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WEAKLY FACTORIAL DOMAINS AND GROUPS OF DIVISIBILITY

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ABSTRACT. An integral domain R is said to be weakly factorial if every nonunit of R is a product of primary elements. We give several conditions equivalent to R being weakly factorial. For example, we show that the following conditions are equivalent: (1) R is weakly factorial; (2) every convex directed subgroup of the group of divisibility of R is a cardinal summand; (3) if P is a prime ideal of R minimal over a proper principal ideal (x), then P has height one and $(x)_P \cap R$ is principal; (4) $R = \cap R_P$, where the intersection runs over the height-one primes of R, is locally finite, and the t-class group of R is trivial.

Throughout this note, R will be a commutative integral domain with identity having quotient field K. Suppose that R is a UFD. If S is a saturated multiplicatively closed subset of R, then $S = \{\lambda p_1 \cdots p_n | \lambda$ a unit of R, $n \geq 0$, each $(p_i) \in Y\}$ for some subset $Y \subseteq X^{(1)}$, where $X^{(1)}$ is the set of height-one prime ideals of R. Conversely, every such subset $Y \subseteq X^{(1)}$ determines a saturated multiplicatively closed subset of R. If we set $T = \{\lambda p_1 \cdots p_n | \lambda$ a unit of R, $n \geq 0$, each $(p_i) \in X^{(1)} - Y\}$, then every nonzero element r of R may be written uniquely up to units in the form $r = st$ where $s \in S$ and $t \in T$. Stated in terms of the group of divisibility $G(R)$ of R, $\langle S \rangle$ is a cardinal summand of $G(R)$. In fact, $\langle S \rangle \oplus_C \langle T \rangle = G(R)$. (See the next paragraph for definitions.) Here $G(R)$ is a cardinal sum of copies of $\mathbb{Z}$, one for each height-one prime, and $\langle S \rangle$ is the sum of the cyclic summands corresponding to the primes from Y. The purpose of this paper is to characterize the integral domains with this property. We show that an integral domain R has the property that every convex directed subgroup of $G(R)$ is a cardinal summand of $G(R)$ if and only if every nonunit of R is a product of primary elements, that is, R is weakly factorial.

Let R be an integral domain with quotient field K. The *group of divisibility of* R is the group $G(R) = K^*/U(R)$, where K^* is the multiplicative group of K and $U(R)$ is the group units of R. $G(R)$ is partially ordered

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by $xU(R) \leq yU(R) \Leftrightarrow x|y$ in R. For results on partially ordered abelian groups and groups of divisibility, the reader is referred to [6, Chapter 3; 10]. If S is a saturated multiplicatively closed subset of R, then it is easily seen that $\langle S \rangle = \{s_1 s_2^{-1} U(R) | s_1, s_2 \in S\}$ is a convex directed subgroup of $G(R)$. Conversely, Mott [10, Theorem 2.1] has shown that every convex directed subgroup of $G(R)$ has this form. A subgroup H of $G(R)$ is called a *cardinal summand* of $G(R)$ if there is a convex directed subgroup L of $G(R)$ with $G(R) = H \oplus_C L$, that is, $G(R)$ is the algebraic direct sum of H and L and the partial order on $G(R) = H \oplus L$ is given by $(h_1, l_1) \leq (h_2, l_2) \Leftrightarrow h_1 \leq h_2$ and $l_1 \leq l_2$. Mott and Schexnayder [11, Proposition 4.1] have shown that a saturated multiplicatively closed subset S of R has $\langle S \rangle$ as a cardinal summand of $G(R)$ if and only if there is a saturated multiplicatively closed subset T of R with $S \cap T = U(R)$ such that every nonzero element r of R can be written uniquely up to units in the form $r = st$, where $s \in S$, $t \in T$ and each element s of S is v-coprime to each element $t \in T$ in the sense that $(s, t)_v = R$ or, equivalently, that $(s) \cap (t) = (st)$. Here v denotes the v-operation, that is, for an ideal I of R, $I_v = (I^{-1})^{-1} = [R: [R: I]]$.

An element $x \in R$ is said to be *primary* if (x) is a primary ideal of R and an integral domain R is said to be *weakly factorial* if every (nonzero) nonunit of R is a finite product of primary elements. Weakly factorial domains were introduced in [2]. Examples of weakly factorial domains include UFD's, one-dimensional semi-quasi-local domains, and GCD domains which are generalized Krull domains. It was shown that, for R weakly factorial, $R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite. Also, a Krull domain is factorial if and only if it is weakly factorial. Hence, a weakly factorial Noetherian domain is factorial if and only if it is integrally closed.

For a Krull domain R, the intersection $R = \bigcap_{P \in X^{(1)}} R_P$ is locally finite, and R is fractorial if and only if $Cl(R) = 0$ where $Cl(R)$ is the divisor class group of R. Since the divisorial ideals form a group under the v-product $(A*B = (AB)_v)$ if and only if R is completely integrally closed, $Cl(R)$ need not be a group for an arbitrary integral domain. However, for any integral domain R, there is a group $Cl_t(R)$ called the t-class group of R which agrees with $Cl(R)$ when R is a Krull domain. A nonzero fractional ideal I of R is said to be *t-invertible* if there is a fractional ideal B of R with $(AB)_t = R$. Here t denotes the t-operation given by $I_t = \cup\{(a_1, \dots, a_n)_v | 0 \neq (a_1, \dots, a_n) \subseteq I\}$. An ideal I is said to be a *t-ideal* if $I_t = I$. The t-operation is an example of a finite-character star operation. For results on star operations, the reader is referred to [6, §§32 and 34]. For any integral domain R, the set of t-invertible t-ideals forms a group under the t-product $A * B = (AB)_t$. The *t-class group* of R is $Cl_t(R)$—the group of t-invertible t-ideals modulo the subgroup of principal fractional ideals. For results on the t-class group, the reader is referred to [4, 5]. We show that, for a domain R, R is weakly factorial if and only if $R = \bigcap_{P \in X^{(1)}} R_P$ is locally finite and $Cl_t(R) = 0$. This result is part of the next theorem, the main result of this paper.

Theorem. *Let R be an integral domain with quotient field K. Then the following statements are equivalent*:

(1) *Every convex directed subgroup of $G(R)$ is a cardinal summand of $G(R)$.*
(2) *For each saturated multiplicatively closed subset S of R, $\langle S\rangle$ is a cardinal summand of $G(R)$.*
(3) *For each prime ideal P of R, $\langle R-P\rangle$ is a cardinal summand of $G(R)$.*
(4) *R is weakly factorial, that is, every nonunit of R is a product of primary elements.*
(5) *$R=\cap_{P\in X^{(1)}}R_P$ where the intersection is locally finite and the natural map $G(R)\to\oplus_{P\in X^{(1)}}G(R_P)$ is surjective (and hence an order isomorphism).*
(6) *If P is a prime ideal of R minimal over a proper principal ideal (x), then $htP=1$ and $(x)_P\cap R$ is principal.*
(7) *$R=\bigcap_{P\in X^{(1)}}R_P$, where the intersection is locally finite and $Cl_t(R)=0$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from the previously mentioned result of Mott [10, Theorem 2.1] that every convex directed subgroup of $G(R)$ has the form $\langle S\rangle$ for some saturated multiplicatively closed subset S of R, and conversely. (2) $\Rightarrow$ (3). Clear. (3) $\Rightarrow$ (6). Let P be a prime ideal minimal over (x). Then P is a t-ideal. Suppose that there is a prime ideal Q with $0\neq Q\subsetneq P$. Let $S=R-Q$. Then $\langle S\rangle$ is a cardinal summand of $G(R)$. By [11, Proposition 4.1] there is a multiplicatively closed subset T of R with $S\cap T=U(R)$, $ST=R-\{0\}$, and, for $s\in S$, and $t\in T$, $(s,t)_v=R$. Now $P\cap S\neq\varnothing$, so there exists an $s_0\in P\cap S$. Let $0\neq q\in Q$, so $q=st$ where $s\in S$ and $t\in T$. Then $s\notin Q$, so $t\in Q$. Thus s_0, $t\in P$ and hence $R=(s_0,t)_v\subset P_t=P$, a contradiction. Therefore, $htP=1$. (This shows that, in a weakly factorial domain, every prime t-ideal has height one.) It follows from [11, Proposition 4.1], taking $S=R-P$, that $(x)_P\cap R$ is principal. (6) $\Rightarrow$ (4). Let S be the multiplicatively closed subset of R consisting of all products of primary elements and units. We first show that S is saturated. Let $x\in S$ be a nonunit and y be a nonunit factor of x. If $x=q_1\cdots q_r$, where (q_i) is P_i-primary, then $\{P_1,\dots,P_r\}$ is the set of height-one primes containing (x). Hence the set of height-one primes containing (y) is a subset of $\{P_1,\dots,P_r\}$, say $\{P_1,\dots,P_n\}$. Now $(y)_{P_1}\cap R$ is principal, say $(y)_{P_1}\cap R=(q_1')$. Since $\dim R_{P_1}=1$, (q_1') is P_1-primary. Now $y\in(q_1')$, so $y=q_1't_1$ where $t_1\notin P_1$, but $t_1\in P_2\cap\cdots\cap P_n$. Continuing in this manner, we get that $y=q_1'\cdots q_n't_n$, where (q_i') is P_i-primary and t_n is contained in no height-one prime ideal, hence a unit. Suppose that some nonunit $a\in R$ is not in S. Then $(a)\cap S=\phi$ since S is saturated. Hence (a) can be enlarged to a prime ideal $P\supseteq(a)$ maximal with respect to missing S. Shrinking P to a prime ideal Q minimal over (a), we have a height-one prime ideal Q containing no primary elements. But this is absurd, for the ideal $(a)_Q\cap R$ is itself a principal primary ideal contained in Q. (4) $\Rightarrow$ (2). Let R be a weakly factorial integral domain and let S be a saturated multiplicatively closed subset of R. Let $x\in S$ be a nonunit and suppose that $x = q_1 \cdots q_n$ where (q_i) is P_i-primary. Let (q) be P_i-primary. Then, for some $s \geq 1$, $q_i^s \in (q)$, so $q_i^s = rq$ for some $r \in R$; hence q is a factor of $x^s \in S$. Since S is saturated, $q \in S$. Thus it is easily seen that $S = \{\lambda q_1 \cdots q_n | \lambda \in U(R),\ n \geq 0$, each (q_i) is P_i-primary where $P_i \in X^{(1)}$ with $P_i \cap S \neq \emptyset\}$. Let $T = \{\lambda q_1 \cdots q_n | \lambda \in U(R),\ n \geq 0$, (q_i) is P_i-primary where $P_i \cap S = \emptyset\}$. Then T is a saturated multiplicatively closed subset of R satisfying the conditions of part (2) of [11, Proposition 4.1]. Hence by this proposition, $\langle S \rangle$ is a cardinal summand of $G(R)$. (4) $\Rightarrow$ (5). By [2, Corollary 14], $R = \bigcap_{P \in X^{(1)}} R_P$ and this intersection is locally finite. Let $0 \neq t \in R_P$ be a nonunit where $P \in X^{(1)}$. Then $t = qu$ where (q) is P-primary and u is a unit of R_P. Thus $qU(R) \to tU(R_P)$. It easily follows that the natural map $G(R) \to \bigoplus_{P \in X^{(1)}} G(R_P)$ is surjective and hence is an order isomorphism. (5) $\Rightarrow$ (4). Let $0 \neq r \in R$ be a nonunit. Let $P_1, \ldots, P_n$ be the height-one prime ideals containing (r). Now, since the natural map $G(R) \to \bigoplus_{P \in X^{(1)}} G(R_P)$ is surjective, for each $i = 1, \ldots, n$, there exists a $k_i \in K$ with $k_i U(R_{P_i}) = rU(R_{P_i})$ and $k_i U(R_P) = 1U(R_P)$ for each $P \in X^{(1)} - \{P_i\}$. Note that each $k_i \in \bigcap_{P \in X^{(1)}} R_P = R$. Moreover, since P_i is the only height-one prime ideal containing (k_i), $(k_i) = \bigcap_{P \in X^{(1)}} (k_i R_P) = k_i R_{P_i} \cap R$ is P_i-primary. Now $rU(R_P) = k_1 \cdots k_n U(R_P)$ for each $P \in X^{(1)}$, so $r = uk_1 \cdots k_n$, where u is a unit of R. (4) $\Rightarrow$ (7). By [2, Corollary 14], $R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite. Let A be a t-invertible ideal of R. We must show that A_t is principal. We may assume that A is an integral ideal. We have already observed that every maximal t-ideal of R has height one. Hence, by a result of Griffin [7, Proposition 4], A_P is principal for each $P \in X^{(1)}$. Moreover, since the intersection is locally finite, $A_P = R_P$ except, say, for $P_1, \ldots, P_n \in X^{(1)}$. But at P_i, $A_{P_i} = q_i R_{P_i}$, where (q_i) is P_i-primary. So $A \subseteq (q_i)_{P_i} \cap R = (q_i)$ and hence $A \subseteq (q_1) \cap \cdots \cap (q_n) = (q_1 \cdots q_n)$. Thus $A = q_1 \cdots q_n A'$, where A' is contained in no height-one prime ideals. Hence $A'_t = R$. So $A_t = (q_1 \cdots q_n A')_t = (q_1 \cdots q_n A'_t)_t = (q_1 \cdots q_n)_t = (q_1 \cdots q_n)$ is principal. (7) $\Rightarrow$ (4). Let $0 \neq x \in R$ be a nonunit. Let $P_1, \ldots, P_n$ be the height-one prime ideals containing (x). Let $Q_i = (x)_{P_i} \cap R$, so $(x) = Q_1 \cap \cdots \cap Q_n$, where Q_i is P_i-primary. Now $x^{-1} Q_1 \cdots Q_n$ is an ideal of R and $x^{-1} Q_1 \cdots Q_n R_P = R_P$ for each height-one prime ideal of R. Hence, by [1, Theorem 1], $(x^{-1} Q_1 \cdots Q_n)^* = R$ where $*$ is the finite character star operation given by $A^* = \bigcap_{P \in X^{(1)}} AR_P$. Since $*$ has finite character, $R = (x^{-1} Q_1 \cdots Q_n)^* \subseteq (x^{-1} Q_1 \cdots Q_n)_t \subseteq R$, so $(x^{-1} Q_1 \cdots Q_n)_t = R$. Thus each Q_i is t-invertible and hence $Q_{it} = (q_i)$ for some $q_i \in R$. Also, $(x) = (Q_1 \cdots Q_n)_t = (Q_{1t} \cdots Q_{nt})_t = ((q_1)) \cdots (q_n))_t = (q_1) \cdots (q_n)$. Now $(q_i) \supseteq Q_i$ and hence $(q_i)_P = R_P$ except possibly at $P = P_i$. So $Q_i = (x)_{P_i} \cap R = (q_i)_{P_i} \cap R = q_i R_{P_i} \cap (\bigcap_{P \in X^{(1)} - \{P_i\}} q_i R_P) = (q_i)$. Hence (q_i) is P_i-primary, so x is a product of primary elements. □

Corollary 1. *Let R be a Noetherian integral domain. Then R is weakly factorial if and only if every grade-one prime ideal has height one and $Cl_t(R) = 0$.*

Proof. ($\Rightarrow$). A grade-one prime ideal is contained in a maximal t-ideal and hence has height one. By the previous theorem, $Cl_t(R) = 0$. ($\Leftarrow$). If every grade-one prime ideal has height one, $R = \bigcap_{P \in X^{(1)}} R_P$, where the intersection is locally finite [8, Theorem 53]. By the previous theorem, R is weakly factorial. □

In the spirit of Mott [9], we use the Theorem and the Krull–Kaplansky–Jaffard–Ohn Theorem to obtain a result of A. Bigard (see, for example, [3, Theoreme 14.4.3, pp. 290–291]) concerning lattice ordered groups.

Corollary 2 (A. Bigard). *A lattice ordered abelian group G is a cardinal sum of Archimedian totally ordered groups if and only if every convex directed subgroup of G is a cardinal summand of G.*

Proof. Let G be a lattice ordered abelian group. By the Krull–Kaplansky–Jaffard–Ohm Theorem ([6, Theorem 18.6] or [9, Theorem 2.1]), G is order isomorphic to the group of divisibility of a Bézout domain R. By the Theorem, $G(R)$, and hence G, has the property that every convex directed subgroup is a cardinal summand if and only if R is weakly factorial. But by [2, Theorem 21] a Bézout domain R is weakly factorial if and only if $G(R)$ is order isomorphic to a cardinal sum of rank one (equivalently, Archimedian) totally ordered abelian groups. □

As previously stated, examples of weakly factorial domains include UFD's and one-dimensional semi-quasi-local domains. Actually, it is easily seen that a one-dimensional domain R is weakly factorial if and only if R is Laskerian (i.e., every ideal has a primary decomposition) and $\text{Pic}(R) = 0$. Also, an integral domain that is both a GCD-domain and a generalized Krull domain is weakly factorial. (An integral domain R is called a *generalized Krull domain* if $R = \cap V_\lambda$, where each V_λ is an essential rank one valuation overring of R and the intersection is locally finite.) In fact, according to [2, Theorem 20], the following conditions on an integral domain R are equivalent: (1) R is a weakly factorial GCD-domain; (2) R is weakly factorial generalized Krull domain; and (3) R is a generalized Krull domain and a GCD-domain. To this list may be added (4) R is weakly factorial and, for each height-one prime ideal P of R (equivalently, maximal t-ideal P of R), R_P is a valuation domain and (5) R is weakly factorial and if p and q are non v-coprime primary elements of R, then $p|q$ or $q|p$. (Use [2, Theorem 18] and its proof). We next give another source of weakly factorial domains.

Example. Let $K \subseteq L$ be fields. Then $R = K + XL[X] = \{f(X) \in L[X] | f(0) \in K\}$ is an atomic weakly factorial domain. It is easily seen that every nonzero nonunit of R may be written uniquely in one of the following two forms: $p_1 \cdots p_n$ or $p_1 \cdots p_n(aX^r)$, where $p_1, \dots, p_n$ are principal primes of R and aX^r $(r \geq 1, a \in L)$ is primary.

It is well known [8, Theorem 5] that an integral domain R is a UFD if and only if every nonzero prime ideal contains a nonzero principal prime ideal. The proof is based on the fact that the set of principal primes forms a saturated multiplicatively closed set. A similar result for weakly factorial domains involving primary elements is not true.

For let R be a Dedekind domain that is not a PID, but has torsion class group. Since some power of each prime ideal is principal, each prime ideal contains a principal primary ideal, but R is not weakly factorial. In this case, the multiplicatively closed set of all products of primary elements is not saturated. As a concrete example, suppose that R has nonprincipal maximal ideals M and N with $M^2 = (x)$, $N^2 = (y)$, and $MN = (z)$. Then $(z^2) = (MN)^2 = M^2N^2 = (xy)$. So z^2 is a product of primary elements, but z itself is not. Note that x, y, and z are all irreducible. Thus a single irreducible element of a Krull domain may be primary without being prime. However, a Krull domain is a UFD if and only if each irreducible element is primary.

However, for any integral domain R and any nonzero prime ideal P of R, it is easily proved that $S_P = \{\lambda p_1 \cdots p_n | \lambda \in U(R), n \geq 0, (p_i) \text{ is } P\text{-primary}\}$ is a saturated multiplicatively closed subset of R. Let S be the multiplicatively closed subset of R consisting of all products of primary elements. Then, in $G(R)$, $\langle S \rangle = \oplus \langle S_P \rangle$ where the direct sum runs over all nonzero primes P of R with P containing a P-primary element. Even though each $\langle S_P \rangle$ is a convex directed subgroup of $G(R)$, $\oplus \langle S_P \rangle$ need not be convex, in fact, $\oplus \langle S_P \rangle$ is convex if and only if S is saturated. Returning to the case where R is a Dedekind domain with torsion class group but is not a PID, we see that each $\langle S_P \rangle$ is order isomorphic to $(\mathbb{Z}, +)$. It is interesting to note that in this case we have $\bigoplus_{P \in X^{(1)}} \langle S_P \rangle \subsetneq G(R) \subsetneq \bigoplus_{P \in X^{(1)}} G(R_P)$, with $\bigoplus_{P \in X^{(1)}} \langle S_P \rangle$ being a directed lattice ordered subgroup of $G(R)$ and $G(R)$ being a directed lattice ordered subgroup of $\bigoplus_{P \in X^{(1)}} G(R_P)$. Moreover, all three groups are (algebraically) isomorphic to a direct sum of $|X^{(1)}|$ copies of $(\mathbb{Z}, +)$.

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