{\mathfrak{p}_x} \arrow[r, two heads] \arrow[d, hook] & \boldsymbol{k}(x) \arrow[d, hook, "u_x"] \\ B \arrow[r] & B_{\mathfrak{p}_x} \arrow[r, two heads] & F_x \end{tikzcd}$$

(1) *L^x is a local ring; more precisely, $L^x \simeq A_{\mathfrak{p}_x} + \mathfrak{J}(B_{\mathfrak{p}_x})$, where $\mathfrak{J}(B_{\mathfrak{p}_x})$ is the Jacobson radical of $B_{\mathfrak{p}_x}$, which coincides (under the embedding $L^x \hookrightarrow B_{\mathfrak{p}_x}$) with the maximal ideal of L^x.*

(2) *$B^x \simeq B'^x$.*

(3) *The ideal $\mathfrak{q}^x = \operatorname{Ker}(B^x \to \boldsymbol{k}(x))$ is the unique prime ideal of B^x such that $\mathfrak{q}^x \cap A = \mathfrak{p}_x$.*

---

(4) B'^x is the largest subring of B, $A \subseteq B'^x$, satisfying the following conditions:

(i) there exists a unique point $x' \in \operatorname{Spec}(B'^x)$ over x;

(ii) the canonical homomorphism $\boldsymbol{k}(x) \hookrightarrow \boldsymbol{k}(x')$ is an isomorphism [38, p. 588].

(4) *If* $\mathfrak{r}_i$ *is the prime ideal of* B *corresponding* (i.e. *logically, coinciding*) *with* $y_i \in Y$ $(i = 1, 2, \dots, n)$, *then the following diagram:*

$$\begin{tikzcd}
B^x \arrow[r, two heads] \arrow[d, hook] & B^x/\mathfrak{q}^x \arrow[d, hook] \\
B \arrow[r, two heads] & B/\bigcap_{i=1}^{n} \mathfrak{r}_i
\end{tikzcd}$$

is a cartesian diagram.

(5) *The conductor of* B *into* B^x *contains the prime ideal* $\mathfrak{q}^x$ *which coincides (under the embedding* $B^x \hookrightarrow B$*) with the ideal* $\bigcap_{i=1}^{n} \mathfrak{r}_i$.

(6) *For every* $f \in \mathfrak{p}_x \subset A$ *or, more generally, for every* $f \in \mathfrak{q}^x \subset B$, *the canonical homomorphism* $B_f^x \rightleftarrows B_f$ *is an isomorphism. Hence, if* Y^x *denotes the* Spec *of* B^x, $Y^x \setminus V(\mathfrak{q}^x)$ *is scheme-theoretically isomorphic to* $Y \setminus \mathrm{Ad}_Y(\Phi_x)$. *In particular, for every prime ideal* $\mathfrak{p} \not\supseteq \mathfrak{p}_x$ *of* A *the canonical homomorphism* $B_{\mathfrak{p}}^x \rightleftarrows B_{\mathfrak{p}}$ *is an isomorphism.*

(7) *Let* $x_1, x_2 \in X$ *and let* $\Phi_{x_i} = \{y_1^{(i)}, y_2^{(i)}, \dots, y_{n_i}^{(i)}\}$ *be the fiber over* x_i *of the map* $Y \to X$, $i = 1, 2$. *Set* $F_{x_i} = \prod_{j=1}^{n_i} \boldsymbol{k}(y_j^{(i)})$, $i = 1, 2$. *All the homomorphisms in the following commutative diagram:*

$$\begin{tikzcd}
B^{x_1,x_2} = L^{x_1,x_2} \times_{B_{\mathfrak{p}_{x_1}} \times B_{\mathfrak{p}_{x_2}}} B \arrow[r] \arrow[d, hook] & L^{x_1,x_2} = (\boldsymbol{k}(x_1) \times \boldsymbol{k}(x_2)) \times_{F_{x_1} \times F_{x_2}} (B_{\mathfrak{p}_{x_1}} \times B_{\mathfrak{p}_{x_2}}) \arrow[r, two heads] \arrow[d, hook] & \boldsymbol{k}(x_1) \times \boldsymbol{k}(x_2) \arrow[d, hook] \\
B \arrow[r] & B_{\mathfrak{p}_{x_1}} \times B_{\mathfrak{p}_{x_2}} \arrow[r, two heads] & F_{x_1} \times F_{x_2}
\end{tikzcd}$$

are canonical. Then:

$$B^{x_1,x_2} \cong B^{x_1} \times_B B^{x_2} \cong (B^{x_2})^{x_1} \cong (B^{x_1})^{x_2}$$

where $(B^{x_i})^{x_j}$ *is the ring obtained by* B^{x_i} *glueing over* x_j $(i, j = 1, 2)$.

(8) *If* A *is seminormal* (⁵) *in* B *then there exists a finite number of points* $x_1, x_2, \dots, x_t \in X$ *in such a way that* $B^{x_1,x_2,\dots,x_t}$ *is isomorphic to* A.

PROOF. – (1): cf. (3.1(2), (10)). (2). If we identify B^x with its image into B and $\boldsymbol{k}(x)$ with its image into $\boldsymbol{k}(y_i)$ $(i = 1, 2, \dots, n)$, then $B^x = \{b \in B \mid b(y_i) \in \boldsymbol{k}(x)$ and $b(y_i) = b(y_j)$ for all $i, j = 1, 2, \dots, n\}$. The conclusion follows from [38, 1.4]. (3). From

(⁵) Cf. [38].

the following commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & A_{\mathfrak{p}_x} \arrow[r] \arrow[d, hook] & \boldsymbol{k}(x) \arrow[d, equal] \\ B^x \arrow[r] & L^x \arrow[r] & \boldsymbol{k}(x) \end{tikzcd}$$

we deduce that $\mathfrak{q}^x = \operatorname{Ker}(B^x \to \boldsymbol{k}(x))$ is a prime ideal such that $\mathfrak{q}^x \cap A = \mathfrak{p}_x = \operatorname{Ker}(A \to \boldsymbol{k}(x))$. The uniqueness follows from (2). The verification of (4) is straightforward. (5) and (6) are particular cases of (1.4(b), (c), (d)), bearing in mind that $\mathrm{Ad}_Y(\Phi_x) = V(\bigcap_{i=1}^{n} \mathfrak{r}_i)$. The statement (7) can be deduced by a direct verification. (8) follows from (7) and [38, Th. 2.1].

(3.4) REMARK. – We note that, if A is an integral domain of dimension $\leqslant 1$ and B the field of quotients of A, then the seminormalization of A in B coincides with the weak normalization in the sense of ENDÔ [16, § 1, p. 341]. A construction quite similar to the seminormalization is the weak normalization introduced in [2], with reference to a problem of topological classification of algebraic varieties.

Another construction, having algebraic-geometrical motivations as well, which can also be treated with the techniques introduces above, is the process of *glucing of prime ideals* (cf. [33], [37]). More precisely, given a noetherian ring B, two prime ideals of it, $\mathfrak{p}_1, \mathfrak{p}_2 \subset B$, and a isomorphism $\varphi: B/\mathfrak{p}_1 \xrightarrow{\sim} B/\mathfrak{p}_2$ such that $\varphi(\mathfrak{p}_1 + \mathfrak{p}_2/\mathfrak{p}_1) = \mathfrak{p}_1 + \mathfrak{p}_2/\mathfrak{p}_2$ and $\varphi: B/\mathfrak{p}_1 + \mathfrak{p}_2 \to B/\mathfrak{p}_1 + \mathfrak{p}_2$ is the identity, then the problem consists in finding a subring A of B, which contains a prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} = \mathfrak{p}_1 \cap \mathfrak{p}_2$, and which can be obtained by glueing over $\mathfrak{p}$ (cf. Note (4)). From the preceeding results, it is easy to verify that the solution of this problem is given by the ring $\boldsymbol{k}(\mathfrak{p}_1) \times_{\boldsymbol{k}(\mathfrak{p}_1) \times \boldsymbol{k}(\mathfrak{p}_2)} B$, where the homomorphism $B \to \boldsymbol{k}(\mathfrak{p}_1) \times \boldsymbol{k}(\mathfrak{p}_2)$ is the composition of the canonical homomorphisms $B \to B_{\mathfrak{p}_1} \times B_{\mathfrak{p}_2}$, $B_{\mathfrak{p}_1} \times B_{\mathfrak{p}_2} \to \boldsymbol{k}(\mathfrak{p}_1) \times \boldsymbol{k}(\mathfrak{p}_2)$ and the homomorphism $\boldsymbol{k}(\mathfrak{p}_1) \to \boldsymbol{k}(\mathfrak{p}_1) \times \boldsymbol{k}(\mathfrak{p}_2)$ is the graph of the isomorphism $\boldsymbol{k}(\mathfrak{p}_1) \xrightarrow{\sim} \boldsymbol{k}(\mathfrak{p}_2)$ deduced from φ. Several geometrical examples are discussed in [33] and [37].

4. – Applications to the CPI-extensions.

This section is essentially devoted to showing how the theory of the CPI-extensions in the sense of BOISEN and SHELDON [5] is included, as a particular case, in the theory developed in Section 1. Further results, especially concerning properties of transfer, related to the topological and ordering structure of the prime spectrum, are given here so as to enlarge and point out knowledge of the theory of the CPI-extensions.

Let R be a ring and $\mathfrak{a}$ an ideal of R, set $A = R/\mathfrak{a}$. We denote by $S(\mathfrak{a})$ the multiplicatively closed subset of R complement in R of the set-theoretic union of the elements of the family $F(\mathfrak{a}) = \{\mathfrak{p}_i \mid i \in I\}$ of all the prime ideals of R, which, modulo $\mathfrak{a}$, consist of zero-divisors an $R/\mathfrak{a}$ [1]. Therefore, the ring $C = S(\mathfrak{a})^{-1}A$ is isomorphic to the total ring of fractions of A. Set $B = S(\mathfrak{a})^{-1}R$, $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $Z = \operatorname{Spec}(C)$. In the following diagram:

$$\begin{tikzcd}
R^{a} = A \times_C B \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
B \arrow[r, "v"] & C
\end{tikzcd}$$

all the homomorphisms are canonical. Set $T = \operatorname{Spec}(R)$, $T^{a} = \operatorname{Spec}(R^{a})$, $\alpha = {}^{a}u\colon Z \to X$, $\beta = {}^{a}v\colon Z \to Y$, $\alpha' = {}^{a}u'\colon Y \to T^{a}$, $\beta' = {}^{a}v'\colon X \to T^{a}$. By the universality property of R^{a}, there exists a unique homomorphism $w\colon R \to R^{a}$ which commutes the following diagram:

$$\begin{tikzcd}
 & A \\
R \arrow[ur, "w_A"] \arrow[r, dashed, "w"] \arrow[dr, "w_B"'] & R^{a} \arrow[u, "v'"'] \arrow[d, "u'"] \\
 & B
\end{tikzcd}$$

w_A and w_B being the canonical homomorphisms. Set $\gamma = {}^{a}w\colon T^{a} \to T$ $\gamma_A = {}^{a}w_A\colon X \to T$, $\gamma_B = {}^{a}w_B\colon Y \to T$, and denote by $V(\mathfrak{a})$ the image of X into T and by $\Delta(\mathfrak{a})$ the image of Y into T.

(4.1) PROPOSITION. – *We preserve the notations of the beginning of this section.*

(1) *$v'\colon R^{a} \to A$ is a surjective homomorphism (i.e. β' is a closed embedding).*

(2) *If $\mathfrak{b} = \operatorname{Ker}(v)$ and $\mathfrak{d} = \operatorname{Ker}(v')$, then $u'|_{\mathfrak{d}}\colon \mathfrak{d} \to \mathfrak{b}$ is a (module-theoretic) isomorphism (we notice that $\mathfrak{b} \simeq S(\mathfrak{a})^{-1}\mathfrak{a}$).*

(3) *$\mathfrak{d}$ is contained in the conductor of $u'\colon R^{a} \to B$ and, hence, for every $f \in \mathfrak{d}$, $R^{a}_f \simeq B_{u'(f)}$ and, for every prime ideal $\mathfrak{q}$ of R^{a}, $\mathfrak{q} \not\supseteq \mathfrak{d}$, $R^{a}_{\mathfrak{q}} \simeq B_{\mathfrak{q}} \simeq R_{\mathfrak{p}}$, where $\mathfrak{p} = \mathfrak{q} \cap R \not\supseteq \mathfrak{a}$, $\mathfrak{p} \subseteq \bigcup_{i \in I} \mathfrak{p}_i$.*

(4) *If we identify X with its image, $V(\mathfrak{d})$, into T^{a} under β', then $\alpha'\colon Y \to T^{a}$ restricted to $\alpha'^{-1}(T^{a} \setminus X) = Y \setminus Z$ establishes a scheme-theoretic isomorphism with $T^{a} \setminus X$.*

(5) *T^{a} is homeomorphic to $X \cup_{\alpha} Y$ or, which is the same, to $V(\mathfrak{a}) \cup_{\gamma_A \circ \alpha} \Delta(\mathfrak{a})$ and, hence, the image of the continuous injective map $T^{a} \to T$ coincides with $V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$ (which is, therefore, a closed set of T^{Const}).*

---

(1) If $\mathfrak{a} = \sqrt{\mathfrak{a}}$, then $F(\mathfrak{a})$ can be taken as equal to the family of all the minimal prime ideals containing $\mathfrak{a}$.

(6) $\Delta(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{p}$ *is a prime ideal of* R *such that* $\mathfrak{p} \subseteq \bigcup_{i \in I} \mathfrak{p}_i\}$.

(7) *If* T *is a noetherian space, then* $T^{\mathfrak{a}}$ *is a noetherian space.*

(8) *The map* $\mathfrak{h} \mapsto v'^{-1}(\mathfrak{h})$ *establishes an isomorphism between the lattice of all the ideals of* R *containing* $\mathfrak{a}$ *and that of all the ideals of* $R^{\mathfrak{a}}$ *containing* $\mathfrak{d}$.

(9) *For every prime ideal* $\mathfrak{p}$ *of* R, $\mathfrak{p} \subseteq \bigcup_{i \in I} \mathfrak{p}_i$, $\mathfrak{p} \not\supseteq \mathfrak{a}$, *the map* $\mathfrak{h} \mapsto u'^{-1}(\mathfrak{h})$ *induces a bijection between the set of all the* $\mathfrak{p}$*-primary ideals of* R *and the set of all* $\mathfrak{q}$*-primary ideals of* $R^{\mathfrak{a}}$, *where* $\mathfrak{q}$ *is the unique prime ideal of* $R^{\mathfrak{a}}$ *such that* $\mathfrak{q} \cap R = \mathfrak{p}$. *In particular, the injective map* $Y \hookrightarrow T^{\mathfrak{a}}$ *(i.e.* $\Delta(\mathfrak{a}) \subset T^{\mathfrak{a}}$*) preserves the ordering.*

(10) *If* $S(\mathfrak{d})$ *is the multiplicatively closed subset of* $R^{\mathfrak{a}}$ *complement in* $R^{\mathfrak{a}}$ *of* $\bigcup_{i \in I} \mathfrak{q}_i$, *with* $\mathfrak{q}_i \cap R = \mathfrak{p}_i$, *and if* $B' = S(\mathfrak{d})^{-1} R^{\mathfrak{a}}$ *then* $R^{\mathfrak{a}} \cong A \times_C B'$.

(11) *If* R *is an integral domain, then* $R^{\mathfrak{a}}$ *is an integral domain with the same field of quotients of* R.

(12) *There exists a unique homomorphism* $\mu\colon (1+\mathfrak{a})^{-1}R \to R^{\mathfrak{a}}$ *which commutes the following diagram:*

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "w"'] & (1+\mathfrak{a})^{-1}R \arrow[d, dashed, "\mu"] \\ & R^{\mathfrak{a}} \end{tikzcd}$$

The subspace of T *image of* $\operatorname{Spec}((1+\mathfrak{a})^{-1}R)$ *coincides with* ${}^{\mathrm{gen}}V(\mathfrak{a})$ *(which is, therefore, a closed subset of* T^{Const}*).*

(13) *If* R *is normal and if* $R/\mathfrak{a}$ *is integrally closed,* $R^{\mathfrak{a}}$ *is normal.*

(14) *If* $\mathfrak{a} = \mathfrak{p}$ *is a prime ideal of* R, *then* $\gamma\colon T^{\mathfrak{p}} \to T$ *establishes a homeomorphism of* $T^{\mathfrak{p}}$ *with its image:* $V(\mathfrak{p}) \cup \Delta(\mathfrak{p})$.

PROOF. – For (1)-(9) cf. (1.4), (1.5) and (1.6). (10). It is sufficient to remark that $S(\mathfrak{d})^{-1}R^{\mathfrak{a}}$ is isomorphic to $S(\mathfrak{a})^{-1}R$. For (11) cf. (1.5(7)). The first part of the statement (12) follows from the universality property of $R^{\mathfrak{a}}$. In fact, there exist two homomorphisms $\varphi\colon (1+\mathfrak{a})^{-1}R \to R/\mathfrak{a}$, $x/1+a \mapsto x+\mathfrak{a}$, $\psi\colon (1+\mathfrak{a})^{-1}R \to S(\mathfrak{a})^{-1}R$, $x/1+a \mapsto x/1+a$ ($S(\mathfrak{a})$ containing $1+\mathfrak{a}$), which commute the following diagram:

$$\begin{tikzcd} (1+\mathfrak{a})^{-1}R \arrow[r, "\varphi"] \arrow[d, "\psi"'] & R/\mathfrak{a} \arrow[d, "u"] \\ S(\mathfrak{a})^{-1}R \arrow[r, "v"'] & C \end{tikzcd}$$

Furthermore, $V(\mathfrak{a})$ is contained in the image of $\operatorname{Spec}((1+\mathfrak{a})^{-1}R)$ into $\operatorname{Spec}(R)$, hence ${}^{\mathrm{gen}}V(\mathfrak{a})$ is also contained in this image. On the other hand, if $\mathfrak{p}$ is a prime ideal such that $\mathfrak{p} \cap (1 + \mathfrak{a}) = \emptyset$ then $\mathfrak{p} + \mathfrak{a} \neq (1)$, hence $V(\mathfrak{p}) \cap V(\mathfrak{a}) = V(\mathfrak{p} + \mathfrak{a}) \neq \emptyset$. (13): cf. (1.5(5)). (14). In this case, $V(\mathfrak{p}) \cup_{\gamma_4 \circ \lambda} \Delta(\mathfrak{p}) \simeq V(\mathfrak{p}) \cup \Delta(\mathfrak{p})$, the closed set Z consisting of one point solely.

(4.2) REMARK. – (*a*) If R is an integral domain, then $R^{\mathfrak{a}}$ coincides with *the CPI-extension*, $C(R, \mathfrak{a})$, *of* R *with respect to* $\mathfrak{a}$ [5].

(*b*) In general, the continuous injective map $T^{\mathfrak{a}} \to T$ does not establish a homeomorphism of $T^{\mathfrak{a}}$ with its image. It is not difficult to build up some explicit examples for which $\gamma: T^{\mathfrak{a}} \to V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$ is neither a homeomorphism nor an isomorphism of ordered sets [5, Ex. 3.14]. Nevertheless, if $\mathfrak{a} = \mathfrak{p}$ is a prime ideal, $T^{\mathfrak{p}}$ is homeomorphic to ${}^{sp}\{\mathfrak{p}\} \cup {}^{gen}\{\mathfrak{p}\}$, but, in general, it is not isomorphic in the scheme-theoretic meaning [5, Ex. 2.11 and Prop. 2.12].

(4.3) LEMMA. – *We preserve the notations of the beginning of this section. The continuous bijective map* $\gamma: T^{\mathfrak{a}} \to V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$ *is a homeomorphism if, and only if, the canonical map* $V(\mathfrak{a}) \coprod \Delta(\mathfrak{a}) \to V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$ *is an identification* (²).

PROOF. – The statement follows easily from the theorem (VI.7.2) of [13], bearing in mind (4.1.(5)) and [13,VI.6.1].

(4.4) REMARK. – It is easy to show that if the family $F(\mathfrak{a})$ is finite, then $\gamma: T^{\mathfrak{a}} \to V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$ is a homeomorphism, $V(\mathfrak{a}) = {}^{sp}F(\mathfrak{a})$, $\Delta(\mathfrak{a}) = {}^{gen}F(\mathfrak{a})$ and $Z \simeq V(\mathfrak{a}) \cap \Delta(\mathfrak{a}) = F(\mathfrak{a})$. This result can be generalized in the following way:

(4.5) PROPOSITION. – *We preserve the notations of the beginning of this section. If the ring* $A = R/\mathfrak{a}$ *is quasi-regular* (³) *and if* $\mathfrak{p} \subseteq \bigcup_{i \in I} \mathfrak{p}_i \subset R$ *implies* $\mathfrak{p} \subseteq \mathfrak{p}_i$ *for some index* i *of* I, *then* $\gamma: T^{\mathfrak{a}} \to V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$ *is a homeomorphism,* $V(\mathfrak{a}) = {}^{sp}F(\mathfrak{a})$, $\Delta(\mathfrak{a}) = {}^{gen}F(\mathfrak{a})$ *and* $Z \cong V(\mathfrak{a}) \cap \Delta(\mathfrak{a}) = F(\mathfrak{a})$.

PROOF. – Since A is quasi-regular, the image of Z into X is homeomorphic to $F(\mathfrak{a})$ (which is a compact subspace of X), hence the image of Z into T, which coincides with $V(\mathfrak{a}) \cap \Delta(\mathfrak{a})$, is still homeomorphic to $F(\mathfrak{a})$. Since $F(\mathfrak{a}) = \mathrm{Ad}_{\mathrm{Const.}}(F(\mathfrak{a}))$, then $V(\mathfrak{a}) = {}^{sp}F(\mathfrak{a})$ (1.1). Furthermore, if $\mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p} \not\supseteq \mathfrak{a}$ and $\mathfrak{p} \subseteq \bigcup_{i \in I} \mathfrak{p}_i$, then $\mathfrak{p} \subseteq \mathfrak{p}_i$, for some $i \in I$. The conclusion follows from (4.1.(5)), because $T^{\mathfrak{a}} \simeq V(\mathfrak{a}) \cup_{\gamma_4 \circ \lambda} \Delta(\mathfrak{a}) \simeq V(\mathfrak{a}) \cup_j \Delta(\mathfrak{a})$, where j is the embedding of $V(\mathfrak{a}) \cap \Delta(\mathfrak{a})$ into $V(\mathfrak{a})$.

(²) A continuous surjective map $f: X \to Y$ is called an *identification map* whenever a subset V of Y is an open set if, and only if, $f^{-1}(V)$ is an open set of X [13, VI.1.3].

(³) A ring is called *quasi-regular* in the sense of Endô [15], if its total ring of fractions is regular (i.e. zero-dimensional and reduced).

(4.6) PROPOSITION. – *We preserve the notations of the beginning of this section. Let $\mathfrak{a}$ be an ideal of R, then*

(1) *R is a G-domain $\Rightarrow R^{\mathfrak{a}}$ is a G-domain.*

(2) *R is a Prüfer domain $\Rightarrow R^{\mathfrak{a}}$ is a Prüfer domain.*

(3) *R is a Q-domain* (⁴) *$\Rightarrow R^{\mathfrak{a}}$ is a Q-domain.*

(4) *R is an i-domain $\Rightarrow R^{\mathfrak{a}}$ is an i-domain.*

(5) *R is an open [resp. propen] domain $\Rightarrow R^{\mathfrak{a}}$ is an open [resp. propen] domain.*

(6) *R is a GD-domain $\Rightarrow R^{\mathfrak{a}}$ is a GD-domain.*

(7) *R is a Goldman ring $\Rightarrow R^{\mathfrak{a}}$ is a Goldman ring.*

If we suppose that the ring $R/\mathfrak{a}$ is quasi-regular (as *in* (4.5)), *then*:

(8) *R is a g-ring [resp. a locally pqr domain] $\Rightarrow R^{\mathfrak{a}}$ is a g-ring [resp. a locally pqr domain].*

(9) *R is a strong G-domain $\Rightarrow R^{\mathfrak{a}}$ is a strong G-domain.*

In particular, if $\mathfrak{a} = \mathfrak{p}$ is a prime ideal, each one of the properties (1)-(9) *transfer from R to $R^{\mathfrak{p}}$.*

PROOF. – (1). We recall that R is a G-domain if, and only if, its field of quotients K is a finitely generated R-algebra [25, Th. 18]. (2). A Prüfer domain is characterized by having all its overrings integrally closed. (3), (4), (5) and (6) are trivial consequences of the definitions. (7). It is rather easy to see that if T is a T_D-space, then $T^{\mathfrak{a}}$ is a T_D-space; in fact, for every ring R, the counter-image in $T^{\mathfrak{a}}$ of the set of all the G-ideals of R, Gold (R), is always contained in the set of all the G-ideals of $R^{\mathfrak{a}}$, Gold $(R^{\mathfrak{a}})$. (8) $R/\mathfrak{a}$ being quasi-regular, $T^{\mathfrak{a}} \simeq V(\mathfrak{a}) \cup \Delta(\mathfrak{a})$. It is easy to show now that if T is an Alexandroff-discrete space then $T^{\mathfrak{a}}$ is also an Alexandroff-discrete space. The statement (9) follow from (2) and (8).

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(⁴) Cf. for instance [34].

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