ON $\star$-POWER CONDUCTOR DOMAINS

D.D. ANDERSON, EVAN HOUSTON, AND MUHAMMAD ZAFRULLAH

ABSTRACT. Let D be an integral domain and $\star$ a star operation defined on D. We say that D is a $\star$-power conductor domain ($\star$-PCD) if for each pair $a, b \in D\backslash(0)$ and for each positive integer n we have $Da^n \cap Db^n = ((Da \cap Db)^n)^\star$. We study $\star$-PCDs and characterize them as root closed domains satisfying $((a,b)^n)^{-1} = (((a,b)^{-1})^n)^\star$ for all nonzero a, b and all natural numbers $n \geq 1$. From this it follows easily that Prüfer domains are d-PCDs (where d denotes the trivial star operation), and v-domains (e.g., Krull domains) are v-PCDs. We also consider when a $\star$-PCD is completely integrally closed, and this leads to new characterizations of Krulll domains. In particular, we show that a Noetherian domain is a Krull domain if and only if it is a w-PCD.

INTRODUCTION

Let D be an integral domain with quotient field K. For $a, b \in D \setminus (0)$ and n a positive integer, it is clear that $Da^n \cap Db^n \supseteq (Da \cap Db)^n$, and it is elementary that we have equality (for all a, b, n) if D is a GCD-domain (e.g., a UFD) or a Prüfer domain. On the other hand, Krull domains, even integrally closed Noetherian domains, may allow $Da^n \cap Db^n \supsetneq (Da \cap Db)^n$ for some nonzero a, b and $n > 1$ (see [3, Section 3]). However, recalling the v-operation on the domain D, given by $A_v = (D : (D : A))$ for nonzero fractional ideals A of D and letting D be a Krull domain, we do have $Da^n \cap Db^n = ((Da \cap Db)^n)_v$ for all nonzero $a, b \in D$ and $n \geq 1$.

Now the v-operation is an example of a star operation. We recall the definition: Denoting by $\mathcal{F}(D)$ the set of nonzero fractional ideals of D, a map $\star : \mathcal{F}(D) \to \mathcal{F}(D)$ is a *star operation* if the following conditions hold for all $A, B \in \mathcal{F}(D)$ and all $c \in K \setminus (0)$:

(1) $(cA)^\star = cA^\star$ and $D^\star = D$;
(2) $A \subseteq A^\star$, and, if $A \subseteq B$, then $A^\star \subseteq B^\star$; and
(3) $A^{\star\star} = A^\star$.

The most frequently used star operations (as well as the most important for our purposes) are the d-operation, given, for $A \in \mathcal{F}(D)$, by $A_d = A$; the v-operation, defined above; the t-operation, given by $A_t = \bigcup B_v$, where the union is taken over all nonzero finitely generated fractional subideals B of A; and the w-operation, given by $A_w = \{x \in K \mid xB \subseteq A \text{ for some finitely generated ideal } B \text{ of } D \text{ with } B_v = D\}$. For any star operation $\star$ on D, we have $d \leq \star \leq v$, in the sense that $A = A_d \subseteq A^\star \subseteq A_v$ for all nonzero fractional ideals A of D. Other basic properties of $\star$-operations may be found in [11, Sections 32, 34] (but we do review much of what we use in the sequel).

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For a star operation $\star$ on D, we say that D is a *$\star$-power conductor domain* ($\star$-PCD) if $Da^n \cap Db^n = ((Da \cap Db)^n)^\star$ for all $a, b \in D \setminus (0)$ and all positive integers n. (The reason for the "conductor" terminology will become clear after Definition 1.2 and Proposition 1.3 below.) As mentioned above, examples of d-PCDs include GCD-domains and Prüfer domains, while Krull domains are v-PCDs. In Section 1, as a consequence of Theorem 1.6, we show that essential domains, i.e., domains possessing a family $\mathcal{P}$ of prime ideals with $D = \bigcap_{P \in \mathcal{P}} D_P$ with each D_P a valuation domain, are v-PCDs. We also show (Proposition 1.11) that if D is a $\star$-PCD, then D must be root closed and that for each maximal ideal M of D, we must have M invertible, $M^{-1} = D$, or $M = (M^2)^\star$. This leads to a characterization of $\star$-PCDs as root closed domains D satisfying $((a, b)^n)^{-1} = (((a, b)^{-1})^n)^\star$ for all nonzero $a, b \in D$. This latter condition is obviously a weakened form of invertibility; indeed, as an easy corollary we obtain that so-called $\star$-Prüfer domains, domains D in which each nonzero finitely generated ideal A satisfies $(AA^{-1})^\star = D$, are $\star$-PCDs.

Section 2 presents examples of the subtleties involved. Among others, we give examples of d-PCDs that are not integrally closed and hence not essential, v-PCDs that are not d-PCDs, and v-PCDs with non-v-PCD localizations. In Section 3 we study complete integral closure in v-PCDs and give several new characterizations of Krull domains. For example, we show that D is a Krull domain if and only if D is a v-PCD in which v-invertible ideals are t-invertible and $\bigcap_{n=1}^\infty (M^n)_v = (0)$ for each maximal t-ideal M of D. In Section 4 we study two notions that have appeared previously in the literature and that are closely related to the d-PCD property. We also study w-PCDs and show that a Noetherian domain is a Krull domain if and only if it is a w-PCD. Finally we characterize Krull domains as w-PCDs in which maximal t-ideals M are divisorial and satisfy $\bigcap_{n=1}^\infty (M^n)_w = (0)$.

There are (at least) two rather natural ways to weaken the $\star$-PCD notion (see Definition 1.1). We show that the three notions are distinct and, where possible (and convenient), prove results in somewhat greater generality than described above.

We use the following notational conventions: The term "local" requires a ring to have a unique maximal ideal but does not require it to be Noetherian; $\subset$ denotes proper inclusion; and for fractional ideals A, B of D, $(A : B) = \{x \in K \mid xA \subseteq B\}$, while $(A :_D B) = \{d \in D \mid dA \subseteq B\}$.

1. Weak $\star$-PCDs

Throughout this section, D denotes a domain and K its quotient field. We begin with our basic definition(s).

Definition 1.1. Let $\star$ be a star operation on D and n a positive integer. We say that a pair $a, b \in D \setminus (0)$ *satisfies* $\star_n$ if $Da^n \cap Db^n = ((Da \cap Db)^n)^\star$. We then say that D

1. is a *weak $\star$-PCD* if for each pair $a, b \in D \setminus (0)$ there is an integer $m > 1$, depending on a, b, such that a, b satisfies $\star_m$;
2. *satisfies* $\star_n$ if each pair of nonzero elements of D satisfies $\star_n$;
3. is a *$\star$-PCD* if D satisfies $\star_n$ for each $n \geq 1$.

With the notation above, since the ideal $Da^n \cap Db^n$ is divisorial and hence a $\star$-ideal, it is clear that the inclusion $Da^n \cap Db^n \supseteq ((Da \cap Db)^n)^\star$ holds automatically. Of course, we have equality when $n = 1$. It is also clear that for any $n > 1$, D is a $\star$-PCD $\Rightarrow$ D satisfies $\star_n$ $\Rightarrow$ D is a weak $\star$-PCD. In Example 2.4 below, we show that these notions are distinct when $\star$ is d or v.

For $x, y \in D \setminus \{0\}$, we have $xy(x,y)^{-1} = xy(Dx^{-1} \cap Dy^{-1}) = Dx \cap Dy = Dx(D \cap D(y/x)) = Dx(Dx :_D Dy) = Dx(D :_D D(y/x))$. Hence for $a, b \in D \setminus (0)$, $u = b/a$, $\star$ a star operation on D, and $n \geq 1$, we have (using the fact that we may cancel nonzero principal ideals in equations involving star operations) $Da^n \cap Db^n = ((Da \cap Db)^n)^\star \Leftrightarrow (Da^n :_D Db^n) = ((Da :_D Db)^n)^\star \Leftrightarrow (D :_D Du^n) = ((D :_D Du)^n)^\star \Leftrightarrow (a^n, b^n)^{-1} = (((a,b)^{-1})^n)^\star$.

Motivated by this, we state the next definition and proposition.

Definition 1.2. Let $\star$ be a star operation on D and n a positive integer. We say that an element $u \in K \setminus (0)$ *satisfies* $\star_n$ if $(D :_D Du^n) = ((D :_D Du)^n)^\star$ (equivalently, $(1, u^n)^{-1} = (((1,u)^{-1})^n)^\star$).

Proposition 1.3. *Let $a, b \in D \setminus (0)$, $u = b/a$, and $n \geq 1$. The following statements are equivalent:*

1. *The pair a, b satisfies $\star_n$.*
2. *The element u satisfies $\star_n$.*
3. $(a^n, b^n)^{-1} = (((a,b)^{-1})^n)^\star$. □

A consequence of the next result is that if D is a $\star$-PCD for any $\star$, then D is a v-PCD.

Lemma 1.4. *Let $\star' \geq \star$ be star operations on D, $a, b \in D \setminus (0)$, and $n \geq 1$.*

1. *If a, b satisfies $\star_n$, then a, b also satisfies $\star'_n$. (Equivalently, if $u \in K \setminus (0)$ satisfies $\star_n$, then u also satisfies $\star'_n$.).*
2. *If D is a weak $\star$-PCD (satisfies $\star_n$, is a $\star$-PCD), then D is a weak $\star'$-PCD, (satisfies $\star'_n$, is a $\star'$-PCD).*
3. *If D is a weak $\star$-PCD (satisfies $\star_n$, is a $\star$-PCD), then D is a weak v-PCD (satisfies v_n, is a v-PCD).*

Proof. (1) Assume that $a, b \in D \setminus (0)$ satisfies $\star_n$ for some n. Then $(Da^n \cap Db^n) = ((Da \cap Db)^n)^\star \subseteq ((Da \cap Db)^n)^{\star'} \subseteq Da^n \cap Db^n$ (since $Da^n \cap Db^n$ is divisorial and hence automatically a $\star'$-ideal). Statement (2) follows from (1), and (3) follows from (2). □

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Proposition 1.5. *Let D be a domain, and let $a, b \in D \setminus (0)$. Then a, b satisfies $\star_m$ for some $m > 1$ if and only if there is a sequence $1 < n_1 < n_2 < \cdots$ such that a, b satisfies $\star_{n_i}$ for each i. Hence D is a weak $\star$-PCD if and only if for each pair $a, b \in D \setminus (0)$, there is a sequence $1 < n_1 < n_2 < \cdots$ such that a, b satisfies $\star_{n_i}$ for each i.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) We have $Da^{n_1} \cap Db^{n_1} = ((Da \cap Db)^{n_1})^\star$ for some $n_1 > 1$. Suppose $1 < n_1 < \cdots < n_k$ have been chosen so that $Da^{n_i} \cap Db^{n_i} = ((Da \cap Db)^{n_i})^\star$ for $i = 1, ..., k$. Choose $n > 1$ so that $(Da^{n_k})^n \cap (Db^{n_k})^n = ((Da^{n_k} \cap Db^{n_k})^n)^\star$. Put $n_{k+1} = n_k n$. Then

$$\begin{aligned} Da^{n_{k+1}} \cap Db^{n_{k+1}} &= (Da^{n_k})^n \cap (Db^{n_k})^n = ((Da^{n_k} \cap Db^{n_k})^n)^\star \\ &= ((((Da \cap Db)^{n_k})^\star)^n)^\star = ((Da \cap Db)^{n_k n})^\star = ((Da \cap Db)^{n_{k+1}})^\star, \end{aligned}$$

as desired. □

Recall that if $D \subseteq R$ is an extension of domains, then R is said to be *LCM-stable over* D if $(Da \cap Db)R = Ra \cap Rb$, for each $a, b \in D$, equivalently, if $(D :_D Du)R = (R :_R Ru)$ for each $u \in K \setminus D$. (LCM-stability was introduced by R. Gilmer [12] and popularized by H. Uda [20, 21].) Each flat overring of D is LCM-stable over D.

Now let $\{D_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of overrings of D with $D = \bigcap_{\alpha \in \mathcal{A}} D_\alpha$, and for each $\alpha \in \mathcal{A}$, let $\star_\alpha$ be a star operation on D_α. For a nonzero fractional ideal I of D, set $I^\star = \bigcap_{\alpha \in \mathcal{A}} (ID_\alpha)^{\star_\alpha}$. Then $\star$ is a star operation on D [1].

Theorem 1.6. *Let $\{D_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of LCM-stable overrings of D with $D = \bigcap_{\alpha \in \mathcal{A}} D_\alpha$, and for each $\alpha \in \mathcal{A}$, let $\star_\alpha$ be a star operation on D_α. Let $n \geq 1$, and let u be a nonzero element of K such that u satisfies $(\star_\alpha)_n$ for each α. Then u satisfies $\star_n$ (where $\star$ is the star operation defined above). In particular, u satisfies v_n.*

Proof. We have

$$\begin{aligned}(D :_D Du^n) &\subseteq \bigcap_\alpha (D_\alpha :_{D_\alpha} D_\alpha u^n) = \bigcap_\alpha ((D_\alpha :_{D_\alpha} D_\alpha u)^n)^{\star_\alpha} \\ &= \bigcap_\alpha (((D :_D Du)^n) D_\alpha)^{\star_\alpha} = ((D :_D Du)^n)^\star.\end{aligned}$$

The "in particular" statement follows from Lemma 1.4. □

Recall that the domain D is said to be *essential* if $D = \bigcap_{P \in \mathcal{P}} D_P$ for some family $\mathcal{P}$ of primes of D with each D_P a valuation domain. Since valuation domains are d-PCDs and localizations are LCM-stable, the next two results are immediate.

Corollary 1.7. *An essential domain D is a v-PCD.* □

Corollary 1.8. *Let $n \geq 1$. If D_M satisfies v_n for each maximal ideal M of D, then D satisfies v_n. In particular, if each D_M is a v-PCD, then D is a v-PCD.* □

The converses of both of these are false–see Examples 2.4(4) and 2.6 below. But the d-PCD property holds if and only if it holds locally:

Corollary 1.9. *Let $n \geq 1$. The following statements are equivalent.*

(1) *D satisfies d_n.*
(2) *D_S satisfies d_n for each multiplicatively closed subset S of D.*
(3) *D_P satisfies d_n for each prime ideal P of D.*
(4) *D_M satisfies d_n for each maximal ideal M of D.*

The statements remain equivalent if "satisfies d_n" is replaced by "is a d-PCD."

Proof. Assume that D_M satisfies d_n for each maximal ideal M of D. By Theorem 1.6, D satisfies $\star_n$ for the star operation given by $A^\star = \bigcap_{M \in \mathrm{Max(D)}} AD_M$. However, $\star = d$ in this case. Hence (4) $\Rightarrow$ (1). Now assume that D satisfies d_n, let S be a multiplicatively closed subset of D, and let $u \in K \setminus (0)$. Then $(D_S :_{D_S} D_S u^n) = (D :_D Du^n)D_S = (D :_D Du)^n D_S = (D_S :_{D_S} D_S u)^n$. This gives (1) $\Rightarrow$ (2). The other implications are trivial. □

In order to get the v-PCD property to pass to quotient rings, we need a finiteness condition. Recall that D is said to be *v-coherent* if I^{-1} is a v-ideal of finite type for each nonzero finitely generated I of D. Obviously, Noetherian domains are v-coherent. More generally, Mori domains, domains satisfying the ascending chain condition on divisorial ideals, are v-coherent.

Corollary 1.10. *Let D be a v-coherent domain and $n \geq 1$. Then the following statements are equivalent.*

1. *D satisfies v_n.*
2. *D_S satisfies v_n for every multiplicatively closed subset S of D.*
3. *D_P satisfies v_n for every prime ideal P of D.*
4. *D_M satisfies v_n for every maximal ideal M of D.*
5. *D_M satisfies v_n for every maximal t-ideal M of D.*
6. *There is a family $\mathcal{P} = \{P\}$ of prime ideals of D such that $D = \cap_{P\in\mathcal{P}} D_P$ and D_P satisfies v_n for every $P \in \mathcal{P}$.*

The statements remain equivalent if "satisfies v_n" is replaced by "is a v-PCD."

Proof. Let $n > 1$ and $u \in K \setminus (0)$. Assume that u satisfies v_n, and let S be a multiplicatively closed subset of D. According to [6, Lemma 2.5], if A is a v-ideal of finite type in the v-coherent domain D, then $A_v D_S = (AD_S)_{v_S}$ (where v_S is the v-operation on D_S). Using this, we have $(D_S :_{D_S} D_S u^n) = (D :_D Du^n)D_S = ((D :_D Du)^n)_v D_S = ((D :_D Du)^n D_S)_{v_S} = ((D_S :_{D_S} D_S u)^n)_{v_S}$. The implication (1) $\Rightarrow$ (2) follows. Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (6) and (3) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial. Finally, (6) $\Rightarrow$ (1) by Theorem 1.6. $\square$

In Corollary 1.9 the existence of a family $\mathcal{P} = \{P\}$ of prime ideals of D such that $D = \cap_{P\in\mathcal{P}} D_P$ and D_P is a d-PCD does not suffice to conclude that D is a d-PCD: A Krull domain D obviously has the property that D_P is a d-PCD for each maximal t-ideal P of D, but such a D is a d-PCD if and only if each P^n is divisorial, and a Krull domain need not have this property ([3, comment following Lemma 3.7]). (We revisit this in Section 4 below.)

We shall make frequent use of the next result. For rings $R \subseteq S$ and a positive integer n, we say that R is *n-root closed in S* if $u \in S \setminus R$ implies $u^n \notin R$, and we say that the domain D is n-root closed if D is n-root closed in K.

Proposition 1.11. *Let D be a domain, and let $\star$ be a star operation on D.*

1. *If $u \in K \setminus D$ satisfies $\star_n$, then $u^n \notin D$.*
2. *If D is a weak $\star$-PCD and M is a maximal ideal of D, then M must satisfy one of the following conditions:*
   - (a) *M is invertible.*
   - (b) *$M^{-1} = D$.*
   - (c) *$M = (M^2)^\star$.*
3. *If D satisfies $\star_n$ for some $n > 1$ (is a $\star$-PCD), then D is n-root closed (is root closed), and each maximal ideal of D must satisfy one of the conditions above.*

Proof. (1) Let $u \in K \setminus D$, and assume that u satisfies $\star_n$, $n > 1$. Then $(D :_D Du^n) = ((D :_D Du)^n)^\star \subseteq (D :_D Du)^\star = (D :_D Du) \subset D$. (The last equality follows since $(D :_D Du)$ is divisorial and hence a $\star$-ideal.)

(2) Assume that D is a weak $\star$-PCD, let M be a maximal ideal of D, and assume that $M^{-1} \neq D$ and that M is not invertible. We may then find $u \in M^{-1} \setminus D$, and we have $M = (D :_D Du)$. By hypothesis, u satisfies $\star_n$ for some $n > 1$. Since M is not invertible, we must have $Mu \subseteq M$, whence $Mu^n \subseteq M$. By (1), $u^n \notin D$, and hence $M = (D :_D Du^n) = ((D :_D Du)^n)^\star = (M^n)^\star$. It then follows rather easily that $M = (M^2)^\star$:

$$M = (M^n)^\star \subseteq (M^2)^\star \subseteq M^\star = ((M^n)^\star)^\star = (M^n)^\star = M.$$

(3) This follows from (1) and (2). ∎

Next, we characterize (n-)root closed domains.

Proposition 1.12. *For $n \geq 1$, a domain D is n-root closed if and only if $(1, u^n)^{-1} = ((1, u)^n)^{-1}$ for all $u \in K$ (equivalently, $(a^n, b^n)^{-1} = ((a, b)^n)^{-1}$ for all nonzero $a, b \in D$).*

Proof. Assume that D is an n-root closed domain, and let $u \in K$. Since $(1, u^n) \subseteq (1, u)^n$, we have $(1, u^n)^{-1} \supseteq ((1, u)^n)^{-1}$. Let $r \in (1, u^n)^{-1}$, that is, let $r, ru^n \in D$. Then for $1 \leq k \leq n$, we have $(ru^k)^n = r^{n-k}(ru^n)^k \in D$, and, since D is n-root closed, we obtain $ru^k \in D$. Hence $r \in ((1, u)^n)^{-1}$, as desired.

Conversely, assume $(1, u^n)^{-1} = ((1, u)^n)^{-1}$ for each $u \in K$. Then if $u^n \in D$, the left side of the equation is equal to D, and the right side then puts $u \in D$. ∎

For $u \in K$, we always have $(1, u^n)^{-1} \supseteq ((1, u)^n)^{-1} \supseteq (((1, u)^{-1})^n)^\star$ for any star operation $\star$. It follows that u satisfies $\star_n$ if and only if both inclusions are equalities. If we combine this observation with Proposition 1.12, we obtain the following characterization of the $\star$-PCD property.

Theorem 1.13. *A domain D satisfies $\star_n$ if and only if it is n-root closed and $((1, u)^n)^{-1} = (((1, u)^{-1})^n)^\star$ for each $u \in K$. Hence D is a $\star$-PCD if and only if D is root closed and $((1, u)^n)^{-1} = (((1, u)^{-1})^n)^\star$ for each $u \in K$ and $n \geq 1$. (Equivalently, D is a $\star$-PCD if and only if D is root closed and $((a, b)^n)^{-1} = (((a, b)^{-1})^n)^\star$ for all nonzero $a, b \in D$ and $n \geq 1$.)*

Proof. If D is n-root closed and satisfies the given equality, then $(D :_D Du^n) = (1, u^n)^{-1} = ((1, u)^n)^{-1} = (((1, u)^{-1})^n)^\star = ((D :_D Du)^n)^\star$. Conversely, if D satisfies $\star_n$, then D is n-root closed, and $((1, u)^n)^{-1} = (1, u^n)^{-1} = (D :_D Du^n) = ((D :_D Du)^n)^\star = (((1, u)^{-1})^n)^\star$. ∎

Let $\star$ be a star operation on D, and let A be a nonzero fractional ideal of D. Then A is said to be *$\star$-invertible* if $(AA^{-1})^\star = D$. It is well-known (and easy to show) that if A is $\star$-invertible and $(AB)^\star = D$ for some fractional ideal B, then we must have $B^\star = A^{-1}$; furthermore, if n is a positive integer and we apply this fact to the equation $(A^n(A^{-1})^n)^\star = D$, we also have $(A^n)^{-1} = ((A^{-1})^n)^\star$. We use this equality in the following theorem. (Note that while the equality holds for $\star$-invertible ideals, it fails in general as we point out in Example 3.5 below.)

Theorem 1.14. *Let $\star$ be a star operation on D, let $u \in K \setminus (0)$, and assume that the fractional ideal $(1, u)$ is $\star$-invertible. Then u satisfies $\star_n$ for each $n \geq 1$. (Equivalently, if $a, b \in D \setminus (0)$ are such that (a, b) is $\star$-invertible, then the pair a, b satisfies $\star_n$ for each $n \geq 1$.)*

Proof. Begin with the equality $(1, u)^{2n} = (1, u)^n(1, u^n)$. Multiplying by $((1, u)^{-1})^n$ and taking $\star$'s yields $((1, u)^n)^\star = (1, u^n)^\star$. Taking inverses then yields $((1, u)^n)^{-1} = (1, u^n)^{-1}$, and combining this with the above-mentioned equality (with $A = (1, u)$), we have $(1, u^n)^{-1} = (((1, u)^{-1})^n)^\star$. This latter equation is equivalent to "u satisfies $\star_n$." ∎

We have the following corollary to (the proof of) Theorem 1.14.

Corollary 1.15. *Let $\star$ be a star operation on D, let $u \in K \setminus (0)$, and assume that the fractional ideal $(1, u)$ is $\star$-invertible. Then $(1, u^n)^\star = ((1, u)^n)^\star$. (Equivalently, if $a, b \in D \setminus (0)$ are such that (a, b) is $\star$-invertible, then $(a^n, b^n)^\star = ((a, b)^n)^\star$).*

If each nonzero finitely generated ideal of D is $\star$-invertible, then D is said to be a *$\star$-Prüfer domain* [2]. (Thus a v-Prüfer domain is a v-domain.) Thus the next result is immediate from Theorem 1.14.

Corollary 1.16. *A $\star$-Prüfer domain is a $\star$-PCD. In particular, a v-domain is a v-PCD.* □

It is well known that essential domains are v-domains; hence Corollary 1.16 strengthens Corollary 1.7. Since a v-PCD need not be integrally closed (see Example 2.4 below), the converse of Corollary 1.16 is false.

Recall [11, Section 32] that to any star operation $\star$ on D, we may associate a star operation $\star_f$ given by $A^{\star_f} = \bigcup B^\star$, where the union is taken over all nonzero finitely generated subideals B of A. If $\star$ is a star operation on D and $\star = \star_f$ (i.e., $\star$ is of *finite type*), it is well known that (1) each nonzero element a of D is contained in a maximal $\star$-ideal, (2) $D = \bigcap D_P$, where the intersection is taken over all maximal $\star$-ideals P of D, and (3) primes minimal over a nonzero element are $\star$-ideals. When $\star = v$, $\star_f$ is the well-studied t-operation. Finally, recall that a *Prüfer $\star$-multiplication domain* (P$\star$MD) is a domain in which each nonzero finitely generated ideal is $\star_f$-invertible. Put another way, a P$\star$MD is a $\star$-Prüfer domain D in which A^{-1} is a finite-type $\star$-ideal for each nonzero finitely generated ideal A of D. (A $\star$-ideal I has *finite type* if $I = J^\star$ for some finitely generated subideal J of I.)

Corollary 1.17. *A P$\star$MD is a $\star$-PCD. In particular, PvMDs are v-PCDs.* □

A domain D is called an *almost GCD-domain* (AGCD-domain) if for each pair $a, b \in D \setminus (0)$ there is a positive integer n for which $Da^n \cap Db^n$ is principal [24]. We end this section by showing that within this class of domains, a v-PCD must be essential.

Proposition 1.18. *For an AGCD domain D the following are equivalent.*

(1) *D is a PvMD.*
(2) *D is essential.*
(3) *D is a v-domain.*
(4) *D is a v-PCD.*
(5) *D is root closed.*
(6) *D is integrally closed.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are well known. For the rest, (3) ⇒ (4) by Corollary 1.16, (4) ⇒ (5) by Proposition 1.11, (5) ⇒ (6) by [24, Theorem 3.1], and (6) ⇒ (1) by [24, Corollary 3.8]. □

2. Pullbacks and examples

Let T be a domain, M a maximal ideal of T, $\varphi : T \to k := T/M$ the natural projection, and D a proper subring of k. Then let $R = \varphi^{-1}(D)$, that is, let R be the domain arising from the following pullback of canonical homomorphisms.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ T \arrow[r, "\varphi"] & T/M = k \end{tikzcd}$$

Since R and T share a nonzero ideal, they have a common quotient field, which, throughout this section, will be denoted by K.

Lemma 2.1. *In the diagram above, assume that D is a field and that $T = (M : M)$. Let $n > 1$. Then:*

(1) *R satisfies d_n if and only if T satisfies d_n, $M = M^2$, and D is n-root closed in k.*
(2) *Suppose that T satisfies v_n locally, $(T_M : MT_M) = T_M$, D is n-root closed in k, and for each nonzero $u \in K$, if $(T_M :_{T_M} T_M u^n)$ is principal in T_M, then $(T_M :_{T_M} T_M u)$ is also principal. Then R satisfies v_n.*

Proof. (1) We begin by assuming that T is local with maximal ideal M. We claim that if $u \in K$ is such that $u, u^{-1} \notin T$, then $(R :_R Ru) = (T :_T Tu)$. To verify this, suppose that $t \in T$ satisfies $tu \in T$. Then $t \in M \subseteq R$ since $u \notin T$, and $tu \in M \subseteq R$ since $t(tu)^{-1} = u^{-1} \notin T$. The claim follows easily. Now assume that R satisfies d_n. It is clear that M cannot be invertible in R and also that $M^{-1} \neq R$. Hence $M = M^2$ by Proposition 1.11. Suppose that $t \in T$ satisfies $\varphi(t)^n \in D$. Then $t^n \in R$, whence $t \in R$ and then $\varphi(t) \in D$. Hence D is n-root closed in k. We next show that T is n-root closed. For this, suppose that $u \in K \setminus T$ and $u^n \in T$. We cannot have $u^n \in R$ since R is n-root closed (Proposition 1.11). Hence $M = (R :_R Ru^n) = (R :_R Ru)^n \subseteq M^n = M$. It follows that $(R :_R Ru) = M$, whence $u \in M^{-1} = (M : M) = T$. Hence T is n-root closed. Finally, let $y \in K \setminus T$. Then $y^n \notin T$. If $y^{-1} \in T$, then $(T :_T Ty^n) = T(y^{-n}) = (T :_T Ty)^n$. If $y^{-1} \notin T$, then from the claim above, we have $(T :_T Ty^n) = (R :_R Ry^n) = (R :_R Ry)^n = (T :_T Ty)^n$. Therefore, T satisfies d_n.

For the converse, assume that T satisfies d_n with $M = M^2$ and D n-root closed in k, and let $u \in K \setminus R$. First suppose that $u \in T$. It is easy to see that D n-root closed in k implies that R is n-root closed in T and hence that $u^n \notin R$. We then have $(R :_R Ru^n) = M = M^n = (R :_R Ru)^n$. Now suppose that $u \notin T$. If $u^{-1} \in R$, then $(R :_R Ru^n) = Ru^{-n} = (R :_R Ru)^n$, as desired. If $u^{-1} \notin R$, then $u^{-1} \notin T$ (lest $u^{-1} \in M \subseteq R$). In this case (again using the claim above), we have $(R :_R Ru^n) = (T :_T Tu^n) = (T :_T Tu)^n = (R :_R Ru)^n$. Hence R satisfies d_n. This proves (1) in local case.

For the general case, note that each maximal ideal of R is of the form $N \cap R$, where N is a maximal ideal of T, and, for $N \neq M$, $R_{N\cap R} = T_N$ (see, e.g., [10, Theorem 1.9]). Localizing the diagram at M yields that R_M satisfies d_n if and only if T_M satisfies d_n, $M = M^2$, and D is n-root closed in k. The general case now follows easily from Corollary 1.9.

For (2), note that $(R : M^2) = ((R : M) : M) = (T : M) = T$, whence $(M^2)_v = T^{-1} = M$. Also, as above, R is n-root closed in T. Now suppose that T is local, and let $u \in K \setminus R$. If $u \in T$, then $u^n \in T \setminus R$ and $(R :_R Ru^n) = M = (M^n)_v = ((R :_R Ru)^n)_v$. Suppose that $u \notin T$. If $u^{-1} \in R$, proceed as before. Assume $u^{-1} \notin R$ and hence (see above) that $u^{-1} \notin T$. Even so, it is possible that $(T :_T Tu) = Tx$ for some $x \in T$. In this case, we have $(R :_R Ru^n) = (T :_T Tu^n) = ((T :_T Tu)^n)_{v_T} = (Tx)^n = (T :_T Tu)^n = (R :_R Ru)^n$. Finally, suppose that $(T :_T Tu)$ is not principal. At this point, it is helpful to observe that if A is a non-principal fractional ideal of T, then (A is a fractional ideal of R and) $A^{-1} = (M : A) = (T : A)$. In particular, $(R :_R Ru)^{-1} = (T : (T :_T Tu))$. By hypothesis, we have $(T :_T Tu^n)$ non-principal, and, as before, $(R :_R Ru^n) = (T :_T Tu^n)$. By the observation, this yields $(R :_R Ru^n)^{-1} = (T : (T :_T Tu^n)) = (T : (T :_T Tu)^n)$. Then, since $(T :_T Tu)^n$ is not principal and is equal to $(R :_R Ru)^n$, we have $(R :_R Ru^n)^{-1} = ((R :_R Ru)^n)^{-1}$, whence $(R :_R Ru^n) = (R :_R Ru^n)_v = ((R :_R Ru)^n)_v$, as desired. This completes the proof of the local case. An easy localization argument, together with Corollary 1.8, then yields the general case. □

Consider the generic pullback diagram above, and assume that k is the quotient field of D. An easy calculation, or an appeal to [9, Proposition 1.8], yields the following facts: For $t \in T$, $(D :_D D\varphi(t)) = \varphi(R :_R Rt)$ and $\varphi^{-1}(D :_D D\varphi(t)) = (R :_R Rt)$. We use these in the next result.

Theorem 2.2. *In the diagram above, assume that $T = (M : M)$, and let $n > 1$. Then R satisfies d_n if and only if T, D both satisfy d_n, D is n-root closed in k, and at least one of the following holds: $M = M^2$ or k is the quotient field of D.*

Proof. The case where D is a field is handled by Lemma 2.1. Suppose that D is not a field but that k is the quotient field of D, and assume that R satisfies d_n. Then each localization of T at a maximal ideal agrees with a localization of R, and hence T satisfies d_n by Corollary 1.9. Now let $t \in T \setminus R$. From the remarks above, we have $(D :_D D\varphi(t)^n) = \varphi(R :_R Rt^n) = \varphi((R :_R Rt)^n) = (D :_D D\varphi(t))^n$. Hence D satisfies d_n. Note that D is automatically n-root closed in k since it satisfies d_n. For the converse, suppose that P is a maximal ideal of R. If P is incomparable to M, then $P = N \cap R$ for some maximal ideal N of T, and hence $R_P = T_N$ satisfies d_n. If $P \supseteq M$, then, localization produces the following pullback diagram:

$$\begin{array}{ccc} R_P & \longrightarrow & D_{\varphi(P)} \\ \downarrow & & \downarrow \\ T_M & \xrightarrow{\varphi} & T/M = k \end{array}$$

At this point, for the remainder of this part of the proof, we change notation and assume that D and T are local with maximal ideals $\varphi(P)$ and M, respectively. Let $u \in K$. If $u \in T$, then $(R :_R Ru^n) = \varphi^{-1}(D :_D D\varphi(u)^n) = \varphi^{-1}((D :_D \varphi(u))^n) = (R :_R Ru)^n$, as required. If $u \notin T$ but $u^{-1} \in R$, then $(R :_R Ru^n) = Ru^{-n} = (R :_R Ru)^n$. If $u^{-1} \notin R$, then $u^{-1} \notin T$, and it is easy to see that $(R :_R Ru^n) = (T :_T Tu^n) = (T :_T Tu)^n = (R :_R Ru)^n$. Therefore, R satisfies d_n.

Finally (and switching back to the original notation), assume that D is not a field and that the quotient field F of D is properly contained in k. Let $S := \varphi^{-1}(F)$. If R satisfies d_n, then by what was proved in the preceding paragraph, S satisfies d_n and (hence) D is n-root closed in F. Lemma 2.1 then yields that T satisfies d_n, $M = M^2$, and that F is n-root closed in k; it follows that D is n-root closed in k. For the converse, assume that D and T satisfy d_n, D is n-root closed in k, and $M = M^2$. To see that F is n-root closed in k, let $x \in k$ with $x^n \in F$. Write $x^n = d/e$ with $d, e \in D$. Then $(ex)^n = e^{n-1}ex^n \in D$, whence $ex \in D$, and we have $x \in F$, as desired. Lemma 2.1 then ensures that S satisfies d_n, and then the preceding paragraph yields that R satisfies d_n. □

Recall that a local domain (R, M) is a *pseudo-valuation domain* (PVD) if M^{-1} is a valuation domain with maximal ideal M [15]; V is then called the canonical valuation overring of R. It follows that a domain R is a PVD if and only if it is a pullback of the type in Lemma 2.1 with T a valuation domain [5, Proposition 2.6]. We specialize Lemma 2.1 to PVDs:

Corollary 2.3. *Let (R, M) be a PVD with canonical valuation overring $V = M^{-1}$, and assume that $R \subsetneq V$. Then:*

(1) *The following statements are equivalent.*
  (a) *R is a weak v-PCD.*
  (b) *$M = M^2$.*
  (c) *R is a weak d-PCD.*

(2) *If $n > 1$, the following statements are equivalent.*
  (a) *R satisfies v_n.*
  (b) *R is n-root closed (equivalently, R/M is n-root closed in V/M) and $M = M^2$.*
  (c) *R satisfies d_n.*

(3) *The following statements are equivalent.*
  (a) *R is a v-PCD.*
  (b) *R is root closed and $M = M^2$.*
  (c) *R is a d-PCD.*

Proof. (1) Assume that R is a weak v-PCD, and choose $x \in V \setminus R$. Then $(R :_R Rx) = M$. For some integer k, we must have $(R :_R Rx^k) = ((R :_R Rx)^k)_v$. If $x^k \in R$, this equality becomes $R = (M^k)_v$, which is impossible since M is divisorial in R. Hence $x^k \notin R$, in which case the equality above becomes $M = (M^k)_v$. This then yields $M = (M^2)_v$, whence $V = M^{-1} = (M^2)^{-1} = ((R : M) : M) = (V : M)$. It follows that M cannot be principal in the valuation domain V, whence $M = M^2$. Thus (a) $\Rightarrow$ (b). Now assume that $M = M^2$, and let $y \in K \setminus R$. If $y^2, y^3 \in R$, then $(y^2)y = y^3 \in R$, whence $y^2 \in M$. However, since $y \in V$, this puts $y \in M \subseteq R$, a contradiction. Therefore, for $m = 2$ or $m = 3$, $y^m \notin R$, whence $(R :_R Ry^m) = M = M^m = (R :_R Ry)^m$, as desired. Finally, suppose $y \notin V$. Then $y^{-1} \in M \subseteq R$, whence for all $s > 1$, we have $(R :_R Ry^s) = Ry^{-s} = (R :_R Ry)^s$. This gives (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) follows from Lemma 1.4.

(2) Assume that R satisfies v_n. Then R is n-root closed by Proposition 1.11, and $M = M^2$ by (1). The implication (b) $\Rightarrow$ (c) follows from Lemma 2.1(1), and (c) $\Rightarrow$ (a) is trivial (Lemma 1.4).

(3) This follows from (2). $\square$

Next, we present examples, several of which were promised above. We begin with PVD examples, where the conclusions are immediate from Corollary 2.3.

Example 2.4. Let (R, M) be a PVD with canonical valuation overring V. Then:

(1) If $M \neq M^2$, then R is not a weak v-PCD (Corollary 2.3). For example, take $R = F + xk[x]_{xk[x]}$, where $F \subset k$ are fields and x is an indeterminate; if, in addition, $[k : F] < \infty$, then R is Noetherian.

(2) If $M = M^2$, but R is not n-root closed for any $n > 1$, then R is a weak d-PCD but does not satisfy v_n for any $n > 1$. (For example, let k be an algebraic closure of $\mathbb{Q}$, let $V = k + M$ be a non-discrete rank-one valuation domain with maximal ideal M, and let $R = \mathbb{Q} + M$.)

(3) If $M = M^2$ and R is 2-root closed but not 3-root closed (e.g., take $V = \mathbb{F}_4 + M$ to be a rank-one non-discrete valuation domain with maximal ideal M, and let $R = \mathbb{F}_2 + M$), then R satisfies d_2 but is not a v-PCD.

(4) If $M = M^2$ and R is root closed, then R is a d-PCD.
  (a) If R/M is not algebraically closed in V/M, then R is not integrally closed. (For example, take $R/M = \mathbb{Q}$ and $V/M = \mathbb{Q}[u]$, where u is a

root of $x^3 - 3x + 1$.) Hence a d-PCD need not be integrally closed and hence need not be an essential domain (or even a v-domain).

(b) If R/M is algebraically closed in V/M, then R is a d-PCD that is integrally closed but not completely integrally closed.

Let $F \subset k$ be fields, X a set of indeterminates, M the maximal ideal of $k[X]$ generated by X, and put $R = F + Xk[X]$. Such rings have often been used to provide interesting examples. We investigate PCD-properties in these rings.

Example 2.5. With the notation above, assume that F root closed in k.

(1) Let $|X| = 1$. Then:
  (a) R is not a d-PCD by Lemma 2.1. In fact, observe that in this case R_M is a PVD that is not a weak v-PCD by Example 2.4. Then, since R is v-coherent (see, e.g. [10, Theorem 3.5]), Corollary 1.10 ensures that R is not even a weak v-PCD.
  (b) If F is algebraically closed in k, then R is an integrally closed domain that is not a weak v-PCD.

(2) If $|X| > 1$, then $k[X]$ satisfies the hypotheses of Lemma 2.1(2). Hence R is a v-PCD but not a weak d-PCD by Proposition 1.11.

(3) If in (2) we take $1 < |X| < \infty$ and $[k : F] < \infty$, then R is a Noetherian v-PCD that is not a weak d-PCD.

Next, we give an example showing that the v-PCD property does not localize.

Example 2.6. Let R be the example given by Heinzer [14]. The domain R is essential, and therefore a v-PCD, but contains a prime ideal P such that R_P is not essential. In fact, it is easy to see that R_P is a PVD with $P \neq P^2$. Hence R_P is not a (weak) v-PCD by Corollary 2.3(1).

3. Completely integrally closed v-PCDs

In this and the next section we return to our convention that D is a domain with quotient field K. Recall that D is *completely integrally closed* if whenever $u \in K$ and a is a nonzero element of D with $au^n \in D$ for all $n \geq 1$, then $u \in D$. Thus the domain D is completely integrally closed if and only if $\bigcap_{n=1}^{\infty}(D :_D Du^n) = (0)$ for each $u \in K \setminus D$. It is well-known that D is completely integrally closed if and only if each nonzero ideal of D is v-invertible. We begin with a characterization of completely integrally closed $\star$-PCDs.

Proposition 3.1. *Let D be a weak $\star$-PCD. Then*

(1) $\bigcap_{n=1}^{\infty}(D :_D Du^n) = \bigcap_{n=1}^{\infty}((D :_D Du)^n)^\star$ *for each* $u \in K \setminus (0)$.

(2) *D is completely integrally closed if and only if* $\bigcap_{n=1}^{\infty}((D :_D Du)^n)^\star = (0)$ *for each* $u \in K \setminus D$.

Proof. (1) Let $u \in K \setminus (0)$, and use Proposition 1.5 to choose $1 < n_1 < n_2 \cdots$ with $(D :_D Du^{n_i}) = ((D :_D Du)^{n_i})^\star$ for each i. Then

$$\bigcap_{n=1}^{\infty}(D :_D Du^n) \subseteq \bigcap_{i=1}^{\infty}(D :_D Du^{n_i}) = \bigcap_{i=1}^{\infty}((D :_D Du)^{n_i})^\star = \bigcap_{n=1}^{\infty}((D :_D Du)^n)^\star,$$

and (1) follows easily.

(2) By definition D is completely integrally closed if and only if $\bigcap_{n=1}^{\infty}(D :_D Du^n) = (0)$ for each $u \in K \setminus D$. Hence the conclusion follows from (1). $\square$

Combining the proposition with Corollary 1.16, we have:

Corollary 3.2. *A $\star$-Prüfer domain D is completely integrally closed if and only if $\bigcap_{n=1}^{\infty}((D :_D Du)^n)^\star = (0)$ for each $u \in K \setminus D$. In particular, a v-domain (and hence an essential domain or a PvMD) is completely integrally closed if and only if $\bigcap_{n=1}^{\infty}((D :_D Du)^n)_v = (0)$ for each $u \in K \setminus D$.* □

Proposition 3.3. *A weak $\star$-PCD D is completely integrally closed if for every maximal $\star_f$-ideal P of D we have $\bigcap_{n=1}^{\infty}(P^n)^\star = (0)$. In particular, a weak d-PCD (weak v-PCD) D is completely integrally closed if for every maximal ideal (maximal t-ideal) P of D we have $\bigcap_{n=1}^{\infty} P^n = (0)$ ($\bigcap_{n=1}^{\infty}(P^n)_v = (0)$).*

Proof. Let D be a weak $\star$-PCD, and let $u \in K \setminus D$. Then $(D :_D Du) \subseteq P$ for some maximal $\star_f$-ideal P of D, and hence $((D :_D Du)^n)^\star \subseteq (P^n)^\star$. The conclusion then follows from Proposition 3.1. □

The condition on the maximal t-ideals in Proposition 3.3 is quite stringent. In particular, the condition requires a maximal t-ideal P to satisfy $P_v \neq D$. Counterexamples to the converse of Proposition 3.3 abound. For example, a non-discrete rank one valuation domain (D, M) is a v-PCD and completely integrally closed but does not satisfy $\cap(M^n)_v = (0)$; in fact, $(M^n)_v = D$ for each n. For another example, $D = k[x, y]$, k a field and x, y indeterminates, is a completely integrally closed v-PCD, but $(x, y)_v = D$.

On the other hand, if we require even more, we can obtain interesting characterizations of Krull domains. Note that if for a maximal t-ideal P we have $\cap(P^n)_v = (0)$, then $P_v \neq D$ and hence $P_v = P$, that is, P is divisorial. In [13] Glaz and Vasconselos called a domain D an *H-domain* if each ideal I of D with $I_v = D$ contains a finitely generated ideal J with $J_v = D$. According to [16, Proposition 2.4], the domain D is an H-domain if and only if every maximal t-ideal of D is divisorial. It is also easy to see that D is an H-domain if and only every v-invertible ideal of D is t-invertible. In particular, if an H-domain D is a v-domain, it must be a PvMD. Finally, Glaz and Vasconcelos showed that a domain D is a Krull domain if and only if it is a completely integrally closed H-domain [13, 3.2(d)].

Corollary 3.4. *The following are equivalent for a domain D.*

1. *D is a Krull domain.*
2. *D is a completely integrally closed H-domain.*
3. *D is an H-domain and a v-domain with t-dimension one.*
4. *D is an H-domain and a PvMD with t-dimension one.*
5. *D is an H-domain and a v-PCD with $\bigcap_{n=1}^{\infty}(P^n)_v = (0)$ for each maximal t-ideal P of D.*
6. *D is an integrally closed H-domain in which $((a, b)^n)^{-1} = (((a, b)^{-1})^n)_v$ for all $a, b \in D \setminus (0)$ and $\bigcap_{n=1}^{\infty}(P^n)_v = (0)$ for each maximal t-ideal P of D.*

Proof. The equivalence of (1) and (2) and the implication (3) ⇒ (4) are discussed above. It is clear that (1) ⇒ (3) and (6). Assume (4), and let P be a minimal prime of a principal ideal. Then P is a maximal t-ideal and is therefore divisorial (see above). Pick $u \in P^{-1} \setminus D$. Then, since D has t-dimension one, $P = (D :_D Du)$. It follows that D is a Krull domain by [16, Proposition 2.4]. Hence (4) ⇒ (1). Finally, we have (6) ⇒ (5) by Theorem 1.13 and (5) ⇒ (2) by Proposition 3.3. □

According to Corollary 3.4, if R is an integrally closed non-Krull H-domain with $\bigcap_{n=1}^{\infty}(P^n)_v = (0)$ for each maximal t-ideal P of R, then we must have $((a,b)^n)^{-1} \neq (((a,b)^{-1})^n)_v$ for some nonzero $a, b \in D$ and $n > 1$. We next give an example of this phenomenon.

Example 3.5. Let T be a PID with a maximal ideal P such that $k := T/P$ admits a field F that is algebraically closed in k (e.g., $T = F(y)[x]$, y, x indeterminates). Let $\varphi : T \to k$ be the natural projection and set $R = \varphi^{-1}(F)$. Then R is an integrally closed non-Krull domain (since R is not completely integrally closed). Of course, P is a divisorial ideal of R. In fact, each maximal ideal of R is divisorial. To see this, let $Q \neq P$ be a maximal ideal of R. Then $Q = N \cap R$ for maximal ideal $N \neq M$ of T. Write $N = Tz$. Then $z^{-1}PQ \subseteq z^{-1}PN \subseteq P \subseteq R$, and $z^{-1}P \nsubseteq R$ (indeed, $z^{-1}P \nsubseteq T$). Hence Q is divisorial. Hence R is (vacuously) an H-domain. Since $Q^n \subseteq N^n = Tz^n$ and Tz^n is divisorial (as an ideal of R), we have $\bigcap_{n=1}^{\infty}(Q^n)_v \subseteq \bigcap_{n=1}^{\infty} Tz^n = (0)$. Write $P = Tc$. Then $P^n = Tc^n$, which is divisorial. Hence $\bigcap_{n=1}^{\infty}(P^n)_v = \bigcap_{n=1}^{\infty} Tc^n = (0)$. Thus R has the required properties. It is not difficult to identify elements a, b as in the preceding paragraph: let $t \in T \setminus R$. Then $(1,t)^{-1} = (R :_R Rt) = P$, whence $(((1,t)^{-1})^n)_v = P^n$. On the other hand, $((1,t)^n)^{-1} = P$. Now take $a = c$ and $b = ct$.

Let D be a Noetherian domain. Then D is certainly an H-domain. Moreover, D is integrally closed if and only if D is a Krull domain. In view of the equivalence (1) $\Leftrightarrow$ (5) of Corollary 3.4, we have:

Corollary 3.6. *A Noetherian domain D is integrally closed if and only if D is a v-PCD and $\bigcap_{n=1}^{\infty}(M^n)_v = (0)$ for each maximal t-ideal M of D.* □

As we saw in Example 2.4(1), a Noetherian domain need not be a v-PCD. What is more interesting here is the fact that not every Noetherian domain has the property that for every maximal t-ideal M we have $\cap(M^n)_v = (0)$. We end this section with an example of this.

Example 3.7. Let $F \subset k$ be a root closed extension of fields with $[k : F]$ finite. Let $T = k[x,y] = k + M$, x, y indeterminates and $M = (x,y)$, and let $R = F + M$. Then R is Noetherian. It is easy to see that $M^{-1} = T$, whence R is a v-PCD by Lemma 2.1(2). (But R is not a weak d-PCD by Proposition 1.11(2)). By direct calculation or Corollary 3.6, we cannot have $\bigcap_{n=1}^{\infty}(M^n)_v = (0)$. (Indeed, $(M^n)_v = M$ for each $n \geq 1$).

4. Connections with other properties

In [26] a domain D was said to have the $\star$*-property* if for $a_1, \ldots, a_m, b_1, , ..., b_n \in D \setminus (0)$ we have $(\bigcap_i Da_i)(\bigcap_j Db_j) = \bigcap_{i,j} Da_ib_j$. The authors of [3] discussed a special case of this, which we call here the $\star\star$*-property*: $(Da \cap Db)(Dc \cap Dd) = Dac \cap Dad \cap Dbc \cap Dbd$ for all $a, b, c, d \in D \setminus (0)$; and they showed that a Noetherian domain satisfying $\star\star$ is locally factorial [3, Corollary 3.9]. Now $\star$ and $\star\star$ are equivalent over a Noetherian domain, for, in this case, $\star\star$ implies locally (factorial and hence) GCD, and it is easy to see that a locally GCD-domain is a locally $\star$-domain and hence a $\star$-domain [26, Theorem 2.1]. Thanks to its efficiency the $\star$-property can be used to provide a more satisfying characterization of integrally closed Noetherian domains than does the v-PCD property, as we shall see below. But the $\star$-property is more potent in that it can be put to use even in v-coherent domains. For example, it was shown in [25, Corollary 1.7] that the $\star$-property makes a v-coherent domain a *generalized GCD-domain* (GGCD-domain): a domain in which $aD \cap bD$ is invertible for each pair $a, b \in D \setminus (0)$. We restate the result as the following proposition.

Proposition 4.1. *An integral domain is a GGCD-domain if and only if D is a v-coherent $\star$-domain.* □

Now recall that PvMDs may be characterized as t-locally valuation domains, that is, D is a PvMD if and only if D_P is a valuation domain for each maximal t-ideal P of D. In [7], Chang used the local version of Proposition 4.1 to characterize PvMDs. Before stating Chang's result, we need some background. We first recall the w-operation: for a nonzero fractional ideal A of D, $A_w = \{x \in K \mid xB \subseteq A$ for some finitely ideal B of D with $B_v = D\}$. It is well known that $A_w = \bigcap AD_P$, where the intersection is taken over the set of maximal t-ideals P of D; moreover, we have $A_w D_P = AD_P$ for each P. Call D a *$\star(w)$-domain* if $((\bigcap_i Da_i)(\bigcap_j Db_j))_w = \bigcap_{i,j} Da_ib_j$ for all $a_i, b_j \in D \setminus (0)$.

Proposition 4.2. ([7, Theorem 3]) *An integral domain D is a PvMD if and only if D is a v-coherent $\star(w)$-domain.* □

In view of [3] we can introduce the notion of a *$\star\star(w)$-domain* as a domain D such that $((Da \cap Db)(Dc \cap Dd))_w = Dac \cap Dad \cap Dbc \cap Dbd$, for all $a, b, c, d \in D \setminus (0)$. We shall show that a Noetherian domain D is integrally closed $\Leftrightarrow$ D is a w-PCD $\Leftrightarrow$ D is a $\star\star(w)$-domain $\Leftrightarrow$ D is a $\star(w)$-domain. This is interesting in light of [3, comment following Lemma 3.7], where it is shown that a Krull domain is a d-PCD if and only if all powers of each maximal t-ideal are divisorial and that an integrally closed Noetherian domain need not have this property and hence need not be a d-PCD.

In fact, we can establish the result in a more general setting. Recall that a domain D is a *strong Mori domain* if it satisfies the ascending chain condition on w-ideals. These domains were introduced and studied by Wang and McCasland [22, 23]. They are characterized as domains D for which (1) D_M is Noetherian for every maximal t-ideal M of D and (2) D has finite t-character (each nonzero element a of D is contained in only finitely many maximal t-ideals of D) [23, Theorem 1.9]. It is well-known (and follows easily from (1)) that an integrally closed strong Mori domain is completely integrally closed and hence a Krull domain.

Theorem 4.3. *The following statements are equivalent for a strong Mori domain D.*

(1) *D is integrally closed.*
(2) *D is a w-PCD.*
(3) *D is a $\star\star(w)$-domain.*
(4) *D is a $\star(w)$-domain.*
(5) *D is completely integrally closed (and hence a Krull domain).*

As mentioned above, items (1) and (5) are equivalent. The rest of the proof is contained in the next two lemmas. For the first, we call a local domain (D, M) *t-local* if its maximal ideal is a t-ideal. (Perhaps a caveat is in order here. Localizing at a maximal t-ideal does not in general produce a t-local domain! However, this is not an issue in the strong Mori setting: for a strong Mori domain D, a prime P of

D is a t-ideal (equivalently, divisorial) if and only if PD_P is a t-ideal [18, Lemma 3.17].)

Lemma 4.4. *For a t-local Noetherian domain (D, M), the following statements are equivalent.*

(1) *D is a $\bigstar$-domain.*
(2) *D is a $\bigstar\bigstar$-domain.*
(3) *D is integrally closed.*
(4) *D is a (rank-one discrete) valuation domain.*
(5) *D is a d-PCD.*
(6) *D is a weak d-PCD.*

Proof. Implications (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), and (4) $\Rightarrow$ (1) are trivial, (3) $\Rightarrow$ (4) is well known, and (2) $\Rightarrow$ (3) is essentially the proof of [3, Corollary 3.9]. Now assume (6). Then M is divisorial, whence $M^{-1} \neq D$, and, clearly, $M \neq M^2$. Therefore, according to Proposition 1.11, M must be invertible, and hence D is a rank-one discrete valuation domain, as desired. Thus (6) $\Rightarrow$ (4), and the proof is complete. $\square$

Lemma 4.5. *A domain is a w-PCD (a $\bigstar\bigstar(w)$-domain, a $\bigstar(w)$-domain, integrally closed) if and only if it is t-locally a d-PCD (a $\bigstar\bigstar$-domain, a $\bigstar$-domain, integrally closed).*

Proof. It is well known that D is integrally closed if and only if it is t-locally integrally closed (and follows easily from the representation $D = \bigcap D_P$, where the intersection is taken over the set of maximal t-ideals P of D). Let D be a w-PCD, and let M be a maximal t-ideal of D. For $u \in K$ we have $(D_M :_{D_M} D_M u^n) = (D :_D Du^n)D_M = ((D :_D Du)^n)_w D_M = (D :_D Du)^n D_M = (D_M :_{D_M} D_M u)^n$. Hence D_M is a d-PCD. Now assume that D is t-locally a d-PCD, and let $\mathcal{P}$ denote the set of maximal t-ideals of D. Then for $u \in K$, $(D :_D Du^n) = (D :_D Du^n)_w = \bigcap_{P \in \mathcal{P}} (D :_D Du^n)D_P = \bigcap_{P \in \mathcal{P}} (D_P :_{D_P} D_P u^n) = \bigcap_{P \in \mathcal{P}} (D :_D Du)^n D_P = ((D :_D Du)^n)_w$. The details in the proofs of the other properties are similar. $\square$

Lemma 4.4 again shows that the $\bigstar$-property is much more potent than the v-PCD-property: according to the lemma, a t-local Noetherian domain satisfying $\bigstar$ must be integrally closed, whereas, if R is as in Example 2.5, then R_M is a non-integrally closed t-local Noetherian v-PCD. For still another example, a PvMD is automatically a v-PCD (Corollary 1.17), but a PvMD with the $\bigstar$-property is a GGCD-domain by Proposition 4.1. On the other hand, the v-PCD property is useful in determining whether a domain is completely integrally closed.

Now let us step back and take another look at (3) of Lemma 4.4 and ask: What if we consider a (not necessarily Noetherian) t-local domain with maximal ideal M divisorial but include the condition that $\bigcap M^n = (0)$? We show that the result would still be a discrete rank one valuation domain:

Proposition 4.6. *Let (D, M) be a local d-PCD such that M is divisorial and $\bigcap_{n=1}^{\infty} M^n = (0)$. Then D is a rank-one discrete valuation domain.*

Proof. By Proposition 3.3, D is completely integrally closed. Thus $(MM^{-1})_v = D$, and then, since M is divisorial, $MM^{-1} = D$. Hence M is principal. The condition $\bigcap M^n = (0)$ then ensures that D is one-dimensional, that is, that D is a rank-one discrete valuation domain. $\square$

We close with yet another characterization of Krull domains.

Theorem 4.7. *A domain D is a Krull domain if and only if it has the folllowing properties:*

(1) *Each maximal t-ideal of D is divisorial.*
(2) $\bigcap_{n=1}^{\infty}(M^n)_w = (0)$ *for each maximal t-ideal M of D.*
(3) *D is a w-PCD.*

Proof. It is well-known that a Krull domain has the first property and that the v-, t- and w-operations coincide. Properties (2) and (3) then follow from Corollary 3.4. Now assume that D is a domain with the properties listed. By (1), R is an H-domain [16, Proposition 2.4]. It is well-known that each maximal w-ideal of D is a maximal t-ideal (and vice versa) and that the w-operation is of finite type. Hence (2) and (3), together with Proposition 3.3, imply that D is completely integrally closed. Therefore, D, being a completely integrally closed H-domain, is a Krull domain [13, 3.2(d)]. □

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Department of Mathematics, The University of Iowa, Iowa City, IA 52242 U.S.A.
E-mail address: dan-anderson@uiowa.edu

Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223 U.S.A.
E-mail address: eghousto@uncc.edu

Department of Mathematic, Idaho State University, Pocatello, ID 83209 U.S.A.
E-mail address: mzafrullah@usa.net