Almost GCD Domains of Finite t-Character

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Let D be an integral domain. Two nonzero elements $x, y \in D$ are v-coprime if $(x) \cap (y) = (xy)$. D is an almost-GCD domain (AGCD domain) if for every pair $x, y \in D$, there exists a natural number $n = n(x, y)$ such that $(x^n) \cap (y^n)$ is principal. We show that if x is a nonzero nonunit element of an almost GCD domain D, then the set $\{M; M$ maximal t-ideal, $x \in M\}$ is finite, if and only if the set $S(x) := \{y \in D; y$ nonunit, y divides x^n for some $n\}$ does not contain an infinite sequence of mutually v-coprime elements, if and only if there exists an integer r such that every sequence of mutually v-coprime elements of $S(x)$ has length $\leq r$. One of the various consequences of this result is that a GCD domain D is a semilocal Bézout domain if and only if D does not contain an infinite sequence of mutually v-coprime nonunit elements. Then, we study integrally closed AGCD domains of finite t-character of the type $A + XB[X]$ and we construct examples of non-integrally closed AGCD of finite t-character by local algebra techniques.

1. INTRODUCTION

Throughout this article, the letter D will denote a commutative integral domain with quotient field K, $\mathbb{N}$ the set of nonnegative integers and $\mathbb{N}^*$ the set of positive integers. The domain D is *local* if D contains only one maximal ideal and *semilocal* if D contains only a finite number of maximal ideals. The domain D is an *almost* GCD *domain* (AGCD domain) if for every pair of elements $x, y \in D$, there exists $n = n(x, y) \in \mathbb{N}^*$ such that $(x^n) \cap (y^n)$ is a principal ideal. AGCD domains were introduced and studied in [27]. Two nonzero elements x, y of a domain D are *v-coprime* if $(x) \cap (y) = (xy)$. As we shall see later (Lemma 2.1) x, y are v-coprime if and only if $(x, y)^{-1} = D$ if and only if $(x, y)_v = D$.

If A is a nonzero fractional ideal of D, the ideal $(A^{-1})^{-1}$ will be denoted by A_v; the ideal $\cup\{F_v; F$ is a nonzero finitely generated subideal of $A\}$ will be denoted by A_t. An ideal A is a ***t**-ideal* if $A = A_t$ and the set of maximal t-ideals of D will be denoted by $\mathrm{Max}_t(D)$. The domain D is of *finite **t**-character* if every nonzero nonunit of D belongs to only finitely many maximal t-ideals of D. The reader in need of review of these concepts may consult [13] and [17]. The class of domains of finite t-character includes Noetherian and Krull domains. The class of AGCD domains of finite t-character includes the almost factorial domains studied by Storch in [23] and Zafrullah in [27].

If x is a nonzero nonunit element of D, the set $\{y \in D; y$ is a nonunit that divides x^n for some $n \in \mathbb{N}^*\}$ will be called the *span* of x. The first objective of Section 2 of this paper is to show that if x is a nonzero nonunit element of an AGCD domain, then the following statements are equivalent:

(i) $\{M \in \mathrm{Max}_t(D); x \in M\}$ is finite.

(ii) The span of x does not contain an infinite sequence of mutually v-coprime elements.
(iii) There exists an integer r such that every sequence of mutually v-coprime elements of the span of x has length $\leq r$.
As an immediate consequence, we obtain that an AGCD domain D is of finite t-character if and only if the following property is satisfied:

(F) For every nonzero nonunit x of D, the span of x does not contain an infinite sequence of mutually v-coprime elements.

The above result is clearly of a local nature. A second objective of Section 2 is to obtain the following global result: If D is an AGCD domain (respectively, a GCD domain) that does not contain an infinite sequence of mutually v-coprime nonunit elements, then D is a semilocal almost Bézout domain (respectively, semilocal Bézout domain). We recall that a domain D is an *almost Bézout* domain if for every $x, y \in D$, there exists $n = n(x, y) \in \mathbb{N}^*$ such that (x^n, y^n) is a principal ideal [1].

In Section 3, we study the integral closure of the AGCD domains of finite t-character. We also study the integrally closed AGCD domains of finite t-character of the type $A + XB[X]$ where $A \subseteq B$ is an extension of domains. The characterizations of the finite t-character property in terms of Property (F) given in Section 2 plays a fundamental role. In Section 4, we construct, by local algebra techniques, examples of non-integrally closed AGCD domains of finite t-character. Finally, in Section 5, we make a few remarks about the divisibility group of an AGCD domain.

2. SEQUENCES OF MUTUALLY V-COPRIME ELEMENTS

The main objective of this section is to prove the following two complementary theorems.

Theorem 2.1. *Let D be an AGCD domain, x an element of D, $S(x)$ the span of x and $\Delta(x)$ the set $\{M \in \mathrm{Max}_t(D); x \in M\}$. Then, the following statements are equivalent:*
(i) $s := \sharp\Delta(x) < \infty$.
(ii) $S(x)$ does not contain any infinite sequence of mutually v-coprime elements.
(iii) $r := \sup\{u;\ \text{there exists } y_1, \dots, y_u \in S(x) \text{ that are mutually } v\text{-coprime}\} < \infty$.
Furthermore, when this occurs, $r = s$.

Before stating the second theorem, we need some definitions.

Definition 2.1. A nonzero nonunit element z of an integral domain D is *almost rigid* if for every $m \in \mathbb{N}^*$ and every pair x, y of divisors of z^m, there exists $n = n(x,y) \in \mathbb{N}^*$ such that x^n divides y^n or y^n divides x^n. Moreover, for an almost rigid element z of an AGCD domain D, the ideal $\{y \in D; y, z \text{ are not } v\text{-coprime}\}$, denoted by $P(z)$, is the *ideal associated with* z.

Theorem 2.2. *Let D, x, $S(x)$ and $\Delta(x)$ be as in Theorem 2.1. Suppose that $\sharp\Delta(x) = r < \infty$. Then*

(a) There exists a sequence of almost rigid elements $z_1, \ldots, z_r \in S(x)$ that are mutually v-coprime.

(b) For any such sequence $z_1, \ldots, z_r$, one has $\Delta(x) = \{P(z_i); i = 1, \ldots, r\}$.

As an immediate consequence of Theorems 2.1 and 2.2, we have:

Corollary 2.1. *. Let D be an AGCD domain. Then*

(a) D is of finite t-character if and only if D satisfies Property (F).

(b) When D is of finite t-character, $\mathrm{Max}_t(D) = \{P(z); z \in D, z \text{ almost rigid}\}$.

For the proofs of Theorems 2.1 and 2.2, we shall need some preliminary lemmas. The first one gathers some useful technical results, some of which are well known, but for the sake of completeness, we give the proofs.

Lemma 2.1. *Let D be an AGCD domain and r, s two nonzero nonunit elements of D. Then, the following statements are equivalent:*

(i) $(r) \cap (s) \neq (rs)$; i.e., r and s are not v-coprime.

(ii) $(r,s)_v \neq (1)$.

(iii) There exists $n \in \mathbb{N}^*$ and d a nonunit of D such that $(r^n, s^n)_v = (d)$.

(iv) There exists $n \in \mathbb{N}^*$ and d a nonunit of D such that $(r^n, s^n)_v \subseteq (d)$.

If furthermore r and s are almost rigid, then (i)–(iv) are also equivalent to:

(v) There exist $n \in \mathbb{N}^*$ such that $r^n | s^n$ or $s^n | r^n$.

Proof. (i)$\Leftrightarrow$(ii). The equivalence between (i) and (ii) is independent of the AGCD hypothesis. We can derive it as in 1.1 of [27].

(ii)$\Rightarrow$(iii). Since A is an AGCD domain, there exists $n \in \mathbb{N}^*$ such that $(r^n) \cap (s^n)$ is principal, say generated by m. Then $(r^n, s^n)_v$ is also principal, generated by $d = r^n s^n / m$. If we suppose that d is a unit, then we have $(r^n, s^n)_v = D$, hence also that $(r,s)_v = D$ since $(r^n, s^n)_v \subseteq (r,s)_v$. This contradicts the hypothesis.

(iii)$\Rightarrow$(iv). Clear.

(iv)⇒(ii). Let $n \in \mathbb{N}^*$ and d a nonunit of D such that $(r^n, s^n) \subseteq (d)$. Then, $(r^n, s^n)_v \subseteq (d) \subset D$ and $(r^n, s^n)_v$ is contained in a proper t-ideal. Let M be a maximal t-ideal containing $(r^n, s^n)_v$; then M must contain r and s, hence contains $(r, s)_v$. Thus, $(r, s)_v \neq D$.

(v)⇒(iv). Clear (and we don't need to suppose that r and s are almost rigid).

Now we suppose that r and s are almost rigid and that (iii) holds. We want to show that (v) is satisfied.

Set $r_1 = r^n$, $s_1 = s^n$. Observe that r_1 and s_1 are almost rigid and that $(r_1, s_1)_v = (d)$. Clearly, $x := d$ and $y := r_1/d$ divide r_1 in D. Since r_1 is almost rigid, there exists $u \in \mathbb{N}^*$ such that d^u divides $(r_1/d)^u$ or $(r_1/d)^u$ divides d^u; hence d^{2u} divides r_1^u or r_1^u divides d^{2u}. Similarly, there exists $k \in \mathbb{N}^*$ such that d^{2k} divides s_1^k or s_1^k divides d^{2k}. By raising these equations to the power k and u, respectively, we have:

(1) d^{2ku} divides r_1^{ku} or (2) r_1^{ku} divides d^{2ku}
and

(3) d^{2ku} divides s_1^{ku} or (4) s_1^{ku} divides d^{2ku}.
If we suppose that (1) and (3) are valid, then we have $(r_1^{ku}, s_1^{ku})_v \subseteq (d^{2ku})$, which is absurd since by Lemma 3.6 of [27], we must have $(r_1^{ku}, s_1^{ku})_v = (d^{ku})$.

If we suppose that (1) and (4) are valid, then we obtain that s_1^{ku} divides r_1^{ku}. If we suppose that (2) and (3) are valid, then r_1^{ku} divides s_1^{ku}.

Finally, if we suppose that (2) and (4) are valid, then both r_1^{ku} and s_1^{ku} divide d^{2ku} and, since d is almost rigid (as a divisor of the almost rigid element r_1), there exists $\ell \in \mathbb{N}^*$ such that $r_1^{ku\ell}$ divides $s_1^{ku\ell}$ or $s_1^{ku\ell}$ divides $r_1^{ku\ell}$.

In any case, we see that there exists $p \in \mathbb{N}^*$ such that r_1^p divides s_1^p or s_1^p divides r_1^p, and we therefore have r^ℓ divides s^ℓ or s^ℓ divides r^ℓ with $\ell = np$. Thus (v) is satisfied. ∎

If D is not an AGCD domain, then two almost rigid elements of D may satisfy condition (i) of Lemma 2.1 without satisfying condition (v).

Example 2.1. If V is a proper valuation domain with quotient field K, then the indeterminates X and Y are almost rigid elements of the ring $D = V + (X, Y)K[X, Y]$ that are not v-coprime, but for every n, neither X^n nor Y^n divides the other.

Lemma 2.2. *Let D be an AGCD domain.*

(a) If r is an almost rigid element of D, then $P(r) := \{y \in D; (y, r)_v \neq (1)\}$ is the only maximal t-ideal of D that contains r.

(b) If r and s are two almost rigid elements of D, then $P(r) = P(s)$ if and only if r and s are not v-coprime.

Proof. (a) Let $y \in D$. Since D is an AGCD domain, then by Lemma 2.1, y belongs to $P(r)$ if and only if there exists a nonunit $d \in D$ and $n \in \mathbb{N}^*$ such that $d|r^n$ and $d|y^n$. Hence $P(r) = \cup\{\sqrt{(d)}; d$ is a nonunit of D such that $r \in \sqrt{(d)}\}$ and this union is directed (actually linearly ordered) because r is almost rigid. Since the radical of a t-ideal is also a t-ideal, $\sqrt{(d)}$ and $P(r)$ are (proper) t-ideals.

Now let M be a maximal t-ideal of D containing r. Then for every $y \in M$, we have $(y, r)_v \subseteq M \neq D$. Hence $M \subseteq P(r)$ and therefore $M = P(r)$.

(b) If r, s are not v-coprime, then by Lemma 2.1 we may assume $r^n|s^n$ for some $n \in \mathbb{N}^*$. Hence $s \in P(r)$ and $P(r) = P(s)$ by (a).

Conversely, if $P(r) = P(s)$ then $s \in P(r)$, $(r, s)_v \neq D$, that is, r and s are not v-coprime. ∎

Lemma 2.3. *Let D be an AGCD domain and r a nonzero element of D. Then the following statements are equivalent:*

(i) r is almost rigid.

(i) The span of r contains no pair of v-coprime elements.

Proof. (i)$\Rightarrow$(ii). Let $S(r)$ be the span of r and x, y two elements of $S(r)$. Then there exist $t_1, t_2 \in \mathbb{N}^*$ such that $x|r^{t_1}$ and $y|r^{t_2}$, hence $x|r^t$ and $y|r^t$ where $t = \max\{t_1, t_2\}$. Since r is almost rigid, r^t is almost rigid. Therefore, there exists $n \in \mathbb{N}^*$ such that $x^n|y^n$ or $y^n|x^n$. Then, by Lemma 2.1 (v)$\rightarrow$(i), x and y are not v-coprime.

(ii)$\Rightarrow$(i). Assume that r is not almost rigid. Then, there exists $t \in \mathbb{N}^*$ and x, y nonunit divisors of r^t such that for every $n \in \mathbb{N}^*$, x^n does not divide y^n and y^n does not divide x^n. Since D is an AGCD domain, there exists $p \in \mathbb{N}^*$ and $m \in D$ such that

$$(x^p) \cap (y^p) = (m). \tag{1}$$

Let

$$d := x^p y^p / m. \tag{2}$$

Then d is a divisor of x^p, y^p and m, and from (1), one easily gets that

$$(x^p/d) \cap (y^p/d) = (m/d)\,. \tag{3}$$

From (2), one gets that

$$m/d = x^p y^p / d^2. \tag{4}$$

The relations (3) and (4) tell us that x/d and y/d are v-coprime.

Now, since x^p does not divide y^p and y^p does not divide x^p, x^p/d and y^p/d are nonunits of D, and since both x^p/d and y^p/d divide r^{tp}, they belong to $S(r)$. This contradicts (i). ▮

LEMMA 2.4. *Let D be an AGCD domain, x a nonzero nonunit element of D, $S(x)$ the span of x, $\Gamma(x)$ the set $\{P(r); r \text{ almost rigid}, x \in P(r)\}$ and $\Delta(x)$ the set $\{M \in \mathrm{Max}_t(D); x \in M\}$. Then*

(a) $\Gamma(x) = \{P(s); s \text{ almost rigid}, s \in S(x)\}$.

(b) If $S(x)$ does not contain any infinite sequence of mutually v-coprime elements, then $\Gamma(x)$ is finite and $\Delta(x) = \Gamma(x)$.

Proof. (a) Let $r \in D$ be an almost rigid element such that $x \in P(r)$; i.e., such that $(x, r)_v \neq D$. Since D is an AGCD domain, Lemma 2.1 implies that there exists $n \in \mathbb{N}^*$ and a nonunit $s \in D$ such that $(x^n, r^n)_v = (s)$. Since s divides x^n, one has $s \in S(x)$. Since $r^n \in (s) \subseteq P(s)$, and since by Lemma 2.2(a) $P(s)$ is prime t-ideal, one has $(r, s)_v \subseteq P(s)$, hence $(r, s)_v \neq D$, and therefore $P(r) = P(s)$ by Lemma 2.2(b). Thus $\Gamma(x) \subseteq \{P(s); s \text{ almost rigid}, s \in S(x)\}$.

Conversely, if s is an almost rigid element in $S(x)$, then there exists $n \in \mathbb{N}^*$ such that $x^m \in (s) \subseteq P(s)$, hence $x \in P(s)$ since $P(s)$ is a prime ideal by Lemma 2.2(a). Thus, $P(s) \in \Gamma(x)$.

(b) We first claim that $S(x)$ contains an almost rigid element. Assume the contrary. In particular, x itself is not almost rigid and, by Lemma 2.3, there exist $y_1, z_1 \in S(x)$ that are v-coprime. By induction, suppose that $y_1, \ldots y_i$, z_i are mutually v-coprime elements of $S(x)$. By our assumption, z_i cannot be almost rigid, hence by Lemma 2.3, there exist y_{i+1}, $z_{i+1} \in S(z_i) \subseteq S(x)$ that are v-coprime. It follows that $y_1, \ldots, y_i; y_{i+1}, z_{i+1}$ are mutually v-coprime elements of $S(x)$. Thus we get an arbitrarily long sequence of mutually v-coprime elements in $S(x)$, contradicting the hypothesis. Thus $S(x)$ contains an almost rigid element r and our claim is proved. As a consequence, $\Gamma(x)$ is not empty by (a).

Now, by hypothesis, $S(x)$ does not contain an infinite sequence of mutually v-coprime elements; thus, by Lemma 2.2(b), $\{P(s); s \text{ almost rigid}, s \in S(x)\}$ is finite. By (a), this means that $\Gamma(x)$ is finite. Thus, $\Gamma(x)$ is not empty and finite, say $\Gamma(x) = \{P_1, \ldots, P_k\}$.

We know that $\Gamma(x) \subseteq \Delta(x)$ by Lemma 2.2(a). Conversely, let $M \in \Delta(x)$ and suppose that $M \notin \Gamma(x)$, hence that $M \not\subseteq P_1 \cup \cdots \cup P_k$. For $i = 1, \ldots, k$, let $x_i \in M \backslash P_i$. Since D is an AGCD domain, then by Remark after Lemma 3.3 of [1], there exist $n \in \mathbb{N}^*$ and $d \in M$ such that $(x^n, x_1^n, \ldots, x_k^n)_v = (d)$. Then d is a nonzero nonunit element such that $\Gamma(d)$ is empty. This is absurd. Indeed, we have $S(d) \subseteq S(x)$; hence $S(d)$ does not contain any infinite sequence of mutually v-coprime elements (since $S(x)$ does not) and therefore $\Gamma(d)$ is not empty by the previous claim applied to the element d instead of x. ∎

We can now give the proofs of Theorems 2.1 and 2.2.

Proof (Proof of Theorem 2.1). (i)$\Rightarrow$(iii). Let $\Delta(x) = \{M_1, \ldots, M_s\}$; then clearly $S(x) \subseteq M_1 \cup \cdots \cup M_s$. If y and z are two v-coprime nonunit elements of D, then $(y, z)_v = (1)$ and a maximal t-ideal M of D cannot contain both y and z. Thus every sequence of mutually v-coprime elements of $S(x)$ has length $\leq s$. (Note that for this implication, we did not use the fact that D was an AGCD domain.)

(iii)$\Rightarrow$(ii). Clear.

(ii)$\Rightarrow$(i). This is given by Lemma 2.4(b).

When properties (i)–(iii) are satisfied, we have $\Delta(x) = \{P(y); y \text{ almost rigid}, y \in S(x)\}$ by Lemma 2.4(a). Furthermore, we have $\sharp\{P(y); y \text{ almost rigid}, y \in S(x)\} = \sup\{u;$ there exists, $y_1, \ldots, y_u \in S(x)$ that are mutually v-coprime$\}$ by Lemma 2.2(b). Thus $s = r$. ∎

Proof (Proof of Theorem 2.2). (a) Since $\sharp\Delta(x) = r < \infty$, then by Theorem 2.1, $S(x)$ does not contain any infinite sequence of mutually v-coprime elements, and by Lemma 2.4, there exist some almost rigid elements $z_1, \ldots, z_r \in S(x)$ such that $\Delta(x) = \{P(z_1), \ldots, P(z_r)\}$. By Lemma 2.2(b), those elements $z_1, \ldots, z_r$ are mutually v-coprime.

(b) If $z_1, \ldots, z_r$ is a sequence of almost rigid elements of $S(x)$ that are mutually v-coprime, then by Lemma 2.2, $P(z_1), \ldots, P(z_r)$ are distinct maximal t-ideals of D. By Lemma 2.4, they belong to $\Delta(x)$. Since $\sharp\Delta(x) = r$, we then have $\Delta(x) = \{P(z_1), \ldots, P(z_r)\}$. ∎

Whereas Theorems 2.1 and 2.2 were of a local nature, the next result is of a global nature.

Proposition 2.1. *Let D be an AGCD domain.*

(a) The following statements are equivalent:

- *(i) D is a semilocal almost Bézout domain.*
- *(ii) D does not contain any infinite sequence of mutualy v-coprime nonunit elements.*
- *(iii) $r := \sup\{u;$ there exist nonunit elements $y_1, \ldots, y_u$ of D that are mutually v-coprime$\} < \infty$.*

(b) When conditions (i)–(iii) occur, then $\sharp\,\mathrm{Max}(D) = r$ and the following statements hold:

- *(i) There exists a sequence of almost rigid elements $z_1, \ldots, z_r$ of D that are mutually v-coprime.*

(ii) For any such sequence $z_1, \ldots, z_r$, $\mathrm{Max}(D) = \{P(z_i); i = 1, \ldots, r\}$.

For the proof of Proposition 2.1, we need a lemma.

Lemma 2.5. *Let D be an AGCD domain and P a prime ideal of D. Then P is a t-ideal if and only if for every $x, y \in P$, there exist $k \in \mathbb{N}^*$ and $p \in P$ such that $(x^k, y^k) \subseteq (p)$.*

Proof. Let P be a proper (not necessarily prime) t-ideal and $x, y \in P$. We have $(x, y) \subseteq P$ hence $(x, y)_v = (x, y)_t \subseteq P_t = P$. Thus, by Lemma 2.1, there exist $k \in \mathbb{N}^*$ and $p \in P$ such that $(x^k, y^k) \subseteq (p)$.

Conversely, suppose that P is a prime ideal and that for every $x, y \in P$, there exist $k \in \mathbb{N}^*$ and $p \in P$ such that $(x^k, y^k) \subseteq (p)$. We want to show that P is a t-ideal; i.e., $(z_1, \ldots, z_n)_v \subseteq P$ for each $z_1, \ldots, z_n \in P$. Before giving the proof, we observe that if $a_1, \ldots, a_n \in \mathbb{N}^*$, $a := \max\{a_1, \ldots, a_n\}$ and $q := an + 1$, then $((z_1, \ldots, z_n)_v)^q \subseteq ((z_1, \ldots, z_n)^q)_v \subseteq (z_1^{a_1}, \ldots, z_n^{a_n})_v$, hence that $(z_1, \ldots, z_n)_v \subseteq P$ if and only if $(z_1^{a_1}, \ldots, z_n^{a_n})_v \subseteq P$.

Now we start the proof. We use induction on n. If $n = 1$, the result is clear. If $n = 2$, then by hypothesis, there exist $k \in \mathbb{N}^*$ and $p \in P$ such that $(z_1^k, z_2^k) \subseteq (p)$. Thus $(z_1^k, z_2^k)_v \subseteq (p)_v = (p) \subseteq P$ and by the previous observation, $(z_1, z_2)_v \subseteq P$. If $n \geq 3$, then by our hypothesis, there exist $\ell \in \mathbb{N}^*$ and $p \in P$ such that $(z_1^\ell, z_2^\ell)_v \subseteq (p)$. Again by the previous observation, $(z_1, z_2, z_3, \ldots, z_n)_v \subseteq P$ if and only if $(z_1^\ell, z_2^\ell, z_3, \ldots, z_n)_v \subseteq P$. Since $(z_1^\ell, z_2^\ell, z_3, \ldots, z_n)_v = ((z_1^\ell, z_2^\ell)_v, z_3, \ldots, z_n)_v \subseteq (p, z_3, \ldots, z_n)_v$, then $(z_1^\ell, z_2^\ell, z_3, \ldots, z_n)_v \subseteq P$ if $(p, z_3, \ldots, z_n)_v \subseteq P$. Since only $(n-1)$ elements of P are involved, then by the induction hypothesis, we have $(p, z_3, \ldots, z_n)_v \subseteq P$. ∎

Proof (Proof of Proposition 2.1). (a) (i)⇒(iii). By Lemma 2.5, in almost Bézout domain D, every prime ideal of D is a t-ideal, hence $\mathrm{Max}_t(D) = \mathrm{Max}(D)$ and $\mathrm{Max}_t(D)$ is finite. As already seen in the first part of the proof of Theorem 2.1, two v-coprime nonunit elements of D cannot be contained in the same maximal t-ideal. Thus every sequence of mutually v-coprime nonunits of D has length $\leq \sharp\,\mathrm{Max}(D)$.

(iii)⇒(ii). Clear.

(ii)⇒(i). By Corollary 2.1(b), $\mathrm{Max}_t(D) = \{P(y); y \in D, y \text{ almost rigid}\}$. Then, by Lemma 2.2(b), $\mathrm{Max}_t(D)$ is finite, hence $\mathrm{Max}(D) = \mathrm{Max}_t(D)$ since $\cup\{M \in \mathrm{Max}(D)\} = \{\text{nonunit elements of } D\} = \cup\{M \in \mathrm{Max}_t(D)\}$. Thus D is semilocal. By [1], Corollary 5.4, D is an almost Bézout domain.

(b) Assume that the statements of (a) are satisfied and let $s := \sharp\,\mathrm{Max}(D)$. By Lemma 2.5, $\mathrm{Max}_t(D) = \mathrm{Max}(D)$. Then by Corollary 2.1, there exists a sequence of almost rigid elements $z_1, \ldots, z_s$ of D such that $\mathrm{Max}(D) =$

$\{P(z_1), \dots, P(z_s)\}$. By Lemma 2.2(b), $z_1, \dots, z_s$ are mutually v-coprime, hence $s \leq r$. It has already been seen during the proof of (a) that $r \leq s$. Thus, $s = r$.

Finally, if $z_1, \dots, z_r$ is any sequence of almost rigid elements of D that are mutually v-coprime, then by Lemma 2.2, $\{P(z_1), \dots, P(z_r)\}$ is a set of r distinct maximal t-ideals. Thus, $\{P(z_1), \dots, P(z_r)\} = \mathrm{Max}(D)$. ∎

Proposition 2.2. *Let D be a GCD domain. Then*

(a) D is a semilocal Bézout domain if and only if D does not contain any infinite sequence of mutually v-coprime nonunit elements.

(b) When this occurs, $\mathrm{Max}(D) = \{P(z); z \in D, z \text{ almost rigid}\}$.

Proof. Apply Proposition 2.1 and Corollary 3.6 of [23]. ∎

Proposition 2.2 can be generalized in the following way.

Proposition 2.3. *Let D be an integrally closed AGCD domain. Then D is a semilocal Bézout domain if and only if D does not contain any infinite sequence of mutually v-coprime nonunit elements.*

Proof. The "only if" part is given by Proposition 2.1. Now, if D does not contain any infinite sequence of mutually v-coprime nonunit elements, then by Proposition 2.1 and its proof, D is semilocal and $\mathrm{Max}(D) = \mathrm{Max}_t(D)$. Then, by Corollary 3.8 and Theorem 3.9 of [27] and Proposition 4.4 of [21], D is a semilocal Prüfer domain, hence a semilocal Bézout domain by Theorems 60 and 107 of [19]. ∎

Remark 2. 1. We have seen in Corollary 2.1 that if D is an AGCD domain of finite t-character, then $\mathrm{Max}_t(D) = \{P(z); z \in D, z \text{ almost rigid}\}$. If D is not of finite character, this need not be true, even if D is Bézout. We give two examples:

(a) Let $\mathbf{A}$ be the ring of all the algebraic integers. Then $\mathbf{A}$ is a Bézout domain, and $\mathrm{Max}(\mathbf{A}) = \mathrm{Max}_t(\mathbf{A})$. By Proposition 42.8 of [13], every nonunit element of $\mathbf{A}$ belongs to uncountably many maximal t-ideals. By Lemma 2.2(a) this implies that $\mathbf{A}$ has no almost rigid element.

(b) Let E be the ring of entire functions. It is a Bézout domain, and $\mathrm{Max}_t(E) = \mathrm{Max}(E)$. By a result on p. 147 of [13], every nonzero nonunit element of E is divisible by some prime element of the type $z-\alpha$ with $\alpha \in \mathbb{C}$. Hence every almost rigid element of E is of the type $y := \varepsilon(z-\alpha)^n$ with ε invertible in E, $\alpha \in \mathbb{C}$, $n \in \mathbb{N}^*$ and, for such y, we have $P(y) = (z-\alpha)E$, a height-one maximal ideal. Since E is infinite dimensional, we obtain that there exist some maximal t-ideals that are not associated with any almost rigid element of E.

3. INTEGRALLY CLOSED AGCD DOMAINS OF FINITE T-CHARACTER

According to [27], Corollary 3.8 and Theorem 3.9, an integrally closed AGCD domain is a Prüfer v-multiplication domain with torsion t-class group. It seems pertinent to give the reader some idea of the concepts mentioned in the above sentence. Note that a fractional ideal A is t-invertible if there exists a fractional ideal B such that $(AB)_t = D$ and in this case $B_t = A^{-1}$. An integral domain D is said to be a Prüfer v-multiplication domain if every finitely generated nonzero ideal of D is t-invertible. For an introduction to the t-class group of D, the reader may consult [27] or [7]. For our purposes, let us note here that the set of t-invertible t-ideals of D forms a group $T(D)$ under the t-product: $A \times_t B = (AB)_t = (A_t B)_t$. Obviously $T(D)$ contains, as a subgroup, the group of nonzero principal fractional ideals $P(D)$. The quotient group $\mathrm{Cl}_t(D) = T(D)/P(D)$ is called the t-class group (a number of authors now prefer to call it the class group of D). This class group, introduced in [6], has the interesting property that if D is a Krull domain, then $\mathrm{Cl}_t(D)$ is just the divisor class group of D and if D is Prüfer $\mathrm{Cl}_t(D)$ is the ideal class group. So, being PVMD's of finite t-character, Krull domains with torsion divisor class group of Storch [24] are examples of AGCD domains with finite t-character.

We first look at the integral closure of an AGCD domain.

Proposition 3.1. *Let D be an AGCD domain and D' its integral closure. Then*

(a) D' is an AGCD domain,

(b) D' is of finite t-character if and only if D is of finite t-character.

Proof. (a) This is given by Theorem 3.4 of [27].

(b) By [1], Theorems 2.1 and 3.1, $D \subset D'$ is a root extension and the canonical map $\alpha : \mathrm{Spec}(D') \to \mathrm{Spec}(D)$ defined by $\alpha(Q) = Q \cap D$ is a homeomorphism. Then, it is sufficient to show that if $A \subseteq B$ is a root extension of AGCD domains, the canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ establishes a bijection between $\mathrm{Max}_t(A)$ and $\mathrm{Max}_t(B)$. By Theorem 2.1 of [1], it suffices to see that for $Q \in \mathrm{Spec}(B)$, Q is a t-ideal if and only if $Q \cap A$ is a t-ideal. That this is indeed true for a root extension of AGCD domains is an easy consequence of Lemma 2.5. ∎

Next, we consider domains R, which are constructed as $A + XB[X]$ where $A \subseteq B$ is an extension of domains. This pullback construction offers more structure than a general pullback. For example, R is the direct sum of A

and $XB[X]$ and, since R lies between the two polynomial rings $A[X]$ and $B[X]$, it can be expected to inherit some properties from these two rings.

The ring $A + XA_S[X]$, where S is a multiplicative system of A, was studied in [10]; it was shown that if A is a GCD domain, then $A+XA_S[X]$ is a GCD domain if and only if $\mathrm{GCD}(a, X)$ exists for every $a \in A$. Since the appearance of [10], the rings of type $A + XA_S[X]$ have served as a source of examples. In [28], it was proved that if A is a GCD domain then $A + XA_S[X]$ is a GCD domain if and only if for all $a \in A\backslash\{0\}$, $a = bs$ where $s \in S$ and b is coprime to every member of S. Later, in [3], this special property of S was used to define a *splitting multiplicative set* of A as a saturated multiplicative set S such that for all $a \in A\backslash\{0\}$, $a = bs$ where $s \in S$ and b is v-coprime to every member of S.

One aim of this section is to investigate properties of A and B that are necessary and sufficient for $A + XB[X]$ to be an integrally closed AGCD domain of finite t-character. Considering that AGCD domains are a generalization of GCD domains, we can expect that a concept generalizing a splitting multiplicative set will surface. The following result proves precisely that.

THEOREM 3.1. *Let $A \subseteq B$ be an extension of domains such that B is an overring of A. Then*

(a) The ring $A + XB[X]$ is an integrally closed AGCD domain if and only if A is an integrally closed AGCD domain and $B = A_S$ where S is a multiplicative system of A satisfying the following property: for every $a \in A\backslash\{0\}$, there exists $n \in \mathbb{N}^*$ such that $a^n = bs$ with $b \in A$ v-coprime to every element of S and $s \in S$.

(b) When $R := A+XA_S[X]$ is an integrally closed AGCD domain, R is of finite t-character if and only if A is of finite t-character and S does not contain any infinite sequence of mutually v-coprime nonunit elements.

We first prove a simple lemma.

LEMMA 3.1. *Let S be a saturated multiplicative set of an integral domain R and let F be a t-invertible (integral) t-ideal of R such that $FR_S = dR_S$ for some $d \in R$. Then there is a t-invertible integral ideal G of R such that for some $s \in S$, $(ds) = (FG)_v$ and $G \cap S \neq \emptyset$.*

Proof. Since $FR_S = dR_S$ for some $d \in R$, then for some $s \in S$ we can assume that $ds \in F$. Let $G = dsF^{-1}$. Then $G \subseteq R$ and as $GR_S = dsF^{-1}R_S = ds(FR_S)^{-1} = R_S$ we conclude that $G \cap S \neq \emptyset$. Multiplying $G = dsF^{-1}$ by F and applying the v-operation we get $(FG)_v = ds(FF^{-1})_v = (ds)$. ∎

Proof (Proof of Theorem 3.1). (a) Let $\mathcal{U}(B)$ denote the set of units of B. As a first step let us note that $S = A \cap \mathcal{U}(B)$ is a saturated multiplicative set in A, and that according to [2], R is integrally closed if and only if A and B are both integrally closed.

Suppose that $R = A + XB[X]$ is an integrally closed AGCD domain. For each $a \in A\backslash\{0\}$, there exists $n = n(a, X)$ such that $(a^n, X^n)_v = dR$ for some $d \in R$. Since $d|a^n$, $d \in A$ and since $d|X^n$, $d \in \mathcal{U}(B)$. Thus $d \in A\cap\mathcal{U}(B)$. Next dividing through by d we get $(a^n/d, X^n/d)_v = R$. Since a^n/d and X^n/d are v-coprime, then a^n/d is v-coprime to every divisor of X^n/d in R and hence to every member of S. Thus we have shown that for each $a \in A\backslash\{0\}$, there is $n \in \mathbb{N}^*$ such that $a^n = hs$ where $s \in A \cap \mathcal{U}(B)$ and $h \in A$ is v-coprime to every member of $A \cap \mathcal{U}(B)$.

Next we show that every element b of B is of the form a/s where $a \in A$ and $s \in A\cap\mathcal{U}(B)$. For this we note that if $b \in A$ we have nothing to prove. So let us assume that $b \in B\backslash A$ and note that, as R is an AGCD domain there exists $n = n(X, bX)$ such that $(X^n, b^nX^n)_v = hR$. Since $b \in B\backslash A$ we can write $b = r/s$ where $r, s \in A$. Since R is an AGCD domain there exists $m = m(r, s)$ such that $(r^mR+s^mR)_v = kR$ where $k \in A$. We cannot have $k = s^m$ because if so, we would then have $b^m = r^m/s^m \in A$ and therefore $b \in A$ since A is integrally closed, which is absurd. Let $r^m = k\alpha$ and $s^m = k\beta$, where $(\alpha, \beta)_v = R$. Since $h^mR = (X^{mn}, (b^m)^nX^{mn})_v = (X^{mn}, \alpha^n/\beta^nX^{mn})_v$, we have $\beta^nh^mR = (\beta^nX^{mn}, \alpha^nX^{mn}_v) = X^{mn}R$ and consequently $h^mR = X^{mn}/\beta^nR$. Thus, $\beta^n \in A \cap \mathcal{U}(B)$, and so $\beta \in A \cap \mathcal{U}(B)$. Now $\beta^nb^{mn} \in A$ implies $\beta b^m \in A$, because A is integrally closed, and hence $b^m \in A_S$. Finally, since A_S is integrally closed, $b \in A_S$.

It still remains to show that A is an AGCD domain. For this, let $a, b \in A\backslash\{0\}$. As $A + XB[X]$ is AGCD there exists $n = n(a, b)$ such that $a^nR \cap b^nR = dR$ for some $d \in R$. But since $d|a^nb^n$ in R we conclude that $d \in A$. Now using the fact that for each $x \in A$, $xR \cap A = xA$ we conclude that $a^nA \cap b^nA = dA$.

Conversely, suppose now that we have:

(i) A is an integrally closed AGCD,

(ii) $B = A_S$,

(iii) S is a multiplicative system of A such that for all $a \in A$, there is $n = n(a)$ such that $a^n = bs$ where $s \in S$ and $b \in A$ is v-coprime to every member of S.

In order to show that $R = A + XA_S[X]$ is an integrally closed AGCD domain, it is sufficient to show that R is a PVMD with torsion t-class group [27]. According to a recent result of [4] $A + XA_S[X]$ is a PVMD if and only if for each $a \in D$, (a, X) is a t-invertible ideal of R. To establish this let us note that for some $n \in \mathbb{N}^*$, $a^n = bs$ where $s \in S$ and b is v-coprime to every member of S. Since b is v-coprime to X in $A_S[X]$ we have $bA_S[X] \cap XA_S[X] = bXA_S[X]$. So for all $h \in bR \cap XR$ there is $s \in S$ such that $sh \in bXR$. Now as X divides h we have $h = Xk$. Thus, $sk \in bR$. This means that b divides the constant term of sk in A. But as b is v-coprime to s we conclude that b divides the constant term of k. Now as every power of s divide the nonconstant terms we see that b divides k. But then $h \in bXR$. This proves that $bR \cap XR = bXR$ and hence $(b, X)_v = R$. Consequently, $(b, X^n/s)_v = R$, or $sR = (bs, X^n)_v = (a^n, X^n)_v$. Using Corollary 3.3 of [27] and the fact that R is integrally closed, we conclude that $sR = ((a, X)^n)_v$. But this makes (a, X) t-invertible, for each $a \in A$.

Finally, to show that R has a torsion t-class group, we show that for every t-invertible t-ideal H of R, $(H^n)_t$ is principal. For this we consider two cases:

(α) when $H \cap S \neq \emptyset$, and

(β) when $H \cap S = \emptyset$.

In case (α) we have $H = H \cap A + XA_S[X]$, according to [10]. Now it can be easily shown that $H \cap A$ is a t-invertible t-ideal of A. Since A is an integrally closed AGCD domain there is $n \in \mathbb{N}^*$ such that $((H \cap A)^n)_v$ is principal. But then so is $(H^n)_v = ((H \cap A)^n)_v + XA_S[X]$.

For the case (β), let H be a t-invertible t-ideal of R. It is well known [7] that HR_S is a t-invertible t-ideal of R_S. Since $H \cap S = \emptyset$, HR_S is nontrivial, and since $R_S = A_S[X]$ is an integrally closed AGCD domain, there is $n \in \mathbb{N}^*$ such that $((HR_S)^n)_v = (H^n R_S)_v = (H^n)_v R_S$ is principal, say $(H^n)_v R_S = dR_S$. But then, by Lemma 3.1 there is a t-invertible integral t-ideal G of $R = A + XA_S[X]$ such that $G \cap S \neq \emptyset$ and for some $s \in S$, $(ds) = ((H^n)_v G)_v = (H^n G)_v$. Now by part ($\alpha$), there is $m \in \mathbb{N}^*$ such that $(G^m)_v = tR$. Now $(ds)^m = ((H^n)_v G)_v = ((H^n G)_v)^m = (H^{mn}(G^m)_v)_v$, and from this it is easy to conclude that $(H^{mn})_v$ is principal.

(b) By Corollary 2.1(a), we may replace the finite t-character assumption by Property (F). Observe that $A \subseteq R$ is a flat extension, hence that two elements $y, z \in A$ are v-coprime in A only if they are v-coprime in R

Suppose that R satisfies Property (F). Let $x \in A$, $T(A, x)$ the span of x in A and $T(R, x)$ the span of x in R. Since $\{\text{units of } A\} = \{\text{units of } R\}$, then $T(A, x) \subseteq T(R, x)$. Since by hypothesis $T(R, x)$ does not contain any infinite sequence of mutually v-coprime nonunit elements of R, then by the above observation, $T(A, x)$ does not contain any infinite sequence of mutually v-coprime elements of A. Thus A satisfies Property (F). Furthermore, every element $s \in S$ divides X in R, hence $\{y \in S; y \text{ is not invertible in } R\}$ is contained in the span of X in R, which, by hypothesis, does not contain any infinite sequence of mutually v-coprime nonunit elements in R. Thus, again by the above observation, S does not contain any infinite sequence of mutually v-coprime nonunit elements of A.

Conversely, suppose that A satisfies Property (F) and that S does not contain any infinite sequence of mutually v-coprime nonunit elements of A.

Without loss of generality, we may suppose that S is saturated. Suppose that R does not satisfy Property (F) and let $f \in R$ be a nonzero nonunit element of R whose span in R contains an infinite sequence $(g_i)_{i\geq 1}$ of mutually v-coprime nonunit elements of R.

If $f \in A$, then looking at the degrees of the polynomials in the equalities of the type $f^n = g_i r_i$, we see that g_i and r_i belong to A. Then, for every $i \geq 1$, g_i belongs to the span of f in A. Furthermore, since g_i, $g_j \in A$ and $y_i R \cap g_j R = g_i g_j R$, we also have $g_i A \cap g_j A = g_i g_j A$. Thus, $(g_i)_{i\geq 1}$ is an infinite sequence of mutually v-coprime nonunit elements of A contained in the span of f in A, contradicting the hypothesis.

If $f \notin A$, we first claim that only finitely many g_i's may not belong to A. Let K be the quotient field of A. The element f is a nonunit of $K[X]$ and $(g_i)_{i\geq 1}$ is a sequence of elements that belong to the span of f in $K[X]$; furthermore, since $K[X]$ is a localization of R, the extension $R \subseteq K[X]$ is flat and by 3.H on p. 23 of [20], the g_i's are mutually v-coprime in $K[X]$. Since $K[X]$ is a PID, the span of f in $K[X]$ cannot contain an infinite sequence of mutually v-coprime nonunit elments. Thus, only a finite number of g_i's may be nonunit elements in $K[X]$; i.e., only a finite number of g_i's may not belong to A. This proves our intermediary claim, and eliminating a finite number of them, we may assume that every g_i belongs to A. By (a), for every $i \geq 1$ there exists $n_i \in \mathbb{N}^*$ and $s_i \in S$ such that $g_i' := g_i^{n_i}/s_i$ belongs to A and is v-coprime to every element of S. Note that since g_i, $g_i \in A$ and $g_i R \cap g_j R = g_i g_j R$, we also have $g_i A \cap g_j A = g_i g_j A$; i.e., g_i, g_j are v-coprime in A; then $g_i^{n_i}, g_j^{n_j}$ are v-coprime in A amd g_i', g_j' are v-coprime in A. Note also that g_i being a nonunit element of R and S being saturated, g_i' is a unit of A if and only if $g_i^{n_i}$ belongs to S. Since by hypothesis S does not contain any infinite sequence of mutually v-copositive nonunit elements of A, then there are only finitely many elements of the set $\{g_i'\}$ that are units in A. Eliminating them, we may assume that every g_i' is a nonunit element of A. Note that g_i belongs to the span of f in R; thus, g_i' belongs to the span of f in R, too. Let $a \in A$, $t \in S$, $m \in \mathbb{N}^*$ such that $(a/t)X^n$ is the leading monomial of f. Then, there exists $r \in \mathbb{N}^*$ such that g_i' divides a^r/t^r in A_S; thus, there exists $s \in S$ such that $sa^r \in sA \cap g_i' A$. Since g_i' is v-coprime with every element of S, we have $sA \cap g_i' A = s g_i' A$. This implies that g_i' divides a^r in A and hence that g_i' belongs to the span of a in A. We are thus reduced to a case that we have already settled; we have seen that it leads to a contradiction. ∎

4. EXAMPLES OF NON-INTEGRALLY CLOSED AGCD DOMAINS OF FINITE T-CHARACTER

To put all the examples of AGCD domains of finite t-character together in this article would be too repetitive, so we point out the relevant references.

Example 4.16 of [1] gives for each n, the ring $R_n = \mathbb{Z} + 2^n i\mathbb{Z}$ where $i = \sqrt{-1}$. The ring R_n has the property that for every subset I of R_n, there is $n = n(I)$ such that the ideal generated by $\{a^n; a \in I\}$ is principal (such rings are called *almost principal ideal* (API) domains) and its integral closure $\mathbb{Z}[i]$ is a PID. This makes the ring one-dimensional with the property that every nonzero nonunit of it belongs to at most a finite number of prime ideals. Noting that each nonzero prime ideal of R_n is of height one, it is easy to deduce that each nonzero prime ideal of R_n is, indeed, a t-ideal.

Theorem 4.17 of [1] provides other examples of API domains that are not integrally closed but whose integral closures are Dedekind domains with torsion class groups. Using similar reasoning as above, these rings can also be shown to be of finite t-character.

Some examples in positive characteristic are given by Remark 4.1 and Proposition 4.1. We shall describe two constructions that provide examples of non-integrally closed AGCD domains. They were suggested by Example 2.13 of [27], whose proof shows, more generally, that for an extension $A \subseteq B$ of domains of charactestic $p > 0$ with $B^p \subseteq A$, A is an almost Bézout domain if and only if B is. First, some introductory comments.

Remark 4. 1. (a) Recall from [25], that an extension $A \subseteq B$ of domains is said to be R_2 *stable* if every two nonzero v-coprime elements of A remain v-coprime in B (equivalently, if $xA \cap yA = fA$, with $x, y, f \in A$, implies $xB \cap yB = fB$). By p. 363 of [25], a flat extension of domains is R_2-stable.

(b) If $A \subseteq B$ is an R_2-stable root extension of domains and A is an AGCD domain, then B is also an AGCD domain. Indeed, if $x, y \in B$, there exist $n \in \mathbb{N}^*$ and $f \in A$ such that $x^n, y^n \in A$ and $x^n A \cap y^n A = fA$, so $x^n B \cap y^n B = fB$.

As our next example shows, (b) does not hold if we drop the R_2-stableness hypothesis. Let k be a field of characteristic two, let B be the subring of $k[X, Y]$ consisting of all the polynomials f that have no term in X, Y and XY (i.e., $f = a + bX^2 + cY^2 + dX^3 + \ldots$) and let A be the UFD subring $k[X^2, Y^2]$ of B. Obviously, $B^2 \subseteq A$. But B is not an AGCD domain, because X^{2n}, Y^{2n} have no nonunit common factor in B and $X^{2n+1}Y^{2n+1} \in (X^{2n}B \cap Y^{2n}B) \backslash X^{2n}Y^{2n}B$ for each $n \geq 1$. Notice that the integral closure of B is $B' = k[X, Y]$ and $B'^2 \subseteq B$. So $B \subseteq B'$ is a root extension, B' is a UFD, but B is not an AGCD domain. This example answers a question posed in the paragraph before Theorem 5.9 in [1].

(c) If D is an integrally closed domain of characteristic $p > 0$, then $B = D[X^p, X^{p+1}, \ldots, X^{2p-1}]$ is free (a fortiori flat) $D[X^p]$-module with basis $1, X^{p+1}, \ldots, X^{2p-1}$.

(d) If D is an integrally closed domain of characteristic $p > 0$, then $D^p[X^p] \subseteq D^p + XD[X] = R$ is an R_2-stable extension. Indeed, assume that f, g are two nonzero v-coprime elements of $D[X]$ (hence that f^p, g^p are v-coprime in $D^p[X^p]$); we may also assume that $f(0) \neq 0$. Let $r \in g^pR :_R f^p$. Since f^p, g^p are also v-coprime in $D[X]$, $r = g^pq$, for some $q \in D[X]$. Since D is integrally closed, we obtain successively: $af^p \in R$, $q(0)(f(0))^p \in D^p$, $q(0) \in Q(D^p) \cap D = D^p$, $q \in R$, so $r \in g^pR$. Consequently, f^p, g^p are v-coprime in R.

(e) By Theorem 5.6 of [27], if D is an integrally closed AGCD domain, then so is $D[X]$.

We now give the promised constructions. Assume that D is an integrally closed AGCD domain (for instance, a GCD domain) of characteristic $p > 0$ such that $D \neq D^p$. Then $D[X^p, X^{p+1}, \ldots, X^{2p-1}]$ and $D^p + XD[X]$ are non-integrally closed AGCD domains. Indeed, it suffices to apply (e), (c) and (b) of Theorem 2.7 of [2] for the first case, and respectively (e), (d) and (b) for the second case. If D is an integrally closed AGCD of finite t-character, then $D[X^p, X^{p+1}, \ldots, X^{2p-1}]$ and $D^p + XD[X]$ are AGCD domains of finite t-character. Indeed, the integral closure $D[X]$ is of finite t-character by Theorem 3.1(b) and therefore the conclusion follows by Proposition 3.1(b).

A certain power series compositum construction used in local algebra, also provides examples of non-integrally closed AGCD domains of finite t-character. According to 0_{IV}, 23.1.1 of [16], a domain D is said to be *japanese*, if for each finite field extension L of $Q(D)$, the integral closure of D in L is a finite D-module.

Proposition 4.1. *Let D be a regular local ring of characteristic $p > 0$. Let C be the power series ring $D[[X]]$, A the subring $D^p[[X]]$ and $B = A[D]$.*

(a) B is a local AGCD domain of finite t-character.

(b) The integral closure B' of B is a Krull domain and $B' = Q(B) \cap C$.

(c) Suppose that D is a DVR. Then B is a local two-dimensional domain, whose maximal ideal can be generated by two elements. Moreover, the following assertions are equivalent:

(i) D is a japanese domain,

(ii) B is a Noetherian domain,

(iii) B is integrally closed,

(iv) B is a UFD,

(v) B is a regular ring,

(vi) the extension $B \subseteq C$ is flat.

(d) If D is a non-japanese DVR, $D^p[[X]][D]$ is a two-dimensional AGCD domain of finite t-character, which is neither integrally closed nor Noetherian. In particular, so is $D^p[[X]][D]$ when $D = k^p[[Y]][k]$, where k is a field of characteristic $p > 0$ with $[k : k^p] = \infty$, or $D = k[[Y]] \cap k(Y, u^2)$, where k is a field characteristic two and $u \in k[[Y]]$ is transcendental over $k(Y)$ (in both cases, Y is an indeterminate over k).

Proof. (a) Since D is regular, $D^p \subseteq D$ is a flat extension by Kunz's Theorem 107 of [20]. By Corollary 1, p. 170 of [20], the power series extension $D^p \subseteq A$ is flat, too. By base change, we obtain the injective flat canonical morphism $A \to A \otimes_{D^p} D = R$. A simple direct limit argument, for instance as in Remark 3.1 of [11], proves the injectivity of the canonical morphism $w : R \to C$, given by $w(g \otimes d) = gd$ for $g \in A$ and $d \in D$. Because $w(A) = A$ and $w(R) = B$, $A \subseteq B$ is a flat extension of domains. A is a UFD because A is a regular ring by p.142 of [20]. By Remark 4.1(b), B is an AGCD domain; by the proof of Proposition 3.1(b), B is of finite t-character.

(b) Since C is a UFD and $C^p \subseteq B$, the integral closure B' of B is exactly $Q(B) \cap C$. Hence B' is a Krull domain Corollary 44.10 of [13].

(c) Since $C^p \subseteq B$, the canonical morphism between the spectra of C and B is bijective. Since C is a two-dimensional local domain (see the proof of Theorem 72 of [19]), so is B. If q is a generator of the maximal ideal of D, then q, X generate the maximal ideal of B, because B/XB is A-isomorphic to A. So (ii)$\Leftrightarrow$(v).

(iii)$\Rightarrow$(ii). If B is integrally closed, then as shown above B is a local Krull domain. But a two-dimensional local Krull domain B with finitely generated maximal ideal is a Noetherian domain by Mori-Nishimura's theorem [20], Theorem 104, (if P is a minimal prime ideal of B, B/P is a one-dimensional local domain with finitely generated maximal ideal, so B/P is Noetherian by Cohen's Theorem, Theorem 8 of [19]).

(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(vi). These equivalences follow at once from [11], Corollary 3.4, as soon as we have noticed the simple fact that, for a DVR D, the 'G-ring' property is equivalent to D being a japanese domain (IV, Remarques 7.6.7 of [16]).

(v)$\Rightarrow$(iv)$\Rightarrow$(iii). These implications are obvious.

(d) The two choices for D of the 'in particular' part are the examples of non-japanese DVRs given by Nagata (Example 3.1, p.206 of [22]), and Kaplansky (Theorem 100 of [19]), respectively. ∎

While AGCD domains of finite t-character that are not integrally closed are somewhat obscure, integrally closed AGCD domains abound as we have already seen. Another class of integrally closed AGCD domains is the GCD domains of finite t-character. From [26], we can see that if D is a GCD domain with Property (F), then every nonzero nonunit element of D belongs to only finitely many maximal t-ideals, each associated with a rigid element. This, in the fourth author's usual terminology, was a translation of Paul Conrad's work [9] on lattice ordered groups, every strictly positive element of which exceeded at most a finite number of mutually disjoint strictly positive elements.

5. THE DIVISIBITY GROUP OF AN AGCD DOMAIN

We make a few remarks about the group of divisibility of an AGCD domain.

Let U denote the set of units of D and let K^* denote the set of nonzero elements of the quotient field K of D. Then K^* is a group under multiplication and U is a subgroup of K^*. The quotient group K^*/U, partially ordered by the relation $xU \leq yU \Leftrightarrow yx^{-1} \in D$ is called the *group of divisibility* of D and commonly denoted by $G(D)$. Note that $1U$ is the identity of $G(D)$ and that the partial order is compatible with the group operation. Clearly the positive cone of $G(D)$ is the set $G^+(D) = \{dU ; d \in D^* = D\backslash\{0\}\}$.

For future reference, we state the following well known result:

Proposition 5.1. *Let D be an integral domain, $G(D) = K^*/U$ the group of divisibility of D and $h, k \in K^*$. Then $hU \wedge kU \in G(D) \Leftrightarrow hU \vee kU \in G(D) \Leftrightarrow hD \cap kD$ is principal $\Leftrightarrow (hD + kD)_v$ is principal.*

Let us call a directed partially ordered abelian group G an *almost lattice ordered group* if, for each pair $x, y \in G$, there exists $n = n(x, y)$ such that $x^n \vee y^n$ exists. Then we have the following statement.

Proposition 5.2. *An integral domain D is an almost GCD domain if and only if $G(D)$ is an almost lattice ordered group.*

We use this introduction to pose some problems, which may be considered at a later time. We know that if G is a lattice ordered group then G is, up to isomorphism, the group of divisibility of a Bézout domain [13], pp. 214–215.

Question 1. Given an almost lattice ordered group $\boldsymbol{G}$, does there exist an AGCD domain whose group of divisibility is $\boldsymbol{G}$?

Question 2. Under what conditions is an almost lattice ordered group a lattice ordered group?

In [9], Paul Conrad studies lattice ordered groups G that satisfy the following condition:

(F) Each strictly positive element $x \in G$ is greater than at most a finite number of mutually disjoint elements of G. (Here, by a strictly positive element x we mean $x > e$, where e denotes the identity of G.)

In the case of an AGCD domain D we have translated condition (F) to: the span of the nonzero nonunit $x \in D$, contains at most a finite number of mutually v-coprime elements. Looking at the definition of the span of x, it appears that the corresponding condition for almost lattice ordered groups G will be:

(F′) For every strictly positive element $x \in G$, $C(x)$, the smallest convex subsemi-group of G^+ containing x, contains at most a finite number of mutually disjoint elements.

This leads to our next question, which we pose as a problem.

Problem 3. Study the structure of almost lattice ordered groups that satisfy condition (F′).

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