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TWO CHARACTERIZATIONS OF MORI DOMAINS

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Abstract. Let R be a commutative integral domain with field of fractions K. Let $\{R_i\}_{i\in I}$ be a defining family of overrings of R of finite character. We show that if R has an infinite strictly ascending chain of integral v-ideals then at least one R_i must allow an infinite strictly ascending chain of integral v-ideals. Thus if each R_i is Mori (satisfies ACC on integral v-ideals) then so must be R. We use this to show that R is a Mori domain if and only if R has a family $\{P(i)\}_{i\in I}$ of prime ideals such that $\{R_{p(i)}\}_{i\in I}$ is a defining family of R of finite character and for each i, $R_{p(i)}$ is Mori. We also show that if R has more than one maximal t-ideals then R is Mori if, and only if, for all non-zero non-units x, R_x is Mori.

Introduction. Throughout this article we use R, with or without subscripts, to denote a commutative integral domain with quotient field K. An integral domain R is a Mori domain if it satisfies ACC on integral v-ideals. Noetherian and Krull domains are the rather obvious examples of Mori domains. Mori domains have been studied in detail in [9], [3] and [4], though the main body of results on them appear in Querre's work. An interested reader may consult [8] and references given there.

In this article we characterize Mori domains using their *overrings* (rings between R and K). The main results may be described as follows. In Section 1, it is shown that if $\{R_i\}_{i\in I}$ is a family of overrings of R with the properties that

(i) $R = \bigcap_i R_i$ and (ii) every non-zero non-unit of R is a non-unit in at most a finite number of R_i, and if R admits an infinite ascending chain of integral v-ideals then so does at least one of R_i. Using this, one concludes that (a) if each of R_i is Mori then so is R and (b) if each of R_i satisfies ACC on principal ideals then so does R. In Section 2, it is shown that if R is a quasi local domain with at least two

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maximal t-ideals (to be defined later) then R is a Mori domain if and only if for all non-zero non-units $x \in R$, R_x is a Mori domain. Here R_x denotes R_S where $S = \{x^i\}_{i=0}^{\infty}$. The terminology used in this section may be of use in rephrasing some known results to get more productive statements.

1. Characterization via overrings. To ensure readibility, we introduce some notions to be used in this article. We also indicate some properties of Mori domains.

Let $F(R)$ denote the set of non-zero fractional ideals of R. Associated to each $A \in F(R)$ there are the fractional ideals $A^{-1} = \{x \in K \mid xA \subseteq R\} = R :_K A$ and $A_v = R :_K (R :_K A) = (A^{-1})^{-1}$. The map $A \mapsto A_v$ on $F(R)$ is a $*$-operation called the v-operation. The reader may consult Section 32 and 34 of [6] for the basic properties of $*$-operations. For our purposes we note that for any $*$-operation $*$ and for any $A \in F(R)$, $A \subseteq A^* \subseteq A_v = (A^*)_v = (A_v)^* = (A_v)_v$ and $A^{-1} = (A^{-1})^* = (A^*)^{-1} = (A_v)^{-1}$. Further, an ideal $A \in F(R)$ is called a v-ideal if $A = A_v$ and a *v-ideal of finite type* if $A = B_v$ for some finitely generated $B \in F(R)$. In particular, A^{-1} is a v-ideal. An ideal $A \in F(R)$ is called a t-ideal if for all finitely generated $B \in F(R)$, $B \subseteq A$ implies $B_v \subseteq A$. An integral ideal maximal w. r. t being a t-ideal is called a *maximal t-ideal* and is prime (Griffin [7]). Further, for $A \in F(R)$, $A_t = \cup B_v$, where B ranges over finitely generated R-submodules of A, is a t-ideal and the operation $A \mapsto A_t$ is another $*$-operation called the *t-operation*.

To indicate the basic properties of Mori domains we quote the following theorem from Raillard [9].

Theorem ([9], *Theorem* A.O). *For R, the following are equivalent.*

(1) *R is a Mori domain.*

(2) *For every strictly descending chain of v-ideals $\{A_n\}_{n=0}^{\infty}$, $\cap_n A_n = (0)$.*

(3) *Every non-empty family of integral v-ideals admits a maximal element.*

(4) *For every integral ideal A of R there is a finitely generated ideal $B \subseteq A$ such that $A^{-1} = B^{-1}$.*

From (4), of the above theorem, one concludes that every v-ideal of a Mori domain is a v-ideal of finite type. From (3), it follows that every integral v-ideal, of a Mori domain, is contained in a maximal v-ideal which is prime according to [3]. Finally, using (2), we can show that every integral v-ideal of a Mori domain is contained in only a finite number of maximal v-ideals ([3]).

Before stating results on characterization of Mori domains via overrings we note that if $\{R_i\}_{i\in I}$ is a family of overrings of R such that $R = \bigcap_i R_i$ then $\{R_i\}_{i\in I}$ is called a defining family of R and it is said to be of finite character if every non-zero non-unit of R is a non-unit in only a finite number of R_i. Further if $\{R_i\}_{i\in I}$ is a defining family of R then the map defined by $A \mapsto A^* = \bigcap_i AR_i$ is a $*$-operation on R induced by the family $\{R_i\}_{i\in I}$.

Theorem 1. Let $\{R_i\}_{i\in I}$ *be a defining family of* R *of finite character. If* R *admits an infinite strictly ascending chain of integral v-ideals then so does at least one of* R_i.

To prove this theorem we need the following two lemmas.

Lemma 2. *Let* $\{R_i\}_{i\in I}$ *be a defining family of* R. *Then for any fractional ideal* A *of* R, $A^{-1} = \bigcap_i (R_i : AR_i) = \bigcap_i (AR_i)^{-1}$ *and* $A_v = \bigcap_i (R_i : (R : A)R_i)$. (*From this point on we shall write* $R : A$ *to mean* $R :_K A$.)

Proof. Let $A \in F(R)$. If $x \in R : A$ then $xA \subseteq R$ and so $xAR_i \subseteq R_i$ for all $i \in I$, and $x \in \bigcap_i (R_i : AR_i)$. Conversely, let $x \in \bigcap_i (R_i : AR_i)$. Then $xAR_i \subseteq R_i$, for all $i \in I$. But $A \subseteq AR_i$. So $xA \subseteq xAR_i \subseteq R_i$ for all $i \in I$. Consequently, $xA \subseteq \bigcap_i R_i$ and $x \in R : A$. Thus we establish $A^{-1} = R : A = \bigcap_i (R_i : AR_i)$. In particular, $A_v = (A^{-1})^{-1} = \bigcap_i (R_i : A^{-1}R_i)$.

Lemma 3. *Let* $\{R_i\}_{i\in I}$ *be a defining family of* R *and let* $A \subset B$ *be two v-ideals of* R. *Then for some* $i \in I$, $R_i : (R : A)R_i \subset R_i : (R : B)R_i$.

Proof. We note that, because A and B are v-ideals, if $A \subset B$ then $R : A \supset R : B$

and so for all $i \in I$, we have $R_i : (R : A)R_i \subseteq R_i : (R : B)R_i$. If we assume that for all $i \in I$, $R_i : (R : A)R_i = R_i : (R : B)R_i$, then by Lemma 2, $A = A_v = \bigcap_i (R_i : (R : A)R_i) = \bigcap_i (R_i : (R : B)R_i) = B_v = B$, a contradiction.

Proof of Theorem 1. Let $A_1 \subset A_2 \subset \ldots$ be an infinite ascending chain of integral v-ideals of R. Then by the finite character of $\{R_i\}_{i \in I}$ there exists only a finite number of R_i for which $R_i : (R : A_1)R_i \neq R_i$. Let $F = \{R_1, R_2, \ldots, R_n\}$ be the set of all the R_i for which $R_i : (R : A_1)R_i \neq R_i$. Now the chain $A_1 \subset A_2 \subset \ldots$ gives rise to the chain $R_i : (R : A_1)R_i \subseteq R_i : (R : A_2)R_i \subseteq \ldots$. for all $R_i \in F$. Because the original chain is strictly ascending, by Lemma 3, for all positive integral m, $R_i : (R : A_m)R_i \subset R_i : (R : A_{m+1})R_i$ for at least one R_i. But then an infinite number of strict inclusions are distributed over a finite number of chains :

$$R_i : (R : A_1)R_i \subseteq R_i : (R : A_2)R_i \subseteq \ldots \quad .$$

Consequently at least one of these chains has an infinite number of strict inclusions. Now to complete the proof all we have to show is that if A is an integral ideal of R then $R_i : (R : A)R_i$ is an integral ideal of R_i. For this we note that $R : A \supseteq R$ and so $(R : A)R_i \supseteq R_i$ which gives $R_i : (R : A)R_i \subseteq R_i$. Being an inverse this ideal is also a v-ideals of R_i.

Corollary 4. *Let* $\{R_i\}_{i \in I}$ *be a defining family of finite character of* R. *Then the following hold.*

(1) *If each of* R_i *is a Mori domain then* R *must be a Mori domain.*

(2) *If each of* R_i *satisfies* ACC *on principal ideals then so does* R.

(3) *If each of* R_i *is a Krull domain then so is* R.

Proof. (1) and (2) are direct. For (3) we note that (a) a Krull domain is a Mori domain which is also completely integrally closed and (b) an arbitrary intersection of completely integrally closed integral domains is completely integrally closed.

It is known that if R is a Mori domain then so are its rings of fractions (see e.g. [3] and [9]). Moreover, we have noted that every non-zero non-unit of a Mori domain belongs to only a finite number of its maximal v-ideals. These observations give rise to the following corollary.

Corollary 5. *An integral domain R is a Mori (Krull) domain if and only if R has a family $\{P(i)\}_{i\in I}$ of prime ideals such that*

(1) *every non-zero non-unit of R belongs to at most a finite number of $P(i)$,*

(2) *for every $i \in I$, $R_{P(i)}$ is a Mori (Krull) domain, and*

(3) $R = \bigcap_i R_{P(i)}$.

2. Characterization via rings of fractions. For the next characterization we include some new terminology in the hope that it would facilitate the statement of results similar to those presented in this section.

Let $x \in R$ be a non-zero non-unit and let $S = \{x^i\}_{i=0}^{\infty}$. Denote R_S by R_x and call it a *localised ring of fractions of R*. Also, call a quasi local domain a *t-local domain* if its maximal ideal is a t-ideal. Obviously, if the maximal ideal is not a t-ideal then we may call R a *non t-local domain.* In fact R is a non t-local domain if and only if there is in R a maximal t-ideal P and a non-zero non-unit $x \in R - P$. Now the promised result.

Theorem 6. *A non t-local domain R is a Mori (Krull) domain if and only if every localised ring of fractions of R is a Mori (Krull) domain.*

The proof of this theorem is direct, but to make the contents of this section more productive we prove it via the following lemma.

Lemma 7. *For an integral domain R the following are equivalent.*

(1) *R is non t-local.*

(2) *There is a set $\{x_1, x_2, \ldots, x_n\}$ of non-units of R such that* $(x_1, x_2, \ldots, x_n)_v = R$.

(3) R *is a finite intersection of localized rings of fractions of itself.*

Proof. (1) $\Rightarrow$ (2). By the remarks before Theorem 6, if R is non t-local then there is a maximal t-ideal P and a non-unit $x_1 \in R - P$. Now $P = P_t \subset (x_1, P)_t$ and as P is maximal $w.r.t$ being an integral t-ideal $(x_1, P)_t = R$. From this it follows that there exist $x_2, x_3, \ldots, x_n$ in P such that $(x_1, x_2, \ldots, x_n)_v = R$.

(2) $\Rightarrow$ (3). Recall that a prime ideal minimal over an ideal of the type $(a) :_R (b) = \{x \in R \mid xb \in (a)\}$ $(\neq (0), R)$ is called an *associated prime* of a (non-zero) principal ideal [5]. Let $P(R)$ denote the set of all associated primes of principal ideals. Since $(x_1, x_2, \ldots, x_n)_v = R$, according to Tang [10], $(x_1, x_2, \ldots, x_n)$ is contained in no member of $P(R)$. That is if
$E_i = \{P_{ai} \in P(R) \mid x_i \in P_{ai}\}$ then $\bigcap_i E_i = \phi$. Thus $P(R) = \cup (P(R) - E_i)$.
Now according to [5], $R = \cap R_P$, where P ranges over $P(R)$ and for a multiplicative set S of R, $R_S = \cap \{R_P \mid P \in P(R), P \cap S = \phi\}$. Now letting $\{x_i^j\}_{j=0}^{j=\infty} = S(i)$ we have
$R_{S(i)} = \cap \{R_P \mid P \in P(R),\ P \cap S(i) = \phi\} = \cap \{R_P \mid P \in P(R) - E_i\}$ and so

$$\begin{aligned}\bigcap_i R_{S(i)} &= \cap \{R_P \mid P \in P(R) - E_i \text{ for some } i = 1, 2, \ldots, n\} \\ &= \cap \{R_P \mid P \in \cup (P(R) - E_i)\} \\ &= \cap \{R_P \mid P \in P(R)\} \\ &= R.\end{aligned}$$

(3) $\Rightarrow$ (1). Let $R = \cap R_{S(i)}$ where $S(i)$ is multiplicatively generated by x_i. Then clearly x_i, $i = 1, 2, \ldots, n$, share no associated prime and so $(x_1, x_2, \ldots, x_n)_v = R$ (see Tang [10]). Now if $(x_1, x_2, \ldots, x_n)_v = R$ then x_i share no maximal t-ideals. Further, as we have defined localised rings of fractions only for non-zero non-units, R has more than one maximal t-ideals.

Proof of Theorem 6. Since R is non t-local, it is a finite intersection of localised rings of fractions each of which is a Mori (Krull) domain by hypothesis. Now by Corollary 4, R must be Mori (Krull). Conversely if R is Mori (Krull) then so are its rings of fractions and combining this with the main hypothesis we get the result.

Remark 8. Lemma 7 may turn out to be a useful result. Useful in that it makes possible the extension of results of multiplicative nature stated for non quasi local domains to non t-local domains. This is a considerable advantage because a non t-local domain may still be quasi local. It may also be a very much needed result at this point of time. An interested reader may compare Lemma 7 with Lemma 2.10 of [2] and with Lemma 2.1 of [1]. For applications, Theorem 2.3 of [1] may be compared with Theorem 6 of this article.

Corollary 9. *Let R be a quasi local domain which is locally factorial in the sense of Fossum (localised rings of fractions are factorial* [1]). *If R has two non-associated principal primes then R is factorial.*

The proof is an obvious application of Theorem 6 (Lemma 7 included) and the celebrated theorem of Nagata on **UFD's**: If R is Krull and S a multiplicative set of R generated by primes then R_S, **UFD** implies R is a **UFD**.

Proof of Corollary 9. Let p and q be two non-associated primes. Then $(p, q)_v = R$ and so $R = R_p \cap R_q$; which means that R is Krull. Now R being Krull and R_p being factorial imply that R is factorial by Nagata's Theorem.

We note that if R is local (Noetherian) it is sufficient to ask for a single prime along with the main hypothesis of Corollary 9 to get a factorial ring.

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