UNIQUE REPRESENTATION DOMAINS, II

SAID EL BAGHDADI, STEFANIA GABELLI, AND MUHAMMAD ZAFRULLAH

Introduction

Let R be an integral domain. Two elements $x, y \in R$ are said to have a greatest common divisor (for short a GCD) if there exists an element $d \in R$ such that $d \mid x, y$ (or $dR \supseteq xR, yR$) and for all $r \in R$ dividing x, y (or $rR \supseteq xR, yR$) we have $r \mid d$ (or $rR \supseteq dR$). A GCD of two elements x, y, if it exists, is unique up to a unit and will be denoted by $\mathrm{GCD}(x, y)$. Following Kaplansky [27, page 32], in part, we say that R is a GCD domain if any two nonzero elements of R have a GCD. For a more detailed treatment of greatest common divisors the reader is referred to [27, page 32].

Of course if d is a GCD of x, y then $x = x_1 d$ and $y = y_1 d$ where $\mathrm{GCD}(x_1, y_1) = 1$. Here $\mathrm{GCD}(x_1, y_1) = 1$ signifies the fact that every common factor of x_1, y_1 is a unit (i.e. if $tR \supseteq x_1 R$, $y_1 R$ then t is a unit). Now a PID R is slightly more refined than a GCD domain in that for every pair of nonzero ideals aR, bR, we have a unique ideal dR with $aR + bR = dR$ where d is a GCD of a, b; $a = a_1 d, b = b_1 d$ and $a_1 R + b_1 R = R$. So dR can be regarded as the GCD of aR, bR. Note that in our PID example the principal ideals aR, bR are invertible and that a Prüfer domain is a domain in which every nonzero finitely generated ideal is invertible. If we regard, for every pair of invertible ideals A and B of a Prüfer domain R, the invertible ideal $C = A + B$ as the GCD of A and B we find that $C \supseteq A, B$. Hence, $R \supseteq AC^{-1} = A_1, BC^{-1} = B_1$ and so $A = A_1 C, B = B_1 C$ where $A_1 + B_1 = R$. Thus, in a Prüfer domain each pair of invertible ideals has the GCD of sorts. Now in GCD domains various types of unique factorization have been studied, see for instance Dan Anderson's recent survey [1]. The aim of this note is to bring to light similar unique factorizations of those ideals that behave like invertible ideals of Prüfer domains.

It was shown in [35] that in a GCD domain R a principal ideal xR has finitely many minimal primes if and only if xR can be uniquely expressed as a product $xR = (x_1 R)(x_2 R) \dots (x_n R)$ where each of $x_i R$ has a unique minimal prime and x_i are coprime in pairs. A GCD domain in which every proper principal ideal has finitely many minimal primes was called a unique representation domain (URD) and a nonzero principal ideal with a unique minimal prime ideal was called a packet in [35]. Following the terminology of [35] an invertible integral ideal I in a Prüfer domain will be called a packet if I has a unique minimal prime and a Prüfer domain R a URD if every proper invertible ideal of R is uniquely expressible as a product of mutually comaximal packets. One aim of this note is to show that if X is an invertible ideal in a Prüfer domain such that the set $\mathrm{Min}(X)$ of minimal primes of X is finite then X is expressible (uniquely) as a product of mutually comaximal packets. This may be used to describe the factorization of invertible ideals of Prüfer domains in which every invertible ideal has at most a finite number of minimal primes. This class includes the Prüfer domains of finite character and the so called generalized Dedekind domains. Our immediate plan is to give a more general definition of URD's, start with a notion that generalizes both Prüfer and GCD domains, prove analogues of results on URD's for this generalization and show that results on GCD and Prüfer domains follow from this. The domains that generalize both GCD and Prüfer domains are called Prüfer v-multiplication domains (PVMD's) by some and t-Prüfer or pseudo Prüfer by others. Once that is done we shall look into generalizations of this kind of unique factorization of ideals. This will make the presentation a bit repetitive but in the presence of motivation the reader will find the paper more readable.

Before we indicate our plan it seems pertinent to provide a working introduction to the notions that we shall use in this paper.

Let R be an integral domain with quotient field K and $F(R)$ the set of nonzero fractional ideals of R. A star operation $*$ on R is a map $F(R) \to F(R)$, $I \mapsto I^*$, such that the following conditions hold for each $0 \neq a \in K$ and for each $I, J \in F(R)$:

(i) $R^* = R$ and $(aI)^* = aI^*$;
(ii) $I \subseteq I^*$, and $I \subseteq J \Rightarrow I^* \subseteq J^*$;
(iii) $I^{**} = I^*$.

For standard material about star operations, see Sections 32 and 34 of [21]. For our purposes we note the following.

Given two ideals $I, J \in F(R)$ we have $(IJ)^* = (I^*J)^* = (I^*J^*)^*$ ($*$-multiplication) and we have $(I + J)^* = (I^* + J)^* = (I^* + J^*)^*$ ($*$-addition).

A nonzero fractional ideal I is a $*$-ideal if $I = I^*$ and it is $*$-finite (or of finite type) if $I^* = J^*$ for some finitely generated ideal $J \in F(R)$. A star operation $*$ is of finite type if $I^* = \bigcup\{J^* : J \subseteq I$ and J is finitely generated$\}$, for each $I \in F(R)$. To each star operation $*$, we can associate a star operation of finite type $*_f$, defined by $I^{*_f} = \bigcup\{J^* : J \subseteq I$ and J is finitely generated$\}$, for each $I \in F(R)$. If I is a finitely generated ideal then $I^* = I^{*_f}$.

Several star operations can be defined on R. The trivial example of a star operation is the identity operation, called the d-operation, $I_d = I$ for each $I \in F(R)$. Two nontrivial star operations which have been intensively studied in the literature are the v-operation and the t -operation. Recall that the v-closure of an ideal $I \in F(R)$ is $I_v = (I^{-1})^{-1}$, where for any $J \in F(R)$ we set $J^{-1} = (R\colon J) = \{x \in K : xJ \subseteq R\}$. A v-ideal is also called a divisorial ideal. The t -operation is the star operation of finite type associated to v. Thus $I = I_t$ if and only if, for every finite set $x_1, \ldots, x_n \in I$ we have $(x_1, \ldots, x_n)_t \subseteq I$. If $\{R_\alpha\}$ is a family of overrings of R such that $R \subseteq R_\alpha \subseteq K$ and $R = \cap R_\alpha$, then, for all $I \in F(R)$, the association $I \mapsto I^* = \cap IR_\alpha$ is a star operation "induced" by $\{R_\alpha\}$.

If $*_1$ and $*_2$ are two star operations defined on R we say that $*_2$ is coarser than $*_1$ (notation $*_1 \leq *_2$) if, for all $I \in F(R)$ we have $I^{*_1} \subseteq I^{*_2}$. If $*_1 \leq *_2$ then for each $I \in F(R)$ we have $(I^{*_1})^{*_2} = (I^{*_2})^{*_1} = I^{*_2}$. The v-operation is the coarsest of all star operations on R and the t-operation is coarsest among star operations of finite type.

A $*$-prime is a prime ideal which is also a $*$-ideal and a $*$-maximal ideal is a $*$-ideal maximal in the set of proper integral $*$-ideals of R. We denote by $*$-Spec(R) (respectively, $*$-Max(R)) the set of $*$-prime (respectively, $*$-maximal) ideals of R. If $*$ is a star operation of finite type, by Zorn's lemma each $*$-ideal is contained in a $*$-maximal ideal, which is prime. In this case, $R = \bigcap_{M \in *\text{-Max}(R)} R_M$. We say that R has $*$-finite character if each nonzero element of R is contained in at most finitely many $*$-maximal ideals.

The star operation induced by $\{R_M\}_{M \in t\text{-Max}(R)}$, denoted by w, was introduced by Wang and McCasland in [34]. The star operation induced by $\{R_M\}_{M \in *\text{-Max}(R)}$, denoted by $*_w$, was studied by Anderson and Cook in [3], where it was shown that $*_w \leq *$ and that $*_w$ is of finite type.

When $*$ is of finite type, a minimal prime of a $*$-ideal is a $*$-prime. In particular, any minimal prime over a nonzero principal ideal is a $*$-prime for any star operation $*$ of finite type [24]. An height-one prime ideal, being minimal over a principal ideal, is a t-ideal. We say that R has t-dimension one if each t-prime ideal has height one.

For any star operation $*$, the set of fractional $*$-ideals is a semigroup under the $*$-multiplication $(I, J) \mapsto (IJ)^*$, with unity R. An ideal $I \in F(R)$ is called $*$-invertible if I^* is invertible with respect to the $*$-multiplication, i.e., $(II^{-1})^* = R$. If $*$ is a star operation of finite type, then a $*$-invertible ideal is $*$-finite. For concepts related to star invertibility and the w-operation the readers may consult [39] and if need arises references there. For our purposes we note that for a finite type star operation $*$, R is a Prüfer $*$-multiplication domain (for short a P$*$MD) if every nonzero finitely generated ideal of R is $*$-invertible. A PtMD is denoted by PVMD for historical reasons.

Finally two $*$-ideals I, J are $*$-comaximal if $(I + J)^* = R$. If $*$ is of finite type, I, J are $*$-comaximal if and only if $I + J$ is not contained in any $*$-maximal ideal. We note that the following statement can be easily proved for $*$ of finite character. If $\{I_\alpha\}$ is a finite family of pairwise $*$-comaximal ideals, then $(\bigcap I_\alpha)^* = (\prod I_\alpha)^*$. In fact we have $(\bigcap I_\alpha)^{*_w} = \bigcap_{M\in *\text{-Max}(R)}(\bigcap I_\alpha)R_M = \bigcap_{M\in *\text{-Max}(R)}(\prod I_\alpha)R_M = (\prod I_\alpha)^{*_w}$, where the second equality holds because at most one I_α survives in R_M. Now the result follows from the fact that $*$ is coarser than $*_w$.

For a finite type star operation $*$, we call a $*$-invertible $*$-ideal $I \subseteq R$ a $*$-packet if I has a unique minimal prime ideal and we call R a $*$-URD if every proper $*$-invertible $*$-ideal of R can be uniquely expressed as a $*$-product of finitely many mutually $*$-comaximal $*$-packets. When $* = t$, a $*$-packet will be called a packet and the $*$-URD will be called a URD.

In the first section we show that if R is a PVMD and I a t-invertible t-ideal then I has finitely many minimal primes if and only if I is expressible as a t-product of finitely many packets, if and only if I is expressible as a (finite) product of mutually t-comaximal packets. We also show that a PVMD R is a URD if and only if every proper principal ideal of R has finitely many minimal primes. The results in this section are a translation of results proved in [35], establishing yet again the folklore observation that most of the multiplicative results proved for GCD domains can be proved for PVMD's.

In Section 2, for a finite type star operation $*$ we prove that a domain R is a $*$-URD if and only if $*$-Spec(R) is treed and every proper principal ideal of R has finitely many minimal prime ideals. Thus Section 2 serves to isolate the main conditions that allowed unique representation in PVMD's. On the other hand the generality of $*$ lets us prove, in one go, results about URD's and about d-URD's which, incidentally, were the main topic in [8].

Finally, in Section 3, we study methods of making new URD's from given ones. We show that if R is a URD and S is a multiplicative set of R then R_S is a URD. Our main result here is the characterization of when a polynomial ring is a URD. We close the section by studying under what conditions we can construct URD's using the $D + XD_S[X]$ construction.

1. From GCD-domains to PVMD's

Let R be a PVMD. If $X, Y \subseteq R$ are t-invertible t-ideals and $\Delta = (X+Y)_t$, then Δ is a t-invertible t-ideal, $\Delta \supseteq X, Y$ and $X_1 = (X\Delta^{-1})_t$, $Y_1 = (Y\Delta^{-1})_t \subseteq R$ are t-comaximal t-invertible t-ideals; thus, as in the case of Prüfer domains, we can say that Δ is the GCD of X, Y. So we can carry out the plans for PVMD's that we made, in the introduction, with the class of Prüfer domains as our model. We intend to prove in this section the following two basic theorems involving packets.

Theorem 1.1. *Let R be a PVMD and $X \subseteq R$ a t-invertible t-ideal. If X is a t-product of a finite number of packets then X can be uniquely expressed as a t-product of a finite number of mutually t-comaximal packets.*

Theorem 1.2. *Let R be a PVMD and $X \subsetneq R$ a t-invertible t-ideal. Then X is a t-product of finitely many packets if and only if X has finitely many minimal primes.*

In a GCD (respectively, Prüfer) domain a t-invertible t-ideal is principal (respectively, invertible) and the t-product is just the ordinary product. Indeed it may be noted that a domain R can have all t-invertible t-ideals as t-products of packets without R being a PVMD as we shall point out in the course of our study.

From this point on we shall use the v and t operations freely. The results of this section are based on the following observations that, if reference or proof is not indicated, can be traced back to Mott and Zafrullah [29] or to Griffin [22].

(I) A nonzero prime ideal P of a PVMD R is essential (i. e., R_P is a valuation domain) if and only if P is a t-ideal.
(II) The set of prime t-ideals of a PVMD R is a tree under inclusion, that is any t-maximal ideal of R cannot contain two incomparable t-primes.
(III) If X is a t-invertible t-ideal and P is a prime t-ideal of a PVMD R then XR_P is principal. (XR_P is an invertible ideal [7] and in a valuation domain invertible ideals are principal.)
(IV) If $X = (X_1X_2)_t$ then in view of the definition of t -multiplication we shall assume that both X_i are t-ideals.

Let us start with some lemmas on packets in a PVMD.

Lemma 1.3. *Let R be a PVMD and let $X, Y \subsetneq R$ be t-invertible t-ideals such that $\Delta = (X+Y)_t \neq R$. Then the following conditions hold:*

(1) *If P is a prime ideal minimal over Δ then P is minimal over X or Y.*
(2) *If P is a prime ideal minimal over X such that $Y \subseteq P$ then P is minimal over Δ.*
(3) *If X and Y are packets then so are Δ and $(XY)_t$.*

Proof. (1). Note that being minimal over Δ, P is a t-ideal. Clearly $X, Y \subseteq P$. So P contains a prime ideal P_1 that is minimal over X and P_2 that is minimal over Y. By (II), P_1 and P_2 are comparable being minimal primes themselves. Say $P_2 \supseteq P_1$. Then both $X, Y \subseteq P_2$. Thus $\Delta = (X+Y)_t \subseteq P_2$. But P is minimal over Δ. So, $P = P_2$ and hence P is minimal over Y. Similarly if P_1 had contained P_2 we would have concluded that P was minimal over X.

(2). Since $P \supseteq X, Y$ we conclude that $P \supseteq (X+Y)_t = \Delta$. So, if P is not minimal over Δ then P contains a prime Q that is minimal over Δ. Again as $Q \supseteq \Delta = (X+Y)_t$, Q contains a prime minimal over X. But Q is contained in P.

(3). Let $P_1 \neq P_2$ be two minimal primes over $\triangle$. By (1), P_i is minimal over X or over Y. Assume that P_1 is (the unique) minimal prime over X, then P_2 is the minimal prime over Y. But $Y \subseteq P_1$, then $P_2 \subsetneq P_1$, which contradicts the minimality of P_1 over Δ. A similar argument applies if P_2 is the minimal prime over X. For $(XY)_t$, assume that P_1 and P_2 are minimal primes over $(XY)_t$, then P_i is minimal over X or over Y. We can assume that P_1 is the minimal prime over X. Then P_2 is the minimal prime over Y. On the other hand, since $\triangle = (X, Y)_t \neq R$ is a packet, the unique minimal prime over $\triangle$ is P_1 or P_2 (by (1)), this forces $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. In either case minimality of both forces $P_1 = P_2$. □

Lemma 1.4. *Let R be a PVMD and let P be a prime ideal minimal over a t-invertible t-ideal X. Then for all t-invertible t-ideals Y contained in P there exists a natural number n such that $XR_P \supsetneq Y^nR_P$.*

Proof. Note that P is a prime t-ideal in a PVMD and so R_P is a valuation domain and XR_P, YR_P are principal. Now suppose on the contrary that $XR_P \subseteq Y^nR_P$ for all natural numbers n. Then $XR_P \subseteq \bigcap_n Y^nR_P$ and $\bigcap_n Y^nR_P$ is a prime ideal properly contained in PR_P and this contradicts the minimality of P over X. Now since in a valuation domain every pair of ideals is comparable we have the result. □

Lemma 1.5. *Let R be a PVMD and let $X_1, X_2 \subsetneq R$ be t-invertible t-ideals. Let W_i be the set of t-maximal ideals containing X_i, for $i = 1, 2$. If $W_2 \subseteq W_1$ and X_2 is not contained in any minimal prime of X_1, then $X_1 \subseteq (X_2^n)_t$, for all natural numbers n.*

Consequently if X, Y are two packets with $(X+Y)_t \neq R$ such that $\operatorname{rad}(X) \supsetneq \operatorname{rad}(Y)$ then $(X^n)_t \supseteq Y$, for all natural numbers n.

Proof. Clearly for every t-maximal ideal P in W_1 (and hence for every t-maximal ideal P) we have $X_1R_P \subseteq X_2^nR_P$ for each n because X_2 is not contained in the minimal prime of X_1R_P. Thus for each n, $\bigcap_P X_1R_P \subseteq \bigcap_P X_2^nR_P$ or $(X_1)_w \subseteq (X_2^n)_w$. Since in a PVMD for every nonzero ideal I we have $I_w = I_t$ [26, Theorem 3.5], and since X_1 is a t-ideal, we conclude that for each n we have $X_1 \subseteq (X_2^n)_t$.

The proof of the consequently part follows from the fact that every t-maximal ideal that contains X contains $\operatorname{rad}(X)$ and hence Y. On the other hand, a t-maximal ideal that contains Y may or may not contain X and X has only one minimal prime ideal which properly contains the minimal prime ideal of Y. □

Lemma 1.6. *Let R be a PVMD and let $X \subsetneq R$ be a t-invertible t-ideal. Then the following conditions are equivalent:*

(i) *X is a packet;*
(ii) *for every t-factorization $X = (X_1X_2)_t$, $X_1 \supseteq (X_2^2)_t$ or $X_2 \supseteq (X_1^2)_t$.*

Proof. (i) $\Rightarrow$ (ii). Let Q be the unique minimal prime of X and let $X = (X_1X_2)_t$. Indeed we can assume $X_i \in \operatorname{Inv_t}(R)$. If either of $X_i = R$ (ii) holds, so let us assume that both X_i are proper. If either of the X_i is not contained in Q the result follows from Lemma 1.5. If both are contained in Q then, because Q is a t-ideal, $\Delta = (X_1 + X_2)_t \subseteq Q$. As described above we can write $X_1 = (Y_1\Delta)_t$ and $X_2 = (Y_2\Delta)_t$ where $Y_i \in \operatorname{Inv_t}(R)$ such that $(Y_1 + Y_2)_t = R$. Now Y_1, Y_2 being t-comaximal cannot share a prime t-ideal. If $Y_1 \nsubseteq Q$ then by Lemma 1.5, $\Delta \subseteq (Y_1^n)_t$ for all n and so $\Delta^2 \subseteq (Y_1^n)_t\Delta \subseteq X_1$, for any n. Combining with $X_2^2 \subseteq \Delta^2$ and applying the t-operation we conclude that $(X_2^2)_t \subseteq X_1$. Same arguments apply if $Y_2 \nsubseteq Q$.

(ii) $\Rightarrow$ (i). Suppose that X satisfies (ii) and let, by way of contradiction, P and Q be two distinct prime ideals minimal over X. Now P and Q are prime t-ideals and so both R_P and R_Q are valuation domains. Let $y \in P\backslash Q$. Then $\Delta = (X + yR)_t \subseteq P$ and $\Delta \nsubseteq Q$. By Lemma 1.4 there exists n such that $\Delta^nR_P \subsetneq XR_P$ and consequently $(\Delta^n)_t \not\supseteq X$. Consider $K = (X + \Delta^n)_t$. We can write $X = (AK)_t$ and $(\Delta^n)_t = (BK)_t$ where $A, B \in \operatorname{Inv_t}(R)$ and $(A + B)_t = R$; thus A and B cannot share a minimal t-prime ideal. Since $\Delta^n \nsubseteq Q$ and since $K \supseteq \Delta^n$ we conclude that $K \nsubseteq Q$. But since $X = (AK)_t \subseteq Q$ we conclude that $A \subseteq Q$. Next, we claim that $A \nsubseteq P$. For if $A \subseteq P$ then since $(A+B)_t = R$, $B \nsubseteq P$ and $BKR_P = \Delta^nR_P \supseteq AKR_P = XR_P$. This contradicts the fact that $\Delta^nR_P \subsetneq XR_P$. Thus we have $X = (AK)_t$ with

(a) $K \subseteq P$ and $K \nsubseteq Q$ and
(b) $A \subseteq Q$ and $A \nsubseteq P$

Now by (ii) $X = (AK)_t$ where $A \supseteq (K^2)_t$ or $K \supseteq (A^2)_t$. If $A \supseteq (K^2)_t$ then as $A \subseteq Q$ we have $K \subseteq Q$ which contradicts (a). Next if $K \supseteq (A^2)_t$ then since $K \subseteq P$ we have $A \subseteq P$ and this contradicts (b). Of course these contradictions arise from the assumption that X satisfying (ii) can have more than one minimal prime. □

Lemma 1.7. *Let A and B be two ideals of a domain R. Then the following statements hold:*

(1) *If $(A+B)_t = R$ and $A_t \supseteq (XB)_t$ for some nonzero ideal X, then $A_t \supseteq X_t$.*
(2) *If $B = B_1B_2$ then $(A+B)_t = R$ if and only if $(A+B_i)_t = R$ for $i = 1, 2$.*

(3) *If R is a PVMD and X is a packet such that $X = (X_1X_2)_t$, with $(X_1 + X_2)_t = R$, then $(X_1)_t = R$ or $(X_2)_t = R$.*

Proof. (1). Note that $A_t \supseteq A_t + (XA)_t \supseteq (XB)_t + (XA)_t$. Applying the t-operation $A_t \supseteq ((XB)_t + (XA)_t)_t = (XB + XA)_t = (X(A + B))_t = X_t$.

(2). Suppose that $(A + B_1B_2)_t = R$. Then $R = (A + B_1B_2)_t \subseteq (A + B_i)_t \subseteq R$ for $i = 1, 2$. Conversely if $(A+B_i)_t = R$ for $i = 1, 2$ then $(A+B_1B_2)_t = (A+AB_1+B_1B_2)_t = (A+B_1(A+B_2))_t = (A + B_1)_t$.

(3). We note that $(X_1)_t \supseteq (X_2^2)_t$ or $(X_2)_t \supseteq (X_1^2)_t$ (Lemma 1.6). If $(X_1)_t \supseteq (X_2^2)_t$ then $(X_1)_t = (X_1 + X_2^2)_t = R$ by (2) because $(X_1 + X_2)_t = R$. □

Lemma 1.8. *Let R be a PVMD and let $X, Y_1, Y_2 \subseteq R$ be t-invertible t-ideals. If $X \supseteq (Y_1Y_2)_t$ then $X = (X_1X_2)_t$ where $X_i \supseteq Y_i$.*

Proof. Let $X_1 = (X + Y_1)_t$. Then $X = (AX_1)_t$ and $Y_1 = (BX_1)_t$ where $(A + B)_t = R$. Now $(AX_1)_t \supseteq (BX_1Y_2)_t$. Multiplying both sides by X_1^{-1} and applying t we get $A = (AX_1X_1^{-1})_t \supseteq (BX_1X_1^{-1}Y_2)_t = (BY_2)_t$. Now $A \supseteq (BY_2)_t$ and $(A+B)_t = R$ and so by Lemma 1.7 $A \supseteq Y_2$. Setting $A = X_2$ gives the result. □

The above result is new in principle but not in spirit. Let us recall that a domain R is a pre-Schreier domain if for $a, b_1, b_2 \in R \setminus \{0\}$, $a \mid b_1b_2$ implies that $a = a_1a_2$ where $a_i \mid b_i$ for each $i = 1, 2$ [37]. Pre-Schreier generalizes the notion of a Schreier domain introduced by Cohn [9] as a generalization of a GCD domain. In simple terms, a Schreier domain is an integrally closed pre-Schreier domain. In [12], a domain R was called quasi-Schreier if, for invertible ideals $X, Y_1, Y_2 \subseteq R$, $X \supseteq Y_1Y_2$ implies that $X = X_1X_2$ where $X_i \supseteq Y_i$. It was shown in [12] that a Prüfer domain is quasi-Schreier, a result which can also be derived from Lemma 1.8 by noting that a Prüfer domain is a PVMD in which every nonzero ideal is a t-ideal. One may tend to define a domain R as t-quasi-Schreier if, for t-invertible t-ideals $X, Y_1, Y_2 \subseteq R$, $X \supseteq (Y_1Y_2)_t$ implies that $X = (X_1X_2)_t$ where $X_i \supseteq Y_i$, but this does not seem to be a place for studying this concept.

Proof of Theorem 1.1. Let R be a PVMD and let $X \subsetneq R$ such that $X = (X_1X_2 \dots X_n)_t$, where X_i are packets. By (3) of Lemma 1.3 the product of two non t-comaximal packets is again a packet. So with a repeated use of (3) of Lemma 1.3 we can reduce the above expression to $X = (Y_1Y_2 \dots Y_r)_t$ where $(Y_i+Y_j)_t = R$ for $i \neq j$. Now let X' be a packet such that $X' \supseteq X = (Y_1Y_2 \dots Y_r)_t$. Then by Lemma 1.8, $X' = (UV)_t$ where $U_t \supseteq Y_1$ and $V_t \supseteq (Y_2 \dots Y_r)_t$. This gives $U_t + V_t \supseteq Y_1 + (Y_2 \dots Y_r)_t$. Applying the t-operation, using the fact that $(Y_i + Y_j)_t = R$ for $i \neq j$ and using Lemma 1.7 we conclude that one of U_t, V_t is R. If $V_t = R$ then $X' \supseteq Y_1$ and X' does not contain any of the other Y_i. If $U_t = R$ then $X' \supseteq (Y_2 \dots Y_r)_t$ and using the above procedure we can in the end conclude that X' contains precisely one of the Y_i. So if X has another expression $X = (B_1B_2 \dots B_s)_t$ where B_i are packets such that $(B_i + B_j)_t = R$ for $i \neq j$ then by the above procedure each of X_i contains precisely one of the B_j and B_j contains precisely one of the X_k. But then $i = k$. For if not then $X_i \supseteq X_k$ so $X_i = (X_i + X_k)_t = R$ which is impossible. Thus $r = s$ and each X_i is equal to precisely one of the B_j.

Proof of Theorem 1.2. Let R be a PVMD and let $X \in \mathrm{Inv_t}(R)$ such that X is a finite product of packets. Then by Theorem 1.1, X can be uniquely expressed as a finite product of mutually t-comaximal packets. That X has only finitely many minimal primes follows from the fact that any minimal prime over X is minimal over some packet in its factorization.

We prove the converse by induction on the number of minimal primes of $X \in \mathrm{Inv_t}(R)$. If X has a single minimal prime ideal then X is a packet and we have nothing to prove. Suppose that if X

has $k-1$ minimal primes then X is expressible as a product of $k-1$ mutually t-comaximal packets. Now suppose that $P_1, P_2, \ldots, P_k$ are all the minimal primes of X. Choose $y \in P_1 \backslash (P_2 \cup P_3 \cup \cdots \cup P_k)$. By Lemma 1.4, there exists a natural number n such that $XR_{P_1} \supsetneq y^n R_{P_1}$. Let $\Delta = (y^n R + X)_t$ with $X = (\Delta X_1)_t$ and $y^n R = (\Delta Y_1)_t$ where $(X_1 + Y_1)_t = R$ and $X_1, Y_1 \in \mathrm{Inv_t}(R)$. Noting that in R_{P_1}, $A_t R_{P_1} = AR_{P_1}$ is principal if A is t -invertible, we convert $XR_{P_1} \supsetneq y^n R_{P_1}$ into $\Delta X_1 R_{P_1} \supsetneq \Delta Y_1 R_{P_1}$ which reduces to $X_1 R_{P_1} \supsetneq Y_1 R_{P_1}$. Now since $(X_1 + Y_1)_t = R$, X_1, Y_1 cannot both be in P_1. Moreover $X_1 R_{P_1} \supsetneq Y_1 R_{P_1}$ ensures that $X_1 \nsubseteq P_1$, otherwise X_1 and Y_1 would be in P_1, and again the strict inclusion implies that Y_1 must be in P_1. From these considerations we have that $X = (X_1 \Delta)_t$ where $X_1 \nsubseteq P_1$, so $\Delta \subseteq P_1$ and because $\Delta \supseteq y^n R$ we conclude that $\Delta \nsubseteq P_2 \cup P_3 \cup \cdots \cup P_k$. As $X = (X_1 \Delta)_t \subseteq P_2 \cap \cdots \cap P_k$ we have $X_1 \subseteq P_2, \ldots, P_k$.

If $(\Delta + X_1)_t = R$ then, since $X = (X_1 \Delta)_t$, the set of minimal primes of X is partitioned into two sets: ones minimal over Δ and ones minimal over X_1. Since $X_1 \subseteq P_2, \ldots, P_k$ we conclude that X_1 has $P_2, \ldots, P_k$ for minimal primes and this leaves us with Δ with a single minimal prime. Consequently Δ is a packet and X_1 is a t-product of $k-1$ mutually t-comaximal packets, by induction hypothesis. So the theorem is proved in this case.

Next, let $U = (\Delta + X_1)_t \neq R$ with $\Delta = (\Delta_0 U)_t$ and $X_1 = (X_0 U)_t$ and note that $(X_0 + \Delta_0)_t = R$. Since $X_1 \nsubseteq P_1$, $U \nsubseteq P_1$ and since Δ is not contained in any of $P_2, P_3, \ldots, P_k$ because $\Delta \supseteq y^n R$ we conclude that $U \nsubseteq P_2, P_3, \ldots, P_k$. Consequently $U \nsubseteq P_1, P_2, P_3, \ldots, P_k$. But since $X_1 = (X_0 U)_t$ is contained in each of $P_2, P_3, \ldots, P_k$ we conclude that $X_0 \subseteq P_2 \cap P_3 \cap \cdots \cap P_k$. Similarly $\Delta_0 \subseteq P_1$. We note here that since P_1 is a minimal prime of X and since $X \subseteq \Delta_0 \subseteq P_1$, P_1 must be a minimal prime of Δ_0. Now we establish that P_1 is indeed the unique minimal prime of Δ_0. For this assume that there is a prime ideal Q that contains Δ_0. Then as $Q \supseteq \Delta_0 \supseteq X$, Q must contain one of $P_1, P_2, P_3, \ldots, P_k$. But since $(X_0 + \Delta_0)_t = R$ and $X_0 \subseteq P_2 \cap P_3 \cap \cdots \cap P_k$ we conclude that Δ_0 is not contained in any of $P_2, P_3, \ldots, P_k$. Thus $Q \supseteq P_1$. Thus P_1 is the unique minimal prime of Δ_0 and Δ_0 is a packet. Thus we have $X = (X_1 \Delta)_t = (X_0 U \Delta_0 U)_t = (X_0 U^2 \Delta_0)_t$. Now $(U^2)_t \supseteq X$ $= (X_0 U^2 \Delta_0)_t$ and U^2 is not contained in any minimal prime of X, so by Lemma 1.5, $X \subseteq (U^{2n})_t$ for each natural n. As U is t-invertible $(X_0 \Delta_0)_t \subseteq (U^2)_t$. By Lemma 1.8, $(U^2)_t = (U_1 U_2)_t$ where $X_0 \subseteq (U_1)_t$ and $\Delta_0 \subseteq (U_2)_t$. As $(X_0 + \Delta_0)_t = R$ we conclude that $(U_1 + U_2)_t = R$. Indeed $(U_1 + U_2)_t = (U_1 + X_0 + \Delta_0 + U_2)_t = (U_1 + (X_0, \Delta_0)_t + U_2)_t = R$. Thus $X = (X_0 U_1 U_2 \Delta_0)_t$ where $(X_0 U_1 + U_2 \Delta_0)_t = R$, P_1 contains, and is minimal over, $(U_2 \Delta_0)_t$ and $P_2, \ldots, P_k$ are minimal over $(X_0 U_1)_t$. It is easy to see that $(U_2 \Delta_0)_t$ is a packet and that by the induction hypothesis $(X_0 U_1)_t$ is a t-product of $k-1$ mutually t-comaximal packets. This establishes the Theorem.

We are now in a position to characterize PVMD URD's.

Theorem 1.9. *A PVMD R is a URD if and only if every nonzero principal ideal of R has at most finitely many minimal primes.*

Proof. If R is a URD then obviously every proper principal ideal of R has finitely many minimal primes. Conversely, suppose that R is a PVMD such that every proper principal ideal of R has finitely many minimal primes and let, by way of a contradiction, I be a t-invertible t-ideal such that I has infinitely many minimal primes. Since I is a t-invertible t-ideal there exist $x_1, x_2, \ldots, x_n \in I$ such that $I = (x_1, x_2, \ldots, x_n)_t$. By Lemma 1.3, each minimal prime of I is a minimal prime of $x_1 R$ or a minimal prime of $(x_2, \ldots, x_n)_t$. Since $x_1 R$ has only finitely many minimal primes we conclude that $(x_2, \ldots, x_n)_t$ has infinitely many minimal primes. Using Lemma 1.3 and the above procedure we can eliminate in turn $x_2, \ldots, x_{n-2}$ and this leaves us with the conclusion that $(x_{n-1}, x_n)_t$ has infinitely many minimal primes. But then $x_{n-1} R$ has infinitely many minimal primes or $x_n R$ does and this, in either case, contradicts the assumption that every proper principal ideal of R has finitely many minimal primes. We conclude by applying Theorem 1.2. □

PVMD's of t-finite character were studied by Griffin [23] under the name of rings of Krull type. Since the t-prime ideals of a PVMD form a tree, in a ring of Krull type every nonzero principal ideal has at most finitely many minimal primes. Thus we have the following corollary.

Corollary 1.10. *A ring of Krull type is a URD.*

Indeed, as a GCD-domain is a PVMD such that every t-invertible t-ideal is principal, we recover the characterization of GCD URD's given in [35].

Corollary 1.11. *A GCD domain R is a URD if and only if every proper principal ideal of R is a finite product of packets.*

Using [35] we can also see how some other generalizations of UFD's can be considered as special cases of PVMD URD's.

2. General Approach

The notion of unique representation of a t-invertible t-ideal $X \subsetneq R$ as a t-product of packets developed in Section 1 is somewhat limited in that it deals with t-invertible t-ideals which have a unique minimal prime ideal. To bring more integral domains under the umbrella of unique representation one may need to concentrate on the properties of packets. We have the following thoughts on this.

(1). We can replace in the definition of a packet "t-invertible t -ideal" with "$*$-invertible $*$-ideal", for a star operation $*$ of finite type, thus getting the notion of $*$-packet (a $*$-invertible $*$-ideal with a unique minimal prime ideal), as we have already defined in the introduction. Since, for $*$ of finite type, every $*$-invertible $*$-ideal is t-invertible, when we define a $*$-URD in the usual fashion as a domain in which every $*$-invertible $*$-ideal $X \subsetneq R$ is expressible uniquely as a $*$-product of finitely many mutually $*$-comaximal $*$-packets, we get a URD albeit of a special kind. We can settle for the fact that a $*$-URD has two special cases: a URD and a d-URD. In a d-URD every invertible ideal is a product of finitely many (invertible) packets and we may note that our results will show that a URD is not necessarily a d-URD. It would be interesting to find a star operation $*$ of finite type, different from d and t, and a URD that is a $*$-URD.

(2). The other choice depends upon another property of packets: a packet X is such that if $X = (X_1X_2)_t$ where X_1, X_2 are proper t-ideals then $(X_1 + X_2)_t \neq R$. More generally, we can take a star operation $*$ of finite type and call an ideal $X \subsetneq R$ $*$-pseudo irreducible if X is a $\star$-ideal such that $X = (X_1X_2)^*$ and $(X_1 + X_2)^* = R$ implies that $(X_1)^* = R$ or $(X_2)^* = R$. This will allow us to develop the theory of factorization of $*$-ideals on the same lines as the Unique Comaximal Factorization domains of Swan and McAdam [1, 28]. In this paper, however, we shall pursue the $*$-URD's.

Given a star operation $*$ of finite type, we start by giving conditions for which a proper $*$-ideal X of a domain R (not necessarily a PVMD) can be written as a $*$-product in the form $X = (X_1X_2 \dots X_n)^*$, where each X_i is a $*$-ideal with prime radical and the X_i's are pairwise $*$-comaximal $*$-ideals. In the case $* = d$, this question was investigated in [8]. Our first result shows that if such a factorization exists, then it is unique (up to order).

Lemma 2.1. *Let R be an integral domain and $*$ a star operation on R. Let X and $\{Y_i\}_{i=1}^n$ be nonzero ideals of R such that $(X + Y_i)^* = R$ for each i. Then $(X + \prod_{i=1}^n Y_i)^* = R$.*

Proof. We give a proof in the case $n = 2$; the general case follows by induction. Assume that $(X + Y_1)^* = (X + Y_2)^* = R$. We have $R = (X + Y_1)^* = (X + Y_1(X + Y_2)^*)^* = (X + Y_1(X + Y_2))^* = (X + Y_1X + Y_1Y_2)^* = (X + Y_1Y_2)^*$, as desired. □

Lemma 2.2. *Let R be an integral domain and $*$ a star operation on R. Let X and Y be two nonzero ideals of R such that $(X+Y)^* = R$. If $X^* \supseteq (ZY)^*$, for some nonzero ideal Z, then $X^* \supseteq Z^*$.*

Proof. If $X^* \supseteq (ZY)^*$, we have $X^* \supseteq X^* + (ZY)^* \supseteq (ZX)^* + (ZY)^*$. Applying the $*$-operation, $X^* \supseteq ((ZX)^* + (ZY)^*)^* = (ZX + ZY)^* = (Z(X+Y))^* = Z^*$. □

Theorem 2.3. *Let R be an integral domain and $*$ a star operation of finite type on R. Let $X \subsetneq R$ be a $*$-ideal and suppose that X has a $*$-factorization of the form $X = (X_1X_2 \dots X_n)^*$, where each X_i is a $*$-ideal with prime radical and the X_i's are pairwise $*$-comaximal $*$-ideals. Then, up to order, such a $*$-factorization is unique.*

Proof. Let $X = (X_1X_2 \dots X_n)^* = (Y_1Y_2 \dots Y_m)^*$ be two $*$-factorizations of X as in the hypotheses. Let $P_1, P_2, \dots, P_n$ be the prime ideals minimal over $X_1, X_2, \dots, X_n$, respectively. Then, up to order, $Y_1, Y_2, \dots, Y_m$ must have the same associated minimal primes. Hence $m = n$. We can assume that $\mathrm{rad}(Y_i) = P_i$ for each $i = 1, \dots, n$. Since the P_i's are $*$-ideals, then $(X_i + Y_j)^* = R$ for $i \neq j$. For a fixed i, set $Z_i = \prod_{j \neq i} Y_j$. We have $(X_i + Z_i)^* = R$ (Lemma 2.1) and $X_i \supseteq X = (Y_iZ_i)^*$. Hence by Lemma 2.2, $X_i \supseteq Y_i$. Similarly, $X_i \subseteq Y_i$. Hence $X_i = Y_i$, for each i. □

We next investigate the decomposition of $*$-ideals into $*$-comaximal $*$-ideals. Recall that $*$-Spec(R) is treed if any two incomparable $*$-primes are $*$-comaximal.

Proposition 2.4. *Let R be a domain and let $*$ be a star operation of finite type on R. If $*$-*Spec(R) *is treed, the following conditions are equivalent for a $*$-ideal $I \subsetneq R$:*

(i) *I has finitely many minimal primes;*
(ii) *$I = (Q_1Q_2 \dots Q_n)^*$, where the Q_i's are $*$-ideals with prime radical;*
(iii) *$I = (Q_1Q_2 \dots Q_n)^*$, where the Q_i's are pairwise $*$-comaximal $*$-ideals with prime radical.*

Under any of these conditions, if I is $*$-invertible, the Q_i's are $*$-packets.

Proof. (i) ⇒ (iii). Let $P_1, P_2, \dots, P_n$ be the prime ideals minimal over I. Since $*$-Spec(R) is treed, $(P_i + P_j)^* = R$ for $i \neq j$. Thus, by Lemma 2.1, $(P_i + \prod_{j \neq i} P_j)^* = R$ for each i. Denote by A_i a finitely generated ideal of R such that $A_i \subseteq \prod_{j \neq i} P_j$ and $(P_i + A_i)^* = R$. For each i, consider the ideal transform of A_i, $T_i = \bigcup_{n \geq 0}(R : A_i^n)$. Since A_i is finitely generated, the set $\mathcal{F}_i$ of the ideals $J \subseteq R$ such that $A_i^n \subseteq J$ for some n is a localizing system of finite type on R (for the definitions, see [19]). Moreover, we have $T_i = R_{\mathcal{F}_i}$. Hence the map $E \mapsto E^{\star_i} = E_{\mathcal{F}_i}$ on the set of nonzero R-submodules of K defines a semistar operation of finite type on R [19, Propositions 2.4 and 3.2], which induces a star operation of finite type on T_i ($T_i^{\star_i} = T_i$), here still denoted by $\star_i$. For more details on the concept of semistar operation, see [30].

Set $I_i = I_{\mathcal{F}_i} \cap R$. We claim that $\mathrm{rad}(I_i) = P_i$; in fact $\mathrm{rad}(I_{\mathcal{F}_i}) = (P_i)_{\mathcal{F}_i}$. To see this, first note that $I_{\mathcal{F}_i} \neq T_i$; otherwise, $A_i^n \subseteq I$ for some $n \geq 1$, which is impossible since $(A_i^n + P_i)^* = R$. Let $\mathcal{P} \subseteq T_i$ be a minimal prime of $I_{\mathcal{F}_i} = (IT_i)^{\star_i}$. Then $\mathcal{P}$ is a $\star_i$-ideal and $\mathcal{P} \cap R$ contains a minimal prime P_j of I. We have $(IT_i)^{\star_i} \subseteq (P_jT_i)^{\star_i} = (P_j)_{\mathcal{F}_i} \subseteq (\mathcal{P} \cap R)^{\star_i} \subseteq \mathcal{P}^{\star_i} = \mathcal{P}$. But for all $i \neq j$, $A_i \subseteq P_j$ and so $(P_j)_{\mathcal{F}_i} = T_i$. Hence P_i is the unique minimal prime of I contained in $\mathcal{P} \cap R$ and it follows that $\mathcal{P} = (P_i)_{\mathcal{F}_i}$. Hence $\mathrm{rad}(I_i) = P_i$. Also $\mathrm{rad}(I_i^*) = P_i$, since P_i is a $*$-ideal.

We next show that $I = (Q_1Q_2 \dots Q_n)^*$, where $Q_i = I_i^*$. For this, since the ideals $I_1, I_2 \dots, I_n$ are pairwise $*$-comaximal (by the $*$-comaximality of their radicals), it is enough to show that $I = (I_1 \cap I_2 \cap \dots \cap I_n)^*$. Let $M \in *$-Max(R). If $I \subseteq M$, then $P_i \subseteq M$ for some i. But since $(P_i + A_i)^* = R$, $A_i \nsubseteq M$ and so $I_{\mathcal{F}_i} \subseteq IR_M$. Hence $I \subseteq I_1 \cap I_2 \cap \dots \cap I_n \subseteq I_{\mathcal{F}_1} \cap I_{\mathcal{F}_2} \cap \dots \cap I_{\mathcal{F}_n} \subseteq \bigcap_{M \in *\text{-Max}(R), I \subseteq M} IR_M = I$ (since I is a $*$-ideal). It follows that $I = (I_1 \cap I_2 \cap \dots \cap I_n)^*$.

(iii) ⇒ (ii) is clear.

(ii) ⇒ (i). If $I = (Q_1Q_2 \ldots Q_n)^*$, a prime minimal over I must be minimal over one of the Q_i's. Hence I has finitely many minimal primes.

To finish, it is enough to observe that if I is $*$-invertible, the Q_i's are also $*$-invertible. □

Proposition 2.5. *Let R be a domain and let $*$ be a star operation of finite type on R. Assume that each nonzero principal ideal $X \subsetneq R$ can be written in the form $X = (X_1X_2 \ldots X_n)^*$, where the X_i's are pairwise $*$-comaximal $*$-packets. Then $*$-Spec(R) is treed.*

Proof. We proceed as in the case of $* = d$ established in [8, Theorem 1]. Assume that P and P' are two incomparable $*$-primes contained in the same $*$-maximal ideal M. Let $x \in P \setminus P'$ and $y \in P' \setminus P$. Write $xyR = (X_1X_2 \ldots X_n)^*$, where the $X_i's$ are pairwise $*$-comaximal $*$-packets. Hence M contains only one of the $X_i's$, say X_1. On the other hand, $xyR \subseteq P \cap P' \subseteq M$; hence $X_1 \subseteq P \cap P'$. Since X_1 has prime radical, then either x or y belongs to $P \cap P'$, a contradiction. It follows that $*$-Spec(R) is treed. □

The following theorem was proven for $* = d$ in [8, Theorem 2].

Theorem 2.6. *Let R be a domain and let $*$ be a star operation of finite type on R. The following conditions are equivalent:*

(i) $*$-Spec(R) *is treed and each proper $*$-ideal of R has finitely many minimal primes;*
(ii) *Each proper $*$-ideal I can be (uniquely) written in the form $I = (Q_1Q_2 \ldots Q_n)^*$, where the Q_i's are pairwise $*$-comaximal $*$-ideals with prime radical.*

Under any of these conditions, R is a $*$-URD.

Proof. If condition (ii) holds, $*$-Spec(R) is treed by Proposition 2.5. Then we can apply Proposition 2.4 and Theorem 2.3. □

If $*$-Spec(R) is treed and R has $*$-finite character, then each $*$-ideal $I \subsetneq R$ has only finitely many minimal primes. Thus the following corollary is immediate:

Corollary 2.7. *Let R be a domain and let $*$ be a star operation of finite type on R such that $*$-Spec(R) is treed and R has $*$-finite character. Then each $*$-ideal $I \subsetneq R$ can be (uniquely) written in the form $I = (Q_1Q_2 \ldots Q_n)^*$, where the Q_i's are pairwise $*$-comaximal $*$-ideals with prime radical.*

In particular R is a $*$-URD.

Since the t-Spectrum of a PVMD is treed, we also have the following generalization of Theorem 1.2.

Theorem 2.8. *Let R be a PVMD. The following conditions are equivalent for a t-ideal $I \subsetneq R$:*

(i) *I has finitely many minimal primes;*
(ii) *$I = (Q_1Q_2 \ldots Q_n)_t$, where the Q_i's are pairwise t-comaximal t-ideals with prime radical.*

We now give a complete characterization of $*$-URD's.

Lemma 2.9. *Let R be an integral domain and $*$ a star operation on R. Let $X = (X_1X_2 \ldots X_n)^*$ and $Y = (Y_1Y_2 \ldots Y_m)^*$ be $*$-factorizations of the $*$-ideals X and Y into $*$-ideals of R such that $(X_i + X_j)^* = (Y_i + Y_j)^* = R$ for $i \neq j$. Then $(X + Y)^* = (\prod_{i,j \geq 1}(X_i + Y_j))^*$.*

Proof. Let X, Y_1, Y_2 be ideals of R such that $(Y_1 + Y_2)^* = R$. Then $(X + Y_1Y_2)^* = ((X + Y_1)(X + Y_2))^*$. In fact we have $(X + Y_1)(X + Y_2) = X^2 + X(Y_1 + Y_2) + Y_1Y_2$. Hence $((X + Y_1)(X + Y_2))^* = ((X^2)^* + (X(Y_1 + Y_2))^* + (Y_1Y_2)^*)^* = ((X^2)^* + X^* + (Y_1Y_2)^*)^* = (X + Y_1Y_2)^*$.

The proof now follows by double induction. □

Lemma 2.10. *Let R be an integral domain and $*$ a star operation of finite type on R such that $*$-Spec(R) is treed. Let $I = (a_1R + a_2R + \ldots + a_nR)^*$ be a $*$-finite $*$-ideal and let P be a prime ideal such that $P \supseteq I$. Then P is minimal over I if and only if P is minimal over some a_iR.*

Proof. Assume that P is a minimal ($*$-)prime of I. Let $P_1, P_2, \ldots, P_n$ be a set of minimal primes of $a_1, a_2, \ldots, a_n$, respectively, such that $P_i \subseteq P$ for each i. Since $*$-Spec(R) is treed, the set $\{P_i\}_i$ is a finite chain. Up to order, we may assume that P_n is the maximal element of this chain. Then $I \subseteq P_n \subseteq P$, and hence $P = P_n$. Thus P is a minimal prime over a_nR. The converse is clear. □

Theorem 2.11. *Let R be an integral domain and $*$ a star operation of finite type on R. Then the following conditions are equivalent:*

(i) *R is a $*$-URD;*
(ii) *Each $*$-ideal of finite type $X \subsetneq R$ can be (uniquely) written in the form $X = (X_1X_2 \ldots X_n)^*$, where each X_i is a $*$-ideal with prime radical and the X_i's are pairwise $*$-comaximal $*$-ideals;*
(iii) *Each nonzero principal ideal $X \subsetneq R$ can be (uniquely) written in the form $X = (X_1X_2 \ldots X_n)^*$, where the X_i's are pairwise $*$-comaximal $*$-packets;*
(iv) *$*$-Spec(R) is treed and each $*$-ideal of finite type $X \subsetneq R$ has only finitely many minimal primes;*
(v) *$*$-Spec(R) is treed and each $*$-invertible $*$-ideal $X \subsetneq R$ has only finitely many minimal primes;*
(vi) *$*$-Spec(R) is treed and each nonzero principal ideal $X \subsetneq R$ has only finitely many minimal primes.*

Proof. The implications (ii) ⇒ (i) ⇒ (iii) and (iv) ⇒ (v) ⇒ (vi) are clear.

(iii) ⇒ (vi). By Proposition 2.5, $*$-Spec(R) is treed. Then, by Proposition 2.4, each nonzero principal ideal X has finitely many minimal primes.

(vi) ⇒ (iv) follows from Lemma 2.10.

(vi) ⇒ (iii). The factorization follows from Proposition 2.4 and the uniqueness follows from Theorem 2.3.

(iii) ⇒ (ii). We restrict ourselves to the case where $X = (aR + bR)^*$; the general case follows by induction on the number of generators of X as a $*$-finite $*$-ideal. Let $aR = (A_1A_2 \ldots A_n)^*$ and $bR = (B_1B_2 \ldots B_m)^*$ be the $*$-factorizations into $*$-ideals as in (iii). By Lemma 2.9, $X = (\prod_{i,j \geq 1}(A_i + B_j)^*)^*$. If $(A_i + B_j)^* \neq R$ for some i, j, then $\text{rad}(A_i) \subseteq \text{rad}(B_j)$ or $\text{rad}(B_j) \subseteq \text{rad}(A_i)$ (since, by (iii)⇒(vi), $*$-Spec(R) is treed), and hence $\text{rad}((A_i + B_j)^*) = \text{rad}(A_i) \vee \text{rad}(B_j)$ is a prime ideal. To conclude, we show that the $*$-ideals $X_{ij} = (A_i + B_j)^*$ are pairwise $*$-comaximal. Let i, j, k and l such that $(X_{ij} + X_{kl})^* \neq R$. Since $*$-Spec(R) is treed, the minimal primes over A_i, B_j, A_k and B_l, respectively, form a chain, this forces that $i = k$ and $j = l$. □

In the case $* = d$, the equivalence (iii) ⇔ (vi) of Theorem 2.11 was proved in [8, Theorem 1], without uniqueness consideration. When $* = t$, we get a complete characterization of URD's.

Corollary 2.12. *Let R be an integral domain. Then the following conditions are equivalent:*

(i) *R is a URD;*
(ii) *Each t-ideal of finite type $X \subsetneq R$ can be (uniquely) written in the form $X = (X_1X_2 \ldots X_n)_t$, where each X_i is a t-ideal with prime radical and the X_i's are pairwise t-comaximal t-ideals;*
(iii) *Each nonzero principal ideal $X \subsetneq R$ can be (uniquely) written in the form $X = (X_1X_2 \ldots X_n)_t$, where the X_i's are pairwise t-comaximal packets;*
(iv) *t-Spec(R) is treed and each t-ideal of finite type $X \subsetneq R$ has only finitely many minimal primes;*

(v) t-Spec(R) *is treed and each t-invertible t-ideal $X \subsetneq R$ has only finitely many minimal primes;*
(vi) t-Spec(R) *is treed and each nonzero principal ideal $X \subsetneq R$ has only finitely many minimal primes.*

Remark 2.13. (1). For a star operation $*$ of finite type, when the $*$-class group is zero (that is each $*$-invertible $*$-ideal is principal) a $*$-URD behaves like a GCD URD, i.e. every proper principal ideal is (uniquely) expressible as a product of finitely many t-comaximal principal ideals with a unique minimal prime. Thus a UFD is a GCD URD. However a UFD is not always a d-URD, since its spectrum need not be treed (just take $\mathbb{Z}[X]$).

(2). By using the characterizations of $*$-URD's given in Theorem 2.11, we get that, for any domain R, t-Spec(R) is treed if and only if w-Spec(R) is treed.

For this, recall that w-Max(R) $= t$-Max(R) and that, if M is a t-maximal ideal, all the primes contained in M are w-primes [34]. Now, assume that t-Spec(R) is treed and let $M \in t$-Max(R). Then t-Spec(R_M) is linearly ordered, because any t-prime of R_M contracts to a t-prime of R contained in M. Hence each principal ideal of R_M has a unique minimal (t-)prime and so is a d-packet. It follows that R_M is trivially a d-URD and in particular, Spec(R_M) is treed. Hence w-Spec(R) is treed. Conversely, since t-ideals are w-ideals, if w-Spec(R) is treed, also t-Spec(R) is treed.

We now show that a $*$-URD R whose $*$-packets are $*$-powers of $*$-prime ideals is indeed a Krull domain. For $* = d$ this implies that R is a Dedekind domain [8, Theorem 9].

Theorem 2.14. *Let R be an integral domain and $*$ a star operation of finite type on R. Then the following conditions are equivalent:*

(i) *R is a $*$-URD and each $*$-packet is a $*$-power of a $*$-prime ideal;*
(ii) *R is a Krull domain and $* = t$.*

Proof. (i) $\Rightarrow$ (ii). We first show that R is a P$*$MD (cf. [17]). Let $P \in *$-Spec(R) and let $0 \neq x \in P$ with $xR = (P_1^{\alpha_1} P_2^{\alpha_2} \dots P_r^{\alpha_r})^*$. Then, there exists i such that $P_i \subseteq P$. The ideal P_i, being a $*$-invertible $*$-prime, is a $*$-maximal ideal. Thus $P = P_i$. Hence each $*$-prime is $*$-maximal. It follows that each $M \in *$-Max(R) has height one. Also, since M is $*$-invertible, MR_M is a principal ideal. Hence R_M is a rank-one discrete valuation domain. Thus R is a P$*$MD.

By [17, Proposition 3.4], R is a PVMD and $* = t$. In particular, R is a URD of t-dimension one. Then $R = \bigcap\{R_M;\ M \in t\text{-Max}(R)\}$ has t-finite character. Whence R is a Krull domain.

(ii) $\Rightarrow$ (i) is well known. □

The conditions of Theorem 2.6 are not equivalent to R being a $*$-URD. In fact in [8] the authors give an example of a Prüfer URD having an ideal (not finitely generated) with infinitely many minimal primes. However we now show that such an example cannot be found among domains $*$-independent of $*$-finite character.

Given a star operation of finite type $*$ on a domain R, as in [6], we say that R is $*$-independent if, for every pair M, N of distinct $*$-maximal ideals, $M \cap N$ does not contain a nonzero ($*$-)prime ideal, equivalently each $*$-prime is contained in a unique $*$-maximal ideal. Domains $*$-independent of $*$-finite character are called h-local domains for $* = d$ and weakly Matlis domains for $* = t$.

A domain R is $*$-independent of $*$-finite character if and only if, for each nonzero nonunit $x \in R$, the ideal xR is expressible as a product $(I_1 I_2 \dots I_n)^*$, where the $I_j's$ are pairwise $*$-comaximal $*$-ideals (not necessarily packets) [6, Proposition 2.7]. Thus weakly Matlis domains do have a kind of factorization as a t-product of mutually t-comaximal ideals.

Theorem 2.15. *Let R be an integral domain and let $*$ be a star operation of finite type on R. Assume that each $*$-prime ideal is contained in a unique $*$-maximal ideal. The following conditions are equivalent:*

(i) $*$-Spec(R) *is treed and R has $*$-finite character;*
(ii) *R is a $*$-URD;*
(iii) *Each $*$-ideal $I \subsetneq R$ is uniquely expressible as a product $I = (Q_1 Q_2 \dots Q_n)^*$, where the Q_i's are pairwise $*$-comaximal $*$-ideals with prime radical.*

Proof. (i) $\Leftrightarrow$ (ii) Since each $*$-prime ideal is contained in a unique $*$-maximal ideal, the $*$-finite character property is equivalent to the condition that each nonzero principal ideal has at most finitely many minimal primes. Hence we can apply Theorem 2.11.

(i) $\Rightarrow$ (iii) is Corollary 2.7.

(iii) $\Rightarrow$ (ii) is clear. □

Corollary 2.16. *Let $*$ be a star operation of finite type on R and assume that R is $*$-independent of $*$-finite character. Then R is a $*$-URD if and only if* $*$-Spec(R) *is treed.*

In particular a weakly Matlis domain R is a URD if and only if t-Spec(R) *is treed.*

We shall give an example of a weakly Matlis domain that is not a URD in the next section (Example 3.2).

If R has t-dimension one, then clearly R is t-independent and t-Spec(R) is treed. If R has t-dimension one and t-finite character, then R is called a weakly Krull domain. The following corollary follows also from [5, Theorem 3.1].

Corollary 2.17. *Let R be a domain of t-dimension one. The following conditions are equivalent:*

(i) *R is a URD;*
(ii) *Each t-ideal $I \subsetneq R$ is expressible as a product $I = (Q_1 Q_2 \dots Q_n)_t$, where the Q_i's are pairwise t-comaximal t-ideals with prime radical;*
(iii) *R is a weakly Krull domain.*

From Corollary 2.17, we see that examples of URD's are abundant in the form of weakly Krull domains as rings of algebraic integers of finite extensions of the field of rational numbers or as extensions $K_1 + XK_2[X]$ where $K_1 \subseteq K_2$ is an extension of fields.

The following proposition is an improvement of Corollary 1.10.

Proposition 2.18. *Let R be a ring of Krull type. Then each proper t-ideal I of R can be uniquely written in the form $I = (Q_1 Q_2 \dots Q_n)_t$, where the Q_i's are pairwise t-comaximal ideals with prime radical. In particular R is a URD.*

Proof. Follows from Corollary 2.7. □

From Theorem 2.15 we also get:

Proposition 2.19. *Let R be a PVMD such that each t-prime ideal is contained in a unique t-maximal ideal. The following conditions are equivalent:*

(i) *R is a ring of Krull type;*
(ii) *R is a weakly Matlis domain;*
(iii) *R is a URD;*
(iv) *For each nonzero nonunit $x \in R$, the ideal xR is uniquely expressible as a product $(I_1 I_2 \dots I_n)_t$, where the I_j's are pairwise t-comaximal ideals;*
(v) *Each t-ideal $I \subsetneq R$ is uniquely expressible as a product $(Q_1 Q_2 \dots Q_n)_t$, where the Q_i's are pairwise t-comaximal ideals with prime radical.*

Recall that a domain R is an almost Krull domain if R_M is a DVR for each t-maximal ideal M. Thus a Krull domain is precisely an almost Krull domain with t-finite character.

Corollary 2.20. *An almost Krull domain is a URD if and only if it is a Krull domain.*

We end this section with a discussion of factorization of t-ideals in generalized Krull domains.

We recall that a Prüfer domain R is called strongly discrete if each nonzero prime ideal P of R is not idempotent and that a generalized Dedekind domain is a strongly discrete Prüfer domain such that every nonzero principal ideal of R has at most finitely many minimal primes [32]. The first author gave a generalization of these domains introducing the class of generalized Krull domains. A generalized Krull domain is defined as a PVMD R such that $(P^2)_t \neq P$, for every prime t-ideal P of R, i. e., R is a strongly discrete PVMD, and every nonzero principal ideal of R has at most finitely many minimal primes [13, 14]. Here it may be noted that this generalized Krull domain is quite different from the generalized Krull domain of Ribenboim [33] and a generalized Dedekind domain of Popescu [32] is quite different from what the third author called a generalized Dedekind domain in [36]. Incidentally these later domains were also studied under the name of pseudo Dedekind domains by Anderson and Kang in [4].

Since a generalized Krull domain is a PVMD such that each t-ideal has at most finitely many minimal primes [13, Theorem 3.9], from Theorem 2.8, we immediately get:

Proposition 2.21. *If R is a generalized Krull domain, then each t-ideal $I \subsetneq R$ is uniquely expressible as a product $I = (Q_1Q_2 \dots Q_n)_t$, where the Q_i's are pairwise t-comaximal ideals with prime radical.*

By the previous proposition, we see that a generalized Krull domain is a URD. As a matter of fact, taking into account Theorem 1.9, we have the following characterization.

Proposition 2.22. *A PVMD R is a generalized Krull domain if and only if R is a URD and $(P^2)_t \neq P$, for each nonzero prime t-ideal P of R.*

For generalized Dedekind domains, Proposition 2.21 was proved with different methods by the second author of this paper and N. Popescu in [20, Proposition 2.4]. They also showed that a nonzero ideal I of a generalized Dedekind domain is divisorial if and only if $I = JP_1 \dots P_n$, where J is a fractional invertible ideal and $P_1, \dots, P_n$ $(n \geq 1)$ are pairwise comaximal prime ideals [20, Proposition 3.2]. This result was sharpened by B. Olberding, who proved that R is a generalized Dedekind domain whose ideals are all divisorial, equivalently an h-local strongly discrete Prüfer domain [15, Corollary 3.6], if and only if each ideal I of R is a product of finitely generated and prime ideals [31, Theorem 2.3]. Passing through the t-Nagata ring, we now extend Olberding's result to generalized Krull domains. A first step in this direction was already done by the first author, who proved that, for each nonzero ideal I of a generalized Krull domain, I is divisorial if and only if $I = (JP_1 \dots P_n)_t$, where $J \subseteq R$ is a finitely generated ideal and $P_1, \dots, P_n$ $(n \geq 1)$ are pairwise t-comaximal t-prime ideals [14, Theorem 3.4].

As in [26], we set $N(t) = \{h \in R[X] \mid h \neq 0 \text{ and } c(h)_t = R\}$, where $c(h)$ denotes the content of h, and call the domain $R\langle X\rangle := R[X]_{N(t)}$ the t-Nagata ring of R. B. G. Kang proved that R is a PvMD if and only if $R\langle X\rangle$ is a Prüfer (indeed a Bezout) domain [26, Theorem 3.7]. In this case, the map $I_t \mapsto IR\langle X\rangle$ is a lattice isomorphism between the lattice of t-ideals of R and the lattice of ideals of $R\langle X\rangle$, whose inverse is the map $J \mapsto J \cap R$, and P is a t-prime (respectively, t-maximal) ideal of R if and only if $PR\langle X\rangle$ is a prime (respectively, maximal) ideal of $R\langle X\rangle$ [26, Theorem 3.4]. We recall that a domain is called w-divisorial if each w-ideal is divisorial [15]. Each w-divisorial domain is a weakly Matlis domain [15, Theorem 1.5] and a w-divisorial generalized Krull domain is precisely a weakly Matlis domain that is a strongly discrete PVMD [15, Theorem 3.5]. We also recall that, when R is a PVMD, $w = t$ [26, Theorem 3.1].

Theorem 2.23. *Let R be a domain. The following conditions are equivalent:*

(i) *R is a w-divisorial generalized Krull domain;*
(ii) *For each nonzero ideal $I \subsetneq R$, $I_w = (JP_1 \dots P_n)_w$, where $J \subseteq R$ is finitely generated and $P_1, \dots, P_n$ $(n \geq 1)$ are pairwise t-comaximal t-prime ideals;*
(iii) *For each nonzero ideal $I \subsetneq R$, $I_w = (J_1 \dots J_m P_1 \dots P_n)_w$, where $J_1, \dots, J_m$ are mutually t-comaximal packets and $P_1, \dots, P_n$ $(n \geq 1)$ are pairwise t-comaximal t-prime ideals.*

Proof. (i) ⇒ (ii) and (iii). Since I_w is divisorial and $w = t$, (ii) follows from [14, Theorem 3.4]. The statement (iii) follows from (ii) and Proposition 2.21.

(iii) ⇒ (ii) is clear.

(ii) ⇒ (i). Assume that the factorization holds. If $M \in t\text{-Max}(R)$, then each nonzero proper ideal of R_M is of type IR_M, for some ideal I of R. Let $I' = IR_M \cap R$, by (ii), $I'_w = (JP_1 \dots P_n)_w$, where $J \subseteq R$ is finitely generated and $P_1, \dots, P_n$ $(n \geq 1)$ are pairwise t-comaximal t-prime ideals. Then $IR_M = I'R_M = (JR_M)(P_1R_M) \dots (P_nR_M)$. We claim that there is an i such that $P_iR_M \neq R_M$. Otherwise, $IR_M = JR_M$, and hence $J \subseteq I' = JR_M \cap R$, so $J_w \subseteq I'_w = (JP_1 \dots P_n)_w \subseteq J_w$. Hence $J_w = (JP_1 \dots P_n)_w$. Let $N \in t\text{-Max}(R)$ such that $P_1 \subseteq N$, then $JR_N = JP_1R_N$, which is impossible by the Nakayama lemma. Hence IR_M is a product of a finitely generated ideal and prime ideals. Thus R_M is a strongly discrete valuation domain by [31, Theorem 2.3]. It follows that R is a strongly discrete PVMD.

To show that R is weakly Matlis, it is enough to show that the t-Nagata ring $R\langle X\rangle$ is h-local [16, Theorem 2.12]. But, since R is a PVMD, each nonzero ideal of $R\langle X\rangle$ is of type $IR\langle X\rangle = I_tR\langle X\rangle = I_wR\langle X\rangle$ for some ideal I of R [26, Theorem 3.14]. Hence each nonzero ideal of $R\langle X\rangle$ is a product of a finitely generated ideal and prime ideals. Again from [31, Theorem 2.3] it follows that $R\langle X\rangle$ is h-local. □

3. Extensions and Examples of URD's

In this section we shall discuss ways of constructing examples of URD's. In addition to showing that a ring of fractions of a URD is again a URD we identify the conditions under which a polynomial ring over a URD is again a URD. Using these two main tools we then delve into the polynomial ring construction $D^{(S)} = D + XD_S[X]$. We show that if D and $D_S[X]$ are URD's then $D^{(S)}$ is a URD if and only if $t\text{-Spec}(D^{(S)})$ is treed. This indeed leads to several interesting results.

Since polynomial ring constructions are our main concern here we start off with a study of polynomial rings over URD's. We recall from [24] and [25] that, for each nonzero ideal I of a domain D, we have that $I_t[X] = I[X]_t$ is a t-ideal of $D[X]$ and that the contraction to D of a t-ideal of $D[X]$ which is not an upper to zero is a t-ideal of D. In addition, each extended t-maximal ideal of $D[X]$ is of type $M[X]$, with $M \in t\text{-Max}(D)$. Note that $f(X) \in M[X]$ if and only if $c(f)_t \subseteq M$, where $c(f)$ denotes the content of $f(X)$, that is the ideal of D generated by the coefficients of $f(X)$.

Proposition 3.1. *Let D be a domain and X an indeterminate over D.*

(1) *If $D[X]$ is a URD, then D is a URD.*
(2) *If D is a URD, then $D[X]$ is a URD if and only if $t\text{-Spec}(D[X])$ is treed.*

Proof. (1). This follows easily from Corollary 2.12, (i) ⇔ (vi) and the fact that, for each $P \in t\text{-Spec}(D)$, $P[X]$ is a t-ideal of $D[X]$.

(2). By Corollary 2.12, (i) $\Leftrightarrow$ (vi), we have only to show that if D is a URD and t-Spec($D[X]$) is treed, then each nonzero polynomial $f(X) \in D[X]$ has at most finitely many minimal primes.

If $f(X) \in D[X]$ is contained in the upper to zero $p(X)K[X] \cap D[X]$ of $D[X]$, then $p(X)$ divides $f(X)$ in $K[X]$. Thus $f(X)$ is contained in at most finitely many uppers to zero.

Assume that $f(X)$ has a minimal prime P such that $P \cap D \neq (0)$ and let N be a t-maximal ideal of $D[X]$ containing P. Then $N = M[X]$, with $M \in t$-Max(D) and $c(f)_t \in M$, in particular $c(f)_t \neq D$. Let $Q \subseteq M$ be a minimal prime of $c(f)_t$. We have $f(X) \in c(f)_t[X] \subseteq Q[X]$ and $Q[X]$ is a t-ideal of $D[X]$. Then $P \subseteq Q[X]$ (since t-Spec($D[X]$) is treed) and P is the only minimal prime of $f(X)$ contained in $Q[X]$. But, since D is a URD, $c(f)_t$ has only finitely many minimal primes (Corollary 2.12). We conclude that $f(X)$ has finitely many minimal primes. □

Example 3.2. An explicit example of a URD D such that $D[X]$ is not a URD is given by $D = \overline{\mathbb{Q}} + T\mathbb{R}[T]_{(T)}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of the rational field $\mathbb{Q}$ in the real field $\mathbb{R}$ and T is an indeterminate over $\mathbb{R}$. Note that D is quasilocal and one dimensional with maximal ideal $M = T\mathbb{R}[T]_{(T)}$. The quotient field of D is $K = \mathbb{R}(T)$ and the t-maximal ideal $M[X]$ of $D[X]$ contains the uppers to zero $P_u = (X - u)K[X] \cap D[X]$, for all $u \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. Indeed, let $u \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. Since D is integrally closed, by [21, Corollary 34.9], $P_u = (X - u)(D : D + uD)[X]$. Clearly, $(D : D + uD)$ is a proper t-ideal of D and $M \subseteq (D : D + uD)$, and hence $M = (D : D + uD)$. Thus $P_u = (X - u)M[X] \subseteq M[X]$, as desired. Now, since uppers to zero are height one t-prime ideals, we conclude that t-Spec($D[X]$) is not treed.

We note that both D and $D[X]$ are weakly Matlis domains. In fact, if a domain D has a unique t-maximal ideal M, then $D[X]$ is a weakly Matlis domain. Indeed, a t-prime of $D[X]$ is either a height one t-maximal upper to zero or it is contained in $M[X]$. In addition, since a nonzero polynomial is contained in at most finitely many uppers to zero, $D[X]$ has t-finite character.

The previous example can be generalized in the sense of the following result. We are thankful to David Dobbs for providing Proposition 3.3 to the third author. Evan Houston also helped.

Proposition 3.3. *[D. Dobbs] Let D be a quasilocal domain with maximal ideal M and X an indeterminate over D. If the integral closure of D is not a Prüfer domain, then $M[X]$ contains infinitely many uppers to zero.*

Proof. Let K be the quotient field of D. By [11, Theorem], there exists an element $u \in K$ such that $D \subset D[u]$ does not satisfy incomparability (INC). On the other hand, for each positive integer n, notice that $D[u^n] \subset D[u]$ is integral and hence satisfies INC. As the composite of INC extensions is an INC extension, it follows that $D \subset D[u^n]$ does not satisfy INC. Since u is not integral over D (because integral implies INC), u is not a root of unity, and so the elements u^n are all distinct, for $n \geq 1$. Thus, there are infinitely many elements $x \in K$ (the various u^n) such that $D \subset D[x]$ does not satisfy INC. Equivalently, there are infinitely many $x \in K$ such that $I_x := \ker(D[X] \to D[x]) \subset M[X]$. Each such I_x is an upper to zero (by the First Isomorphism Theorem). Moreover, distinct x always lead to distinct I_x. Indeed, if x and y are distinct nonzero elements of K, then I_x and I_y are distinct, since if $y = a/b$ for some nonzero elements a and b of D, we have $bX - a$ in I_y but not in I_x. □

Dobbs' result allows us to characterize unique representation polynomial rings.

We recall that a domain D is called a UMT-domain if each upper to zero P of $D[X]$ is a t-maximal ideal [25]; this is equivalent to say that D_P has Prüfer integral closure for each $P \in t$-Spec(D) [18, Theorem 1.5]. A PVMD is precisely an integrally closed UMT-domain [25, Proposition 3.2]. The UMT-property is preserved by polynomial extensions [18, Theorem 2.4].

Theorem 3.4. *Let D be a domain and X an indeterminate over D. The following conditions are equivalent:*

(i) *$D[X]$ is a URD;*
(ii) *D is a UMT-domain and a URD.*

Proof. (i) $\Rightarrow$ (ii). Assume that $D[X]$ is a URD and let $P \in t$-Spec(D). We claim that D_P has Prüfer integral closure. Deny, by Proposition 3.3, the ideal $PD_P[X]$ in $D_P[X]$ contains infinitely many uppers to zero, say $\{Q_\alpha\}$. For each α, set $P_\alpha = Q_\alpha \cap D[X]$. Then the P_α's are distinct uppers to zero of $D[X]$ contained in $P[X]$, which is impossible since t-Spec$(D[X])$ is treed. It follows that D is a UMT-domain. The second assertion follows from Proposition 3.1.

(ii) $\Rightarrow$ (i). Assume that D is a URD. By Proposition 3.1, we have only to show that t-Spec$(D[X])$ is treed. Since D is a UMT-domain, uppers to zero of D are t-maximal ideals of height one. Also, if Q is a prime t-ideal of $D[X]$ with $Q \cap D = q \neq (0)$, then $Q = q[X]$ (cf. proof of Theorem 3.1 in [25]). Since q does not contain any two incomparable prime t-ideals, Q does not contain any two incomparable prime t-ideals. □

Corollary 3.5. *Let D be a domain and X an indeterminate over D. D is a PVMD and a URD if and only if $D[X]$ is a PVMD and a URD.*

To give more examples of URD's, we will consider domains of the form $D^{(S)} = D + XD_S[X]$, where S is a multiplicative set of D, in particular of the form $D + XK[X]$, where K is the quotient field of D [10]. It was shown in [35, Proposition 7] that if D is a GCD domain and S is a multiplicative set such that $D^{(S)}$ is a GCD domain, then $D^{(S)}$ is a URD if and only if D is a URD. It is not clear how the $D^{(S)}$ construction will fare in the general case; however the case when $D^{(S)}$ is a PVMD appears to be somewhat straightforward.

Let us call a domain R a pre-Krull domain if every nonzero principal ideal of R has at most finitely many minimal primes. So R is a URD if and only if R is pre-Krull and t-Spec(R) is treed (Corollary 2.12).

Lemma 3.6. *Let R be an integral domain and S a multiplicative set of R.*

(1) *If R is pre-Krull then R_S is pre-Krull.*
(2) *If R is a URD then R_S is a URD.*

Proof. (1). A nonzero principal ideal of R_S can be written as xR_S, where $x \in R\backslash\{0\}$ and $xR \cap S = \emptyset$. If Q is a minimal prime of xR_S, then $Q \cap R$ is a minimal prime of xR.

(2). By part (1) and Corollary 2.12 it is enough to show that t-Spec(R_S) is treed. Note that if M is a prime t-ideal in R_S then $M \cap R$ is a prime t-ideal in R [39, page 436]. Now any two prime t-ideals P and Q of R_S contained in M are comparable because $P \cap R$ and $Q \cap R$ are comparable. □

Next we investigate other examples of URD's using the $D + XD_S[X]$ construction.

Lemma 3.7. *Let D be a domain, S a multiplicative subset of D and X an indeterminate over D. Let I be an ideal of $D^{(S)}$ such that $I \cap S \neq \emptyset$. Then $I = JD^{(S)} = J + XD_S[X]$ for some ideal J of D with $J \cap S \neq \emptyset$. Moreover, $I_t = J_t + XD_S[X]$.*

Proof. Since $I \cap S \neq \emptyset$, we have $XD_S[X] \subseteq I$, hence $I = J + XD_S[X] = JD^{(S)}$ for some ideal J of D with $J \cap S \neq \emptyset$.

For the second statement, we first show that if A is a finitely generated ideal of D such that $A \cap S \neq \emptyset$, then $(AD^{(S)})_v = A_v D^{(S)}$. By [38, Lemma 3.1], $(AD^{(S)})^{-1} = A^{-1}D^{(S)}$, so $A_v D^{(S)}(AD^{(S)})^{-1} \subseteq D^{(S)}$, and hence $A_v D^{(S)} \subseteq (AD^{(S)})_v$. For the reverse inclusion, let $f = a_0 + Xg(X) \in (AD^{(S)})_v$.

Then $f(AD^{(S)})^{-1} \subseteq D^{(S)}$, that is, $fA^{-1}D^{(S)} \subseteq D^{(S)}$. In particular, $a_0A^{-1} \subseteq D$. So $a_0 \in A_v$. Thus $f \in A_v + XD_S[X] = A_vD^{(S)}$. Hence $(AD^{(S)})_v = A_vD^{(S)}$.

Next, note that if F is a nonzero finitely generated subideal of $I = J + XD_S[X]$ such that $F \cap S = \emptyset$ then for any $s \in J \cap S$ we have $F_v \subseteq (F, s)_v = (A + XD_S[X])_v = A_v + XD_S[X] = A_vD^{(S)}$, for some finitely generated ideal $A \subseteq J$ of D. So $I_t = (JD^{(S)})_t = \bigcup\{(AD^{(S)})_v;\ A \subseteq J$ finitely generated and $A \cap S \neq \emptyset\} = \bigcup\{A_vD^{(S)};\ A \subseteq J$ finitely generated and $A \cap S \neq \emptyset\} = J_tD^{(S)} = J_t + XD_S[X]$. □

Theorem 3.8. *Let D be a domain, S a multiplicative subset of D and X an indeterminate over D. Then $D^{(S)} = D + XD_S[X]$ is a pre-Krull domain if and only if D and $D_S[X]$ are pre-Krull domains.*

Proof. Suppose that D and $D_S[X]$ are pre-Krull domains. Note that each prime ideal of $D^{(S)}$ comes from either of the following disjoint sets of primes: $\mathcal{L} = \{P \in \text{Spec}(D^{(S)}) : P \cap S \neq \emptyset\}$ and $\mathcal{M} = \{P \in \text{Spec}(D^{(S)}) : P \cap S = \emptyset\}$. Now let $f(X) \in D^{(S)}$, $f(X) \neq 0$. Then $f(X) = a_0 + Xg(X)$ where $g(X) \in D_S[X]$ and $a_0 \in D$. We have two cases: (i) $a_0 \neq 0$ and (ii) $a_0 = 0$.

(i) Assume that $a_0 \neq 0$ and let P be a prime ideal minimal over $f(X)$. (a) Suppose first that $P \in \mathcal{L}$. We claim that $P = p + XD_S[X]$, where p is a prime minimal over a_0. This is because $P \cap S \neq \emptyset$ and so $P = p + XD_S[X]$ (Lemma 3.7) and in addition $Xg(X) \in XD_S[X]$, which leaves $a_0 \in p$. If p is not minimal over a_0D and say $q \subsetneq p$ is, then $q + XD_S[X]$ would contain $f(X)$ and be properly contained in P contradicting the minimality of P. Now as a_0D has finitely many minimal primes we conclude that a finite number of minimal primes of f come from $\mathcal{L}$. (b) Suppose then that $P \in \mathcal{M}$. Since $P \cap S = \emptyset$ we have $P = PD_S[X] \cap D^{(S)}$ [10, pp. 426-427]. So P is a minimal prime of $f(X)$ if and only if $PD_S[X]$ is a minimal prime of $f(X)D_S[X]$. Since $D_S[X]$ is pre-Krull, $f(X)$ has finitely many minimal primes from $\mathcal{M}$. Combining (a) and (b), we conclude that $f(X)$ has finitely many minimal primes.

(ii) If $a_0 = 0$ then $f(X) = \frac{X^r}{s}g(X)$ where $g(X) \in D^{(S)}$ and $s \in S$. Now as $f(X) \in XD_S[X]$, no prime ideal that intersects S nontrivially can be minimal over $f(X)$. So the number of prime ideals minimal over $f(X)$ is the same as the number of prime ideals minimal over $fD_S[X]$ which is finite because $D_S[X]$ is pre-Krull.

Combining (i) and (ii) we have the conclusion.

For the converse suppose that $D^{(S)}$ is pre-Krull. Then, as S is a multiplicative set in $D^{(S)}$ and as $D_S[X] = D^{(S)}_S$, we conclude that $D_S[X]$ is pre-Krull (Lemma 3.6). Now let d be a nonzero nonunit of D. We claim that dD has finitely many minimal primes. Suppose on the contrary that dD has infinitely many minimal primes: for such a minimal prime Q, we have two cases (i) $Q \cap S \neq \emptyset$ or (ii) $Q \cap S = \emptyset$.

In case (i) for every minimal prime Q of dD, $QD^{(S)} = Q + XD_S[X]$ is a minimal prime over $dD^{(S)}$. But as $D^{(S)}$ is pre-Krull, there can be only finitely many minimal primes of $dD^{(S)}$. Consequently, there are only finitely many minimal primes Q of dD with $Q \cap S \neq \emptyset$. In case (ii) $Q \cap S = \emptyset$, then QD_S is a minimal prime of dD_S and so $QD_S[X]$ is a minimal prime of $dD_S[X]$. But $dD_S[X]$ has a finite number of minimal primes, because $D_S[X]$ is pre-Krull, a contradiction. □

Corollary 3.9. *Let D be a domain, S a multiplicative subset of D and X an indeterminate over D. If D and $D_S[X]$ are URD's, then $D^{(S)}$ is a URD if and only if t-Spec$(D^{(S)})$ is treed.*

Proof. By Corollary 2.12 and Theorem 3.8. □

Corollary 3.10. *Let D be a domain with quotient field K and X an indeterminate over D. Then $R = D + XK[X]$ is a URD if and only if D is a URD.*

Proof. This follows Lemma 3.7, Theorem 3.8 and Corollary 3.9. □

The following is an example of D and $D_S[X]$ both URD's without $D^{(S)}$ being a URD.

Example 3.11. Let V be a discrete rank two valuation domain with principal maximal ideal m. Set $p = \bigcap m^n$ (such that V_p is a DVR). Then $R = V + XV_p[X]$ is a non-URD pre-Krull domain with V and $V_p[X]$ both URD's.

We show that t-Spec(R) is not treed. For this we note that the maximal ideal $m + XV_p[X]$ of R contains two prime ideals $P_1 = XV_p[X]$ and $P_2 = pV_p[X] \cap R$. Both P_1 and P_2 are t-ideals because they are contractions of two t-ideals of $V_p[X] = R_{V \setminus p}$. Since, in $V_p[X]$, both $pV_p[X]$ and $XV_p[X]$ are distinct principal prime ideals and hence incomparable, P_1 and P_2 are incomparable.

Remark 3.12. Example 3.11 shows that if D and $D_S[X]$ are URD's, $D^{(S)}$ is not in general a URD. We next give a partial answer to when the converse is true.

Recall that a t-prime ideal P of a domain R is well-behaved if PR_P is a t-prime of R_P and that R is well-behaved if each t-prime of R is well-behaved [38]. Clearly, if P is a well behaved t-prime such that $P \cap S = \emptyset$, then PD_S is a t-prime.

Now, suppose that $D^{(S)}$ is a URD. Since URD's are stable under quotients (Lemma 3.6), $D_S[X] = D_S^{(S)}$ is a URD. We know that D is pre-Krull (Theorem 3.8). If moreover D is a well behaved domain, then t-Spec(D) is treed and so D is a URD. Indeed, let $M \in t$-Max(D) and let P, Q be two t-primes contained in M. If $M \cap S = \emptyset$, as D is well behaved, then MD_S is a prime t-ideal and so are PD_S, $QD_S \subseteq MD_S$. Since $D_S[X]$ is a URD, D_S is also a URD (Proposition 3.1). Hence PD_S, QD_S are comparable and so are P and Q. If $M \cap S \neq \emptyset$ then $M + XD_S[X]$ is a t-ideal by Lemma 3.7. Now $P + XD_S[X]$ and $Q + XD_S[X]$ are prime ideals contained in $M + XD_S[X]$ and because w-Spec$(D^{(S)})$ is treed (Remark 2.13(2)) we conclude that $P + XD_S[X]$ and $Q + XD_S[X]$ are comparable. Hence P and Q are comparable. So t-Spec(D) is treed.

Corollary 3.13. *Let D be a PVMD, X an indeterminate over D and S a multiplicative subset of D such that $D^{(S)}$ is a PVMD. Then $D^{(S)}$ is a URD if and only if D is a URD.*

Proof. Suppose that $D^{(S)}$ is a URD. Since a PVMD is a well behaved domain [38], that D is a URD follows from Remark 3.12. Conversely, if D is a URD, $D_S[X]$ is a URD by Corollary 3.5. Applying Corollary 3.9 we have that $D^{(S)}$ a URD. □

We have seen in Proposition 2.21 that a generalized Krull domain D is a URD. Thus if D is a generalized Krull domain and $D^{(S)}$ is a PVMD, then $D^{(S)}$ is a URD. We now show that if $D^{(S)}$ is a PVMD then it is actually a generalized Krull domain.

Proposition 3.14. *Let D be a generalized Krull domain and let S be a multiplicative set in D such that $D^{(S)}$ is a PVMD. Then $D^{(S)}$ is a generalized Krull domain.*

Proof. Let P be a prime t-ideal of $D^{(S)} = D + XD_S[X]$ where D is a generalized Krull domain and S is such that $D^{(S)}$ is a PVMD. Let us assume that $D^{(S)}$ is not a generalized Krull domain. Now every principal ideal of $D^{(S)}$ has finitely many minimal primes (Theorem 3.8); in fact both D and $D_S[X]$ are generalized Krull domains [13, Theorem 4.1] and a generalized Krull domain is pre-Krull [13, Theorem 3.9]. So $D^{(S)}$ not being a generalized Krull domain would require that there is a prime t-ideal P such that $(P^2)_t = P$. If $P \cap S \neq \emptyset$ then $P = p + XD_S[X] = pD^{(S)}$ for some t-prime ideal p of D with $p \cap S \neq \emptyset$ (Lemma 3.7). Now $P^2 = p^2 + XD_S[X]$ and again by Lemma 3.7 $(P^2)_t = (p^2)_t + XD_S[X]$. Now $(p^2)_t + XD_S[X] = p + XD_S[X]$ forces, by degree considerations, $(p^2)_t = p$. But p is a prime t-ideal of D which is assumed to be a generalized Krull domain, contradiction. Hence P does not intersect S nontrivially. Next, since $D^{(S)}$ is a PVMD for every nonzero ideal A of $D^{(S)}$, with $A \cap S = \emptyset$, we have $(A(D^{(S)})_S)_t = A_t(D^{(S)})_S = A_t D_S[X]$. Setting $A = P^2$, we get $(P^2(D^{(S)})_S)_t = (P^2)_t(D^{(S)})_S = (P^2)_t D_S[X] = PD_S[X]$. As $P^2(D^{(S)})_S = (P(D^{(S)})_S)^2 = (PD_S[X])^2$ we have $((PD_S[X])^2)_t = PD_S[X]$. But as $D_S[X]$ is a generalized Krull domain [13, Theorem 4.1] and $PD_S[X]$ a prime t-ideal of $D_S[X]$ this is impossible. Having exhausted all the cases we conclude that $D^{(S)}$ is a generalized Krull domain. □

Since a Krull domain is a (Ribenboim) generalized Krull domain, by Corollary 2.7 of [2] for any multiplicative set S of a Krull domain D we have that $D^{(S)}$ is a PVMD. Hence we have the following consequence.

Corollary 3.15. *If D is a Krull domain and S is any multiplicative set of D, then $D^{(S)}$ is a generalized Krull domain, and hence a URD.*

Acknowledgments

Part of this work was done while the first named author was visiting the Department of Mathematics of Università degli Studi "Roma Tre", supported by an INdAM visiting grant.

We are grateful to the referee for making a number of insightful suggestions which helped improve the presentation of the paper a great deal.

References

[1] D.D. Anderson, Non-atomic unique factorization in integral domains, Arithmetical properties of commutative rings and monoids, 1–21, Lect. Notes Pure Appl. Math., 241, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[2] D.D. Anderson, D.F. Anderson and M. Zafrullah, The ring $D + XD[1/S][X]$ and t-splitting sets, Commutative algebra. Arab. J. Sci. Eng. Sect. C Theme Issues 26 no. 1 (2001), 3–16.

[3] D.D. Anderson and S. Cook, Two star-operations and their induced lattices, Comm. Algebra 28 (2000), 2461–2475.

[4] D.D. Anderson and B. Kang, Pseudo-Dedekind domains and divisorial ideals in $R[X]_T$, J. Algebra 122 (1989), 323–336.

[5] D.D. Anderson, J. Mott and M. Zafrullah, Finite character representations for integral domains, Bollettino U.M.I. (7) 6-B (1992), 613–630.

[6] D.D. Anderson and M. Zafrullah, Independent locally finite intersections of localizations, Houston J. Math. 25(1999) 433-452

[7] A. Bouvier and M. Zafrullah, On some class groups of an integral domain, Bull. Soc. Math. Grèce (N.S.) 29 (1988), 45–59

[8] J. W. Brewer and W. J. Heinzer, On decomposing ideals into products of comaximal ideals, Comm. Algebra 30 (2002), 5999–6010.

[9] P. Cohn, Bezout rings and their subrings, Proc. Cambridge Philos. Soc. 64 (1968), 251–264.

[10] D. Costa, J. Mott and M. Zafrullah, The construction $D + XD_S[X]$, J. Algebra 53 (1978), 423–439.

[11] D. Dobbs, On INC-extensions and polynomials with unit content, Canad. Math. Bull. 23 (1980), 37–42.

[12] T. Dumitrescu and R. Moldovan, Quasi-Schreier domains, Math. Rep. (Bucur.) 5(55) (2003), 121–126.

[13] S. El Baghdadi, On a class of Prüfer v-multiplication domains, Comm. Algebra 30 (2002), 3723–3742.

[14] S. El Baghdadi, Factorization of divisorial ideals in a generalized Krull domain, Rings, modules, algebras, and abelian groups, 149–160, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, 2004.

[15] S. El Baghdadi and S. Gabelli, w-divisorial domains, J. Algebra 285 (2005), 335–355.

[16] S. El Baghdadi and S. Gabelli, Ring-theoretic properties of PvMDs, Comm. Algebra, to appear.

[17] M. Fontana, P. Jara and E. Santos, Prüfer $\star$-multiplication domains and semistar operations, J. Algebra Appl. 2 (2003), 21–50.

[18] M. Fontana, S. Gabelli and E. Houston, UMT-domains and domains with Prüfer integral closure, Comm. Algebra 26 (1998), 1017–1039.

[19] M. Fontana and J.A. Huckaba, Localizing systems and semistar operations, Non-Noetherian commutative ring theory, 169–197, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[20] S. Gabelli and N. Popescu, Invertible and divisorial ideals of generalized Dedekind domains, J. Pure Appl. Algebra 135 (1999), 237–251.

[21] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, 1972.

[22] M. Griffin, Some results on Prüfer v-multiplication rings, Canad. J. Math. 19 (1967), 710–722.

[23] M. Griffin, Rings of Krull type, J. Reine Angew. Math. 229 (1968), 1–27.

[24] J. Hedstrom and E. Houston, Some remarks on star operations, J. Pure Appl. Algebra 18 (1980), 37–44.

[25] E. Houston and M. Zafrullah, On t-invertibility II, Comm. Algebra 17 (1989), 1955–1969.

[26] B. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra 123 (1989), 151–170.

[27] I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.

[28] S. McAdam and R. G. Swan, Unique Comaximal factorization, J. Algebra 276(1) (2004), 180–192.

[29] J. Mott and M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35 (1981), 1–26.

[30] A. Okabe and R. Matsuda, Semistar-operations on integral domains, Math. J. Toyama Univ. 17 (1994), 1–21.

[31] B. Olberding, Factorization into prime and invertible ideals, J. London Math. Soc. 62 (2000), 336–344.

[32] N. Popescu, On a class of Prüfer domains, Rev. Roumaine Math. Pures Appl. 29 (1984), 777–786.

[33] P. Ribenboim, Anneaux normaux réels a caractère fini, Summa Brasil. Math. 3 (1956), 213–253.

[34] F. Wang and R. McCasland, On w-modules over strong Mori domains, Comm. Algebra 25 (1997), 1285–1306.

[35] M. Zafrullah, On unique representation domains, J. Natur. Sci. and Math. 18 (1978), 19–29.

[36] M. Zafrullah, On generalized Dedekind domains, Mathematika 33 (1986), no. 2(1987), 285–295.

[37] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15 (1987), no. 9, 1895–1920

[38] M. Zafrullah, Well behaved prime t-ideals, J. Pure Appl. Algebra 65 (1990) 199–207.

[39] M. Zafrullah, Putting t-invertibility to use, Non-Noetherian commutative ring theory, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

Department of Mathematics, Faculté des Sciences et Techniques, P.O. Box 523, Beni Mellal, Morocco

E-mail address: baghdadi@fstbm.ac.ma

Dipartimento di Matematica, Università degli Studi Roma Tre, Largo S. L. Murialdo, 1, 00146 Roma, Italy

E-mail address: gabelli@mat.uniroma3.it

57 Colgate Street, Pocatello, ID 83201, USA

E-mail address: zafrullah@lohar.com