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WELL BEHAVED PRIME t-IDEALS

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We call a prime t-ideal P, of an integral domain D, *well behaved* if PD_P is a t-ideal of D_P. It is known that some integral domains have prime t-ideals that are not well behaved. In this article, we characterize integral domains in which every prime t-ideal is well behaved and construct examples of those with badly behaved prime t-ideals.

Let D be an integral domain. A fractional ideal A of D is called a *t-ideal* if $A = \bigcup (F^{-1})^{-1}$ where F ranges over finitely generated non-zero subideals of A. In [14], it was shown that if P is a prime t-ideal of D it is not necessary that PD_P should also be a t-ideal of D_P. Let us call a prime t-ideal *well behaved* if PD_P is also a t-ideal. It is logical to ask, "Under what conditions is a prime t-ideal well behaved? and what is the significance of the information that a prime t-ideal is well behaved?" In this article we answer the first part of this question by characterizing the well behaved prime t-ideals. For the second part of the above question we study the integral domains in which every prime t-ideal is well behaved and show that most of the integral domains of current interest have all prime t-ideals well behaved. Such integral domains may obviously be called *well behaved domains*. This study leads to some other interesting questions and constructions, but for that we need some familiarity with the t-ideals, etc.

Throughout this article the letter D denotes a commutative integral domain with quotient field K, and $F(D)$ denotes the set of all non-zero fractional ideals of D. We shall use the word ideal to mean a non-zero fractional ideal and will distinguish the ideals $A \subseteq D$ by calling them *integral ideals*.

Associated to each $A \in F(D)$ is the fractional ideal $(A^{-1})^{-1} = A_v$. The function $A \to A_v$ on $F(D)$ is a *star operation* called the *v-operation*. The reader may consult [4, Sections 32 and 34] for the definition and properties of star operations. For our purposes we note the following. Let $A, B \in F(D)$ and let $x \in K - \{0\}$. Then:

(1) $(xD)_v = xD$, $(xA)_v = xA_v$.

(2) $A \subseteq A_v$ and if $A \subseteq B$, $A_v \subseteq B_v$.

(3) $(A_v)_v = A_v$.

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(4) $(AB)_v = (A_v B)_v = (A_v B_v)_v$, we shall refer to these equations as defining *v-multiplication*.

(5) $A^{-1} = (A_v)^{-1} = (A^{-1})_v$.

(6) $A_v = \bigcap \{xD \mid x \in K - \{0\}$ and $A \subseteq xD\}$.

An ideal $A \in F(D)$ is called a *v-ideal* if $A = A_v$ and a *v-ideal of finite type* if $A = B_v$ for some finitely generated $B \in F(D)$. For every $A \in F(D)$, A^{-1} is a v-ideal ((5) above) and a non-zero intersection of principal ideals is a v-ideal ((6) and (3)). A v-ideal is also called *divisorial*. As indicated in the leading paragraph, an ideal $A \in F(D)$ is a t-ideal if $A = \bigcup \{F_v \mid F$ is a finitely generated subideal of $A\}$. A v-ideal is a t-ideal ((2) above). An integral ideal maximal w.r.t. being a t-ideal is a prime ideal called *maximal t-ideal*. An integral domain D is called a *Prüfer v-multiplication domain* (*PVMD*) if the set $H(D)$ of v-ideals of finite type is a group under v-multiplication. According to [5], D is a PVMD if, and only if, for every maximal t-ideal M of D, D_M is a valuation domain.

It was shown in [14] that if D is locally PVMD (i.e. if for every maximal ideal P, D_P is a PVMD), then D is a PVMD if and only if every maximal t-ideal of D is well behaved. Let us call D *conditionally well behaved* if each maximal t-ideal of D is well behaved. It is again logical to ask if a conditionally well behaved domain is well behaved. We shall construct an example to prove that the answer to this question is no. We then prove, with reference to [14] and the above mentioned example, that if D is a PVMD the integral domain $D^{(S)} = D + xD_S[x] = \{a_0 + \sum_{i=1}^{n} a_i x^i \mid a_0 \in D,\ a_i \in D_S\}$ is a PVMD if, and only if, $D^{(S)}$ is well behaved. We shall use this result to establish that badly behaved integral domains abound.

We split this article into three sections. In the first section we characterize well behaved prime t-ideals and study some properties of well behaved domains. In the second section we construct the example and in the third we study the $D + xD_S[x]$ construction from PVMDs.

1. Well behaved prime t-ideals

Proposition 1.1. *A non-zero prime ideal P of D is a well behaved prime t-ideal if, and only if, for every finitely generated subideal F of P there exist elements $a \in P$ and $b \in D$, with $a \nmid bs$ for any $s \in D - P$,* such that $F \subseteq \frac{a}{b}D$.

Proof. Let P be a prime ideal and suppose that the given condition holds for P. Let $x_1, x_2, \ldots, x_n \in PD_P$. We can assume that $x_i \in D$. So $x_1, \ldots, x_n \in P$ and by the condition there are $a \in P$ and $b \in D$, where a does not divide bs for all $s \in D - P$, such that $(x_1, x_2, \ldots, x_n) \subseteq \frac{a}{b}D$. Extending to D_P, we have $(x_1, x_2, \ldots, x_n)D_P \subseteq \frac{a}{b}D_P$ where a does not divide b; by the condition. So, obviously, for all $x_1, x_2, \ldots, x_n \in PD_P$, $(x_1, x_2, \ldots, x_n)_v \subseteq PD_P$ and PD_P is a t-ideal. To see that P is also a t-ideal let $a_1, a_2, \ldots, a_n \in P$. Then $(a_1, a_2, \ldots, a_n)D_P \subseteq PD_P$ and so $((a_1, a_2, \ldots, a_n)D_P)_v \subseteq PD_P$; because PD_P is a t-ideal. But according to [12, Lemma 4], $((a_1, a_2, \ldots, a_n)D_P)_v =$

$((a_1, a_2, \ldots, a_n)_v D_P)_v$. Consequently, $(a_1, a_2, \ldots, a_n)_v D_P \subseteq PD_P$, which is possible only if $(a_1, a_2, \ldots, a_n)_v \subseteq P$.

Conversely, let $x_1, x_2, \ldots, x_n \in P$, $((x_1, x_2, \ldots, x_n)D_P)_v \subseteq PD_P$. This means that there exist $a, b \in D$, with a does not divide b in D_P, such that $(x_1, x_2, \ldots, x_n)D_P \subseteq \frac{a}{b} D_P$. But then $a \nmid bx_i$ in D_P, or $a \nmid btx_i$ in D and a does not divide bt so that $(x_1, x_2, \ldots, x_n) \subseteq \frac{a}{bt} D$ where $a \in P$ and a does not divide bts for any $s \in D - P$. □

Following the line adopted in the introduction let us call D well behaved if every prime t-ideal of D is well behaved.

Proposition 1.2. *An integral domain D is well behaved if, and only if, for every multiplicative set S of D, D_S is well behaved.*

Proof. For sufficiency it suffices to note that a multiplicative set could also be the set of units, so we have only to deal with the necessity. Suppose that D is well behaved, let S be a multiplicative set in D and M be a prime t-ideal of D_S. The argument used in the proof of the first part of Proposition 1.1, leads to the conclusion that $m = M \cap D$ is a t-ideal. But then, as D is well behaved; mD_m is a t-ideal. On the other hand $mD_m = M(D_S)_M$. Thus for every prime t-ideal M of D_S, $M(D_S)_M$ is a t-ideal of $(D_S)_M = D_m$; where $m = M \cap D$. □

Corollary 1.3. *Let D be well behaved, S a multiplicative set in D and let P be a prime t-ideal of D with $P \cap S = \emptyset$. Then PD_S is a prime t-ideal.* □

Although worded as a characterization, Proposition 1.2 does not provide a useful sufficient condition. The following proposition gives a useful sufficient condition which does not seem to be necessary, but this author is unable to find a suitable example.

Proposition 1.4. *Let D be an integral domain such that for every finitely generated ideal A, and for every multiplicative set S of D, $A_v D_S$ is divisorial. Then D is a well behaved domain.*

Proof. Let A be a finitely generated ideal of D. According to [12, Lemma 4], $(AD_S)_v = (A_v D_S)_v$. But as $A_v D_S$ is a divisorial ideal, we have $(AD_S)_v = A_v D_S$. Now if P is a prime t-ideal of D and if $\mathscr{A}$ is a finitely generated ideal contained in PD_P, then $\mathscr{A} = (a_1, a_2, \ldots, a_n)D_P$, where $a_i \in P$. Because P is a t-ideal we have $(a_1, a_2, \ldots, a_n)_v \subseteq P$ and so $(a_1, a_2, \ldots, a_n)_v D_P \subseteq PD_P$. But $(a_1, a_2, \ldots, a_n)_v D_P = ((a_1, a_2, \ldots, a_n)D_P)_v = \mathscr{A}_v$ and from this we conclude that PD_P is a t-ideal. □

Recall from [1, Lemma 2.5] that if $x_1, x_2, \ldots, x_n \in D$ such that $(x_1, x_2, \ldots, x_n)^{-1}$ is of finite type, then $(x_1, x_2, \ldots, x_n)_v D_S = ((x_1, x_2, \ldots, x_n)D_S)_v$. So if D is an integral domain in which the inverse of every finitely generated ideal is a v-ideal of finite type then D is well behaved. Now this includes among well behaved integral domains, Mori domains (ones that satisfy ACC on integral v-ideals) because in a Mori domain, for every $A \in F(D)$, there is a finitely generated ideal B of D such that $A_v = B_v$. Because both noetherian and Krull domains are Mori, they are also well behaved. On the other hand, coherent domains and PVMDs are also well behaved because in a coherent domain the inverse of every finitely generated ideal is finitely generated and in a PVMD the inverse of a finitely generated ideal is a v-ideal of finite type. The GCD-domains being a special case of PVMDs also follow through. The badly behaved domains include integral domains that are not PVMDs but whose localizations at primes are PVMDs. Now these badly behaved domains D have the property that for some finitely generated ideal A, A^{-1} is not a v-ideal of finite type. This may lead one to think that for an integral domain D to be well behaved it is necessary that for every finitely generated ideal A, A^{-1} should be of finite type. This, however, is not the case as the following example indicates:

Example 1.5. Let D be a quasilocal one dimensional domain completely integrally closed which is not a valuation domain (see [10, 11]). Now being one-dimensional D is obviously well behaved. But for some finitely generated ideal A, A^{-1} is not of finite type. For if for every finitely generated ideal A, A^{-1} were of finite type then D would be a PVMD. This is because D being completely integrally closed for every non-zero fractional ideal A of D, $(AA^{-1})_v = D$ (see [4, Theorem 34.3]).

2. A conditionally well behaved domain that is not well behaved

Let V be a valuation domain of rank >1 and let Q be a non-zero non-maximal prime ideal of V. Throughout this section we shall use the letter R to denote $R = V + xV_Q[x]$. To show that R is conditionally well behaved but not well behaved we need some preparation.

Recall from [2] that $x \in D$ is *primal* if $x \mid ab$ in D implies that $x = x_1x_2$ where $x_1 \mid a$ and $x_2 \mid b$ in D. Moreover an integrally closed integral domain D is a *Schreier domain* if every non-zero non-unit of D is primal. A GCD-domain, for example, is a Schreier domain [2]. Now if D is a GCD-domain and S is a multiplicative set in D then, according to [3], $D + xD_S[x]$ is Schreier. We shall also need the following lemmata:

Lemma 2.1. *Let D be a Schreier domain and let $x_1, x_2, \ldots, x_n$ be non-zero elements of D. Then $(x_1, x_2, \ldots, x_n)_v = D$ if and only if x_i have no non-unit common factor.*

Proof. If $(x_1, x_2, \ldots, x_n)_v = D$, then obviously the x_i's have no non-unit common factor. Now suppose that $A = (x_1, x_2, \ldots, x_n)$. Then

$$A^{-1} = \bigcap_{j=1}^{n} \left(\frac{1}{x_j}\right) = \frac{1}{\prod x_j}\left(\bigcap_{i=1}^{n} (a_i)\right) \quad \text{where } a_i = \frac{\prod x_j}{x_i}.$$

Suppose that $A^{-1} \neq D$. Then there exists $h \in \bigcap (a_i)$ such that $\prod x_j \nmid h$. But then, because both $\prod x_j, h \in \bigcap (a_i)$, there exists $d \in \bigcap (a_i)$ such that $d \mid \prod x_j$ and $d \mid h$ [13, Theorem 1.1]. Consequently $d \mid \prod x_j$ properly. Now let $\prod x_j = ba_i d_i$ where $d = a_i d_i$. But then $\prod x_j = b(\prod x_j/x_i)d_i$ or $x_i = bd_i$. That is $b \mid x_i$ for all $i = 1, 2, \ldots, n$. Thus if $(x_1, x_2, \ldots, x_n)^{-1} \neq D$ then x_i must have a non-unit common factor. □

Let us call an element $f \in R$ *discrete if* $f(0)$ is a unit in V.

Lemma 2.2. *Every non-zero non-unit f of the domain R can be uniquely written as a product $f = gd$ where d is a discrete element and g is not divisible by any non-unit discrete element of R.*

Proof. Note that: (a) R is a Schreier domain, because V is a GCD-domain [3]; (b) every irreducible element of a Schreier domain is a prime [2]; and (c) because of the degree considerations, only a finite product of non-unit discrete elements can divide a non-zero element of R.

Now take $f \in R - \{0\}$ and let p_1 be a discrete prime dividing f. Then $f = f_1 p_1$. If $f_1 p_1$ is the claimed factorization, stop. If not, write $f_1 = f_2 p_2$ where p_2 is a discrete prime. If no discrete prime divides f_2 then $f = f_2 p_2 p_1$. If not, repeat the process. At a general stage we shall have $f = f_n p_n p_{n-1} \cdots p_2 p_1$. By (c) this process must stop for some value of n. But then $f = (f_n)(p_n p_{n-1} \cdots p_2 p_1)$ where $\prod p_i$ is discrete and f_n is not divisible by any discrete primes. □

It is easy to see that every principal prime P of an integral domain is a maximal t-ideal such that PD_P is a t-ideal [8]. But in $R = V + xV_Q[x]$ every discrete prime P is also a rank one prime because $R_P \supseteq L[x]$ where L is the quotient field of V. This proves the following lemma:

Lemma 2.3. *Every principal prime P of R generated by a discrete prime is well behaved such that R_P is also well behaved.* □

Now if we can see that there is only one prime t-ideal which avoids all the discrete elements we have singled out the one possible prime ideal P such that PR_P is a t-ideal but R_P is not a well behaved domain.

Lemma 2.4. *Let $M = \{f \in R \mid f$ is not discrete$\}$. Then M is a well behaved prime t-ideal.*

Proof. Indeed the set of discrete elements is multiplicative and saturated. So M is the complement of a multiplicative set and to show that M is a prime ideal it is sufficient to show that it is closed under addition.

Let $f_1, f_2 \in M$. Then, $f_1(0)$ and $f_2(0)$ are non-units in V. Now because V is a valuation domain $f_1(0) + f_2(0)$ is a non-unit of V, and so $f_1 + f_2$ is a non-discrete element.

To show that M is in fact a t-ideal we first note that as $M \cap (V - Q) \neq \emptyset$,

$M=m+xV_Q[x]$ where m is a prime ideal of V [3] and it is easy to see that m is the maximal ideal of V. Now let $f_1, f_2, \ldots, f_n \in M-\{0\}$. Then $f_i(0)\in m$ and so there is $h\in M-Q$ such that $h \mid f_i(0)$. But then $h \mid f_i$, because $h \mid x$, in R. Consequently $(f_1, f_2, \ldots, f_n)\subseteq hR$ which leads to $(f_1, f_2, \ldots, f_n)_v \subseteq hR$.

Finally to show that M is well behaved, we use similar reasoning for MR_M; because the non-discrete elements stay non-discrete after localization (in this case). □

Proposition 2.5. *R is conditionally well behaved but not well behaved.*

Proof. According to the above lemmas there are two types of maximal t-ideals: (1) the principal rank one prime ideals generated by discrete primes, and (2) the prime ideal M consisting of all non-discrete elements of R. Moreover, these maximal t-ideals are well behaved.

Obviously if P is a prime t-ideal that contains a discrete element then P is of type (1) and if P contains no discrete element then $P\subseteq M$. Now to complete the proof we show that $N=Q+xV_Q[x]$ is a prime t-ideal that is not well behaved. For this we first need to show that N is in fact a t-ideal. We note that in $V, Q=\bigcap_{a\in m-Q} aV$, where m is the maximal ideal of V. To establish that N is a prime t-ideal we show that

$$N=Q+xV_Q[x]=\bigcap_{a\in m-Q} aR.$$

Obviously $N\subseteq \bigcap_{a\in m-Q} aR$. Let $f\in \bigcap_{a\in m-Q} aR$. Then $f\in aR$ for all $a\in m-Q$. If $f=f_0+\sum f_i x^i$, where $f_0\in V$, $f_i\in V_Q$, then $a \mid f_0$ for all $a\in m-Q$ and this forces $f_0\in Q$. Now $\sum f_i X^i$ being already in $xV_Q[x]$ we conclude that $f\in Q+xV_Q[x]=N$. But then N being an intersection of principal ideals, is a v-ideal itself, and hence is a t-ideal.

Now to see that N is not well behaved we consider R_N. Obviously

$$R_N=(V+xV_Q[x])_{Q+xV_Q[x]}=(V_Q[x])_{NV_Q[x]}$$

because $N\bigcap (V-Q)=\emptyset$. But $NV_Q[X]=QV_Q+xV_Q[x]$ which is not a prime t-ideal of $V_Q[x]$ because for every $q\in Q-\{0\}$, q and x are coprime in $V_Q[x]$ and so

$$qV_Q[x]\cap xV_Q[x]=qxV_Q[x].$$

This equation extends to $R_N=V_Q[x]_{NV_Q[x]}$ so that N contains two elements q and x such that q and x are coprime in the GCD-domain R_N. So qR_N and xR_N do not belong to the same maximal t-ideal of R_N (Lemma 2.1). Because NR_N is the maximal ideal of R_N, it cannot be a t-ideal. □

3. Badly behaved domains abound

In this section we show that if D is a PVMD and S a multiplicative set in D and if $D^{(S)}=D+xD_S[x]$ is well behaved then $D^{(S)}$ is a PVMD. An immediate conclusion from this result is that if D is a PVMD and for some multiplicative set S, $D+xD_S[x]$ is not a PVMD then $D+xD_S[x]$ is not well behaved; hence the title of this section. Another conclusion from this result is that if D is a noetherian Krull domain then for any multiplicative set S of D, $D^{(S)}$ is a PVMD. This result could, in some ways be interesting. Because if D is a noetherian Krull domain then $D+xD_S[x]$ will be a coherent integrally closed integral domain. To simplify our study of the $D+xD_S[x]$ construction we prove the following lemmata.

Lemma 3.1. *Let A be a finitely generated ideal of D and let S be a multiplicative set of D. Then $(AD^{(S)})^{-1}=A^{-1}D^{(S)}$.*

Proof. Let $A=(a_1,a_2,\dots,a_n)$ then as $A^{-1}=\bigcap_{i=1}^n(1/a_i)$ we conclude that $A^{-1}D^{(S)}=(\bigcap_{i=1}^n(1/a_i))D^{(S)}\subseteq\bigcap(1/a_i)D^{(S)}=(AD^{(S)})^{-1}$. Conversely let $f\in(AD^{(S)})^{-1}$. Then as $f\in\bigcap_{i=1}^n(1/a_i)D^{(S)}$, $f=f_0+\sum_{i=1}^m f_ix^j$ where $f_ja_i\in D_S$ and $f_0a_i\in D$. Consequently $f_0A\subseteq D$ and $f_jAD_S\subseteq D_S$. Therefore $f_0\in A^{-1}$ and $f_j\in(AD_S)^{-1}=A^{-1}D_S$ [12, Lemma 4]. But then $f_0+\sum f_jx^j\in A^{-1}+A^{-1}xD_S[x]\subseteq A^{-1}D^{(S)}$. □

Lemma 3.2. *Let A be a v-ideal of finite type of D and let S be a multiplicative set in D. Then $(AD^{(S)})^{-1}=A^{-1}D^{(S)}$.*

Proof. Let $A=B_v$ where $B=(b_1,b_2,\dots,b_n)$. Then as $A\supseteq B$ we have $AD^{(S)}\supseteq BD^{(S)}$ and so $(AD^{(S)})^{-1}\subseteq(BD^{(S)})^{-1}=B^{-1}D^{(S)}$ (Lemma 3.1) $=A^{-1}D^{(S)}$. Thus $(AD^{(S)})^{-1}\subseteq A^{-1}D^{(S)}$. For the reverse containment let $y\in A^{-1}D^{(S)}$. It is easy to see that $y=y_0+\sum y_jx^j$ where $y_0\in A^{-1}$ and $y_j\in A^{-1}D_S=(AD_S)^{-1}$ (using [12, Lemma 4] it can be easily shown that if A is a v-ideal of finite type then $(AD_S)^{-1}=A^{-1}D_S$). Consequently $y_0A\subseteq D$ and $y_jAD_S\subseteq D_S$ so

$$(y_0+\sum y_jx^j)AD^{(S)}\subseteq y_0AD^{(S)}+\sum_{j=1}^n y_jx^jAD^{(S)}\subseteq D^{(S)}+xD_S[x]=D^{(S)}.$$

Thus $A^{-1}D^{(S)}\subseteq(AD^{(S)})^{-1}$. □

Proposition 3.3. *Let D be a PVMD and let S be a multiplicative set in D. Then $D^{(S)}=D+xD_S[x]$ is a PVMD if and only if $D^{(S)}$ is a well behaved domain.*

Proof. Obviously if $D^{(S)}$ is a PVMD it is well behaved; because PVMDs are well behaved. Conversely let $D^{(S)}$ be well behaved. We prove that $D^{(S)}$ is a PVMD by showing that for every maximal t-ideal $\bar{P}$ of $D^{(S)}$, $D^{(S)}_{\bar{P}}$ is a valuation domain.

Let $\bar{P}$ be a maximal t-ideal of $D^{(S)}$. If $\bar{P}\cap S=\emptyset$ then by Corollary 1.3 $\bar{P}(D^{(S)})_S=\bar{P}D_S[x]$ is a prime t-ideal. But as D_S is a PVMD, and so is $D_S[x]$, we conclude that $D^{(S)}_{\bar{P}}=(D_S[x])_{\bar{P}D_S[x]}$ is a valuation domain [9, Proposition 4.1]. If, on the other hand, $\bar{P}\cap S\neq\emptyset$ then $\bar{P}=P+xD_S[x]$ where $P=\bar{P}\cap D$ [3]. We claim that P is a t-ideal. For if $A=(x_1,x_2,\dots,x_n)\subseteq P$. Then by Lemmata 3.1 and 3.2,

$$(AD^{(S)})_v=((AD^{(S)})^{-1})^{-1}=(A^{-1}D^{(S)})^{-1}=(A^{-1})^{-1}D^{(S)},$$

because in a PVMD, for every finitely generated ideal A, A^{-1} is a v-ideal of finite type. So

$$(AD^{(S)})_v = A_v D^{(S)} \subseteq P + xD_S[x]$$

and this forces A_v in P. Having seen that P is indeed a prime t-ideal we conclude that D_P is a valuation domain, because D is a PVMD [9, Proposition 4.1]. Now according to [14, Lemma 1.2], to show that $D_P^{(S)}$ is a valuation domain we also have to show that P intersects S in *detail*; that is, every non-zero prime ideal contained in P intersects S. To show this let us note that $D-P$ is multiplicatively closed in $D^{(S)}$ and that, because $D^{(S)}$ is well behaved, $D^{(S)}_{D-P}$ is also well behaved (Proposition 1.2). But $D^{(S)}_{D-P} = D_P + xD_{S(D-P)}[x]$. If $D_{S(D-P)} \neq K$ then P must contain a non-zero prime ideal Q such that $D_{S(D-P)} = (D_P)_Q$, because D_P is a valuation domain. But then $D^{(S)}_{D-P} = D_P + xD_Q[x]$ is not well behaved (Proposition 2.5). So the requirement that $D^{(S)}$ is well behaved forces $D_{S(D-P)}$ to be the quotient field of D. This is possible only if every non-zero prime ideal contained in P intersects S. □

Corollary 3.4. *If D is a noetherian Krull domain and S a multiplicative set in D then $D^{(S)}$ is a* PVMD.

Proof. Note that for D noetherian $D^{(S)}$ is coherent [3, Theorem 4.32], that a coherent domain is well behaved, and that a Krull domain is a PVMD. □

Using Corollary 3.4, we can construct rings of Krull type [6], without any mention of valuation. A reader familiar with the area can easily check that if, in Corollary 3.4, S were assumed to meet only a finite numbers of rank one primes of the Krull domain D, then the resulting $D^{(S)}$ construction will be a ring of Krull type. In fact, if D is a ring of Krull type and S is a multiplicative set of D such that $D^{(S)}$ is a PVMD and S meets only a finite number of maximal t-ideals of D even then $D^{(S)}$ is a ring of Krull type. It is plausible to conjecture that if D is Krull then for any multiplicative set S, $D^{(S)}$ is a PVMD but this author cannot find a proof based on the available techniques.

Another consequence of the above results is that if starting with a PVMD, D, we construct $D^{(s)}$ which is not a PVMD then we have constructed a badly behaved domain. Now a GCD-domain is a PVMD and it was shown in [14] that with D a GCD-domain, $D^{(S)}$ is a GCD-domain if and only if it is a PVMD. Now according to [14], again, if D is a GCD-domain whose group of divisibility is not a cardinal sum of totally ordered archimedean groups then D contains a multiplicative set S such that $D^{(S)}$ is not a GCD-domain and hence is a badly behaved domain.

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