ALMOST SPLITTING SETS AND AGCD DOMAINS

D.D. ANDERSON, TIBERIU DUMITRESCU, AND MUHAMMAD ZAFRULLAH

Abstract. Let D be an integral domain. A multiplicative set S of D is an *almost splitting set* if for each $0 \neq d \in D$, there exists an $n = n(d)$ with $d^n = st$ where $s \in S$ and t is v-coprime to each element of S. An integral domain D is an *almost GCD* (AGCD) domain if for every $x, y \in D$, there exists a positive integer $n = n(x, y)$ such that $x^n D \cap y^n D$ is a principal ideal. We prove that the polynomial ring $D[X]$ is an AGCD domain if and only if D is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension, where D' is the integral closure of D. We also show that $D + XD_S[X]$ is an AGCD domain if and only if D and $D_S[X]$ are AGCD domains and S is an almost splitting set.

1. Introduction

Let D be an integral domain with quotient field K and D' the integral closure of D. By an overring of D we mean a ring between D and K. D is said to be an *almost GCD* (AGCD) *domain* if for every $x, y \in D$, there exists $n = n(x, y) \in \mathbb{N}^*$ such that $x^n D \cap y^n D$ is a principal ideal. AGCD domains were introduced by the third author in [1] as a generalization of GCD domains (also see [2] and [3]). If D is an AGCD domain, then D' is an AGCD domain [1, Theorem 3.4] and $D \subseteq D'$ is a root extension (i.e., for each $x \in D'$ there exists a positive integer n such that $x^n \in D$) [1, Theorem 3.1]. By [1, Theorem 5.6], an integrally closed domain D is an AGCD domain if and only if the polynomial ring $D[X]$ is. The primary aim of this paper is to extend this result for arbitrary domains. We prove that $D[X]$ is an AGCD domain if and only if D is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension.

Recall that for a nonzero fractional ideal I of D, $I_v = (I^{-1})^{-1} = (D : I) : I = \bigcap\{xD \mid xD \supseteq I,\ x \in K\}$ and $I_t = \bigcup\{J_v \mid 0 \neq J \subseteq I \text{ is finitely generated}\}$. Hence if I is finitely generated, $I_t = I_v$. It is well known that for $x, y \in D^* = D - \{0\}$, $xD \cap yD$ is a principal ideal if and only if $(x, y)_v$ is (indeed, $(x, y)_v$ is principal $\Leftrightarrow (x, y)^{-1} = x^{-1}D \cap y^{-1}D = \frac{1}{xy}(xD \cap yD)$ is principal $\Leftrightarrow xD \cap yD$ is principal). Call two nonzero elements $x, y \in D$ *v-coprime* if $(x, y)_v = D$, or equivalently, if $xD \cap yD = xyD$ (see Proposition 2.2 for several other equivalences).

A saturated multiplicative set S of D is called an *almost splitting set* if for each nonzero $x \in D$, there is a natural number $n = n(x)$ such that $x^n = ds$, where $s \in S$ and d is v-coprime to every member of S. We also prove that for a saturated multiplicative set S, the composite polynomial ring $D + XD_S[X] = \{f(X) \in D_S[X] \mid f(0) \in D\}$ is an AGCD domain if and only if D and $D_S[X]$ are AGCD domains and S is an almost splitting set.

1991 *Mathematics Subject Classification.* Primary 13A05, 13A15; Secondary 13B22, 13B25.
Key words and phrases. almost GCD domain, polynomial ring, root extension.

2. Almost Splitting Sets

In this section we investigate almost splitting sets. However, we begin by reviewing the notion of a splitting set. A saturated multiplicative set S of D is said to be a *splitting set* if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$, i.e., s' and a are v-coprime. The set $T = \{t \in D^* \mid sD \cap tD = stD \text{ for all } s \in S\}$ is also a splitting set, called the *m-complement of* S. Each $d \in D^*$ has a unique representation (up to unit factors) $d = st$, where $s \in S$ and $t \in T$. If $d = st$ ($s \in S$, $t \in T$), then $dD_S \cap D = tD$. In fact, a saturated multiplicative set S of D is a splitting set if and only if $dD_S \cap D$ is principal for each $d \in D^*$. For these, and other, results on splitting sets, see [4]. Splitting sets are investigated further in [5].

Splitting sets can also be viewed in the context of the group of divisibility $G(D) = K^*/U(D)$ of D. Here $U(D)$ denotes the group of units of D and $G(D)$ is partially ordered by $aU(D) \leq bU(D) \Leftrightarrow a|b$ in D. Note that $G(D)$ is order-isomorphic to $P(D)$ the multiplicative group of nonzero principal fractional ideals of D ordered by inverse inclusion: $aU(D) \leftrightarrow aD$. Mott [6, Theorem 2.1] showed that there is a one-to-one correspondence between the set of convex directed subgroups of $P(D) \cong G(D)$ and the set of saturated multiplicative closed subsets of D. The correspondence is given as follows. If S is a saturated multiplicative closed subset of D, then $\langle S\rangle = \left\{s_1s_2^{-1}D \mid s_1, s_2 \in S\right\}$ is a convex directed subgroup of $P(D)$ with positive cone $\langle S\rangle_+ = \{sD \mid s \in S\}$. In $G(D)$, we may identify $\langle S\rangle$ with $U(D_S)/U(D)$. In [7], Mott and Schexnayder considered the question of when $\langle S\rangle \cong U(D_S)/U(D)$ is a cardinal summand of $P(D) \cong G(D)$, that is, when there is a subgroup H of $P(D)$ with $\langle S\rangle \oplus_c H = P(D)$. In our terminology, they showed that $\langle S\rangle$ is a cardinal summand if and only if S is a splitting set.

A splitting set S is said to be an *lcm splitting set* if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal, or equivalently D_T, where T is the m-complement of S, is a GCD-domain. Perhaps the most important example of an lcm splitting set is as follows. A set $\{p_\alpha\}$ of nonzero principal primes is a *splitting set of principal primes* if (a) for each α, $\bigcap_{n=1}^{\infty} p_\alpha^n D = 0$ (or equivalently, $ht\ p_\alpha D = 1$), and (b) for any sequence $\{p_{\alpha_n}\}$ of nonassociate members of $\{p_\alpha\}$, $\bigcap_{n=1}^{\infty} p_{\alpha_n} D = 0$. Then $S = \{up_{\alpha_1} \cdots p_{\alpha_n} \mid u \in U(D),\ p_{\alpha_i} \in \{p_\alpha\},\ n \geq 0\}$ is an lcm splitting set [4, Proposition 2.6].

We next introduce the notion of an almost splitting set and an almost lcm splitting set.

Definition 2.1. Let S be a saturated multiplicative set of an integral domain D. Then S is an *almost splitting set* if for each $d \in D^*$, there is an $n = n(d)$ with $d^n = st$ where $s \in S$ and t is v-coprime to every element of S. An almost splitting set S is an *almost lcm splitting set* if for all $s \in S$ and $d \in D^*$, there is an $n = n(s, d)$ such that $d^n D \cap s^n D$ is principal.

But first, we investigate the notions of v-coprimeness and m-complements more closely. The proof of the next proposition is straightforward and left to the reader. Proofs of several of the implications may be found in [4] and [5].

Proposition 2.2. *Let S be a* (*not necessarily saturated*) *multiplicative set of the domain D. Then for $t \in D^*$, the following are equivalent.*

(1) $(s,t)_v = D$ *for all* $s \in S$,
(2) $sD \cap tD = stD$ *for all* $s \in S$,

(3) $tD_S \cap D = tD$,
(4) $D_S \cap D_t = D$ *where* $D_t = \{d/t^n \mid d \in D, n \geq 0\}$, *and*
(5) $Dt : s = Dt$ *for all* $s \in S$.

Definition 2.3. For a nonempty subset S of an integral domain D, let $S^\perp = \{x \in D^* \mid (x, s)_t = D \text{ for all } s \in S\}$.

Note that by Proposition 2.2, $D_S \cap D_{S^\perp} = D$. It is easily checked that for an integral domain D and $\emptyset \neq S_1 \subseteq S_2 \subseteq D^*$, $S_1^\perp \supseteq S_2^\perp$, and $S_1 \subseteq S_1^{\perp\perp}$.

Proposition 2.4. *Let S be a nonempty subset of nonzero elements of an integral domain D. Then $S^\perp$ is a saturated multiplicative set of D with $\bar{S} \subseteq S^{\perp\perp}$ and $\bar{S} \cap S^\perp = U(D)$ where $\bar{S}$ is the saturation of the multiplicative set generated by S. Moreover, $S^\perp = S^{\perp\perp\perp}$. If S is an (almost) splitting set, then $S = S^{\perp\perp}$ and hence $S^\perp$ is an (almost) splitting set with $S^{\perp\perp} = S$.*

Proof. Let $x, y \in S^\perp$ and $s \in S$. Then $(x, s)_v = (y, s)_v = D$, so $D = ((x, s)(y, s))_v = (xy, xs, ys, s^2)_v \subseteq (xy, s)_v \subseteq D$ and hence $(xy, s)_v = D$. Thus $xy \in S^\perp$. Clearly, $S^\perp$ is saturated. Thus $S^{\perp\perp}$ is a saturated multiplicative set with $S \subseteq S^{\perp\perp}$ and so $\bar{S} \subseteq S^{\perp\perp}$. Certainly $\bar{S} \cap S^\perp \supseteq U(D)$. Suppose $x \in \bar{S} \cap S^\perp$, so $xy = s_1 \cdots s_n$ for some $y \in D$ and $s_1, \ldots, s_n \in S$. Now $(x) = (x, xy)_v = (x, s_1 \cdots s_n)_v = D$, so $x \in U(D)$. Note that $S \subseteq S^{\perp\perp}$ gives $S^\perp \supseteq S^{\perp\perp\perp}$ and $S^\perp \subseteq (S^\perp)^{\perp\perp}$, so $S^\perp = S^{\perp\perp\perp}$.

Suppose that S is an almost splitting set. Let $x \in S^{\perp\perp}$, so there exists an $n \geq 1$ with $x^n = st$ where $s \in S$ and $t \in S^\perp$. Then $st = x^n \in S^{\perp\perp} \Rightarrow t \in S^{\perp\perp} \Rightarrow t \in S^{\perp\perp} \cap S^\perp = U(D)$. So $x^n \in S$ and hence $x \in S$. Thus $S = S^{\perp\perp}$. □

However, in general we need not have $S = S^{\perp\perp}$ even when S is a saturated multiplicative set generated by principal primes.

Example 2.5. Let $(V, (p))$ be a discrete valuation ring of rank greater than one. Then $S = \{up^n \mid n \geq 0, u \in U(V)\}$ is a saturated multiplicative set of V. Here $S^\perp = U(D)$ and hence $S \subsetneq S^{\perp\perp} = D^*$.

The use of $S^\perp$ may remind some readers of the set of orthogonal elements of a set S in a partially ordered group or in a Riesz space (i.e., the set of all positive elements a with $a \wedge s = 0$ for each $s \in S$). Viewing an integral domain in the context of its group of divisibility, which is a partially ordered group, we see that the notion of v-coprimality is precisely the same as that of orthogonality.

We next give an example of an almost splitting set which generalizes the notion of a splitting set of primes.

Example 2.6. Let $\{P_\alpha\}_{\alpha \in \Lambda}$ be a nonempty collection of height-one prime ideals of an integral domain D with $\bigcap_{n=1}^\infty P_{\alpha_n} = 0$ for any countable subcollection. For each $\alpha \in \Lambda$, assume that some $(P_\alpha^n)_t$ is principal; say $(P_\alpha^{n_\alpha})_t = (q_\alpha)$. Let $S = \{q_{\alpha_1}^{l_{\alpha_1}} \cdots q_{\alpha_n}^{l_{\alpha_n}} \mid \alpha_i \in \Lambda, \text{ each } l_{\alpha_i} \geq 0\}$ and let $\bar{S}$ be the saturation of S. Then for $0 \neq d \in D$, there exists an $n \geq 1$ with $d^n = st$ where $s \in S$ and $t \in S^\perp$. Hence $\bar{S}$ is an almost splitting set.

Since each P_α is t-invertible, if I is a nonzero ideal contained in P_α, we get $I_t = (P_\alpha J)_t$ with $J = P^{-1}I$. We repeatedly use this factorization property starting with $I = dD$. By our height-one and intersection assumptions on the P_α's, we get $dD = (P_{\alpha_1} \cdots P_{\alpha_n} J)_t$ for some $\alpha_1, \ldots, \alpha_n$, $n \geq 0$ and some ideal J contained in no P_α. As $(P_\alpha^{n_\alpha})_t = q_\alpha D$ and $q_\alpha \in S$, $d^k D = s(J^k)_t$ for some $k \geq 1$ and $s \in S$. So $(J^k)_t = fD$ for some $f \in D$. Then $f \notin \bigcup P_\alpha$, hence $(P_\alpha, f)_t = D$ for each α, because P_α being t-invertible is a maximal t-ideal [8, Lemma 1]. As $(P_\alpha^{n_\alpha})_t = q_\alpha D$, we get $(q_\alpha, f)_t = D$ for each α. Hence $f \in S^\perp$, because the q_α's generate S. Thus $d^k \in SS^\perp$. Hence $\bar{S}$ is an almost splitting set. (Note: $\hat{S} = \{us \mid u \in U(D),\ s \in S\}$ need *not* be saturated as is seen by taking a Dedekind domain D with class group $\mathrm{Cl}(D) = \mathbb{Z}_2$. Suppose that M and N are nonprincipal maximal ideals of D. Let $M^2 = (a)$, $N^2 = (b)$ and $MN = (c)$, and let $S = \{a^n b^m \mid n,\ m \geq 0\}$. Then $(c^2) = M^2 N^2 = (a)(b)$, so $c^2 = uab$ for some $u \in U(D)$. Thus $c^2 \in \hat{S}$, but $c \notin \hat{S}$.)

The following characterization of almost splitting sets similar to a characterization of splitting sets [4, Theorem 2.2] will be used.

Proposition 2.7. *For a saturated multiplicative set S of an integral domain D, the following are equivalent.*

(1) *S is an almost splitting set.*
(2) *For $d \in D$, there exists an $n = n(d) \geq 1$ with $d^n D_S \cap D$ principal.*

Proof. (1) $\Rightarrow$ (2) Let $d^n = st$ where $s \in S$ and $t \in S^\perp$. Then $d^n D_S \cap D = stD_S \cap D = tD_S \cap D = tD$ with the last equality following from Proposition 2.2.

(2) $\Rightarrow$ (1) Let $0 \neq d \in D$. Suppose that $d^n D_S \cap D = tD$. Then $tD_S \cap D = d^n D_S \cap D = tD$; so by Proposition 2.2, $t \in S^\perp$. Now $d^n = rt$ for some $r \in D$. Hence $rtD_S = d^n D_S = tD_S$, so $rD_S = D_S$. Since S is saturated, $r \in S$. □

We next give a characterization of almost lcm splitting sets similar to the characterization of lcm splitting given in [4].

Theorem 2.8. *For an almost splitting set S of an integral domain D, the following conditions are equivalent.*

(1) *S is an almost lcm splitting set.*
(2) *For $s_1, s_2 \in S$, there exists an $n = n(s_1, s_2) \geq 1$ with $s_1^n D \cap s_2^n D$ principal.*
(3) *For $s_1, s_2 \in S$, there exists an $n = n(s_1, s_2) \geq 1$ and $s \in S$ with $s_1^n D \cap s_2^n D = sD$.*
(4) *$D_{S^\perp}$ is an AGCD-domain.*

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) Let $s_1^n D \cap s_2^n D = xD$. Write $x^m = st$ where $s \in S$ and $t \in S^\perp$. Then $s_1^{nm} D \cap s_2^{nm} D = x^m D = stD$. Now $tD = tD_S \cap D = stD_S \cap D = x^m D_S \cap D = (s_1^{nm} D \cap s_2^{nm} D) D_S \cap D = D_S \cap D = D$ implies $t \in U(D)$. So $s_1^{nm} D \cap s_2^{nm} D = sD$. (3) $\Rightarrow$ (4) Let $xD_{S^\perp}, yD_{S^\perp}$ be principal ideals of $D_{S^\perp}$ where $x, y \in D^*$. Now $x^n = s_1 t_1$ where $s_1 \in S$ and $t_1 \in S^\perp$ implies $x^n D_{S^\perp} = s_1 D_{S^\perp}$ and likewise $y^m D_{S^\perp} = s_2 D_{S^\perp}$ where $s_2 \in S$. Hence $x^{nm} D_{S^\perp} = s_1^m D_{S^\perp}$ and $y^{nm} = s_2^n D_{S^\perp}$. Choose l with $(s_1^m)^l D \cap (s_2^n)^l D = sD$ where $s \in S$. Then $x^{nml} D_{S^\perp} \cap y^{nml} D_{S^\perp} = s_1^{ml} D_{S^\perp} \cap s_2^{nl} D_{S^\perp} = (s_1^{ml} D \cap s_2^{nl} D) D_{S^\perp} = sD_{S^\perp}$ is principal. So $D_{S^\perp}$ is an AGCD-domain. (4) $\Rightarrow$ (1) Let $s \in S$ and $d \in D^*$. So for some $n \geq 1$, $d^n = s_1 t_1$ where $s_1 \in S$ and $t_1 \in S^\perp$. Choose $m \geq 1$ with $(s^n)^m D_{S^\perp} \cap s_1^m D_{S^\perp}$ principal. As in the proof of (2) $\Rightarrow$ (3), we can assume that $(s^n)^m D_{S^\perp} \cap s_1^m D_{S^\perp} = s'' D_{S^\perp}$ for some $s'' \in S$. Then $s^{nm} D \cap d^{nm} D = s^{nm} D \cap s_1^m t_1^m D = s^{nm} D \cap s_1^m D \cap t_1^m D = (s^{nm} D_{S^\perp} \cap D) \cap (s_1^m D_{S^\perp} \cap D) \cap t_1^m D = ((s^{nm} D_{S^\perp} \cap s_1^m D_{S^\perp}) \cap D) \cap t_1^m D = (s'' D_{S^\perp} \cap D) \cap t_1^m D = s'' D \cap t_1^m D = s'' t_1^m D$ is principal. □

Recall that for an integral domain D, the *t-class group of D* is $\mathrm{Cl}_t(D) = TI(D)/P(D)$ where $TI(D)$ is the group of t-invertible t-ideals of D and $P(D)$ is its subgroup of nonzero principal fractional ideals. When S is a splitting set, there is a natural isomorphism $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_{S^\perp})$ given by $[I] \to ([ID_S], [ID_{S^\perp}])$ where $[\ ]$ denotes the class of an ideal. See [4]. For D a Krull domain, $\mathrm{Cl}_t(D)$ is the usual divisor class group.

Theorem 2.9. *Let S be an almost splitting set in an integral domain D. Then the kernel of the canonical homomorphism $\theta\colon \mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_{S^\perp})$ given by $\theta([I]) = ([ID_S], [ID_{S^\perp}])$ is a torsion subgroup of $\mathrm{Cl}_t(D)$.*

Proof. It suffices to show that if I is a nonzero integral ideal of D with ID_S and $ID_{S^\perp}$ principal, there is a $k \geq 1$ with $(I^k)_v$ principal. Suppose that $ID_S = a_1 D_S$ and $ID_{S^\perp} = a_2 D_{S^\perp}$ where $a_1, a_2 \in D$. Since S is an almost splitting set, we can write $a_i^k = s_i t_i$ where $s_i \in S$ and $t_i \in S^\perp$, $i = 1, 2$. Then $I^k D_S = a_1^k D_S = s_1 t_1 D_S = t_1 D_S = s_2 t_1 D_S$. Likewise, $I^k D_{S^\perp} = s_2 t_1 D_{S^\perp}$. So $I^k D_S = a D_S$ and $I^k D_{S^\perp} = a D_{S^\perp}$ where $a = s_2 t_1 \in D$. But then $I^k \subseteq I^k D_S \cap I^k D_{S^\perp} = a D_S \cap a D_{S^\perp} = a(D_S \cap D_{S^\perp}) = aD$. So $I^k = Ja$ for some ideal J of D. Then $aD_S = I^k D_S = J D_S a D_S$, so $J D_S = D_S$. Likewise, $J D_{S^\perp} = D_{S^\perp}$. Let $x \in J^{-1}$; so $xJ \subseteq D$. Then $xJD_S \subseteq D_S$ implies $xD_S \subseteq D_S$, so $x \in D_S$. Likewise, $x \in D_{S^\perp}$, so $x \in D_S \cap D_{S^\perp} = D$. Thus $J^{-1} \subseteq D$ and hence $J_v = D$. So $(I^k)_v = aD$ is principal. □

Recall that a Krull domain D is said to be *almost factorial* if $\mathrm{Cl}_t(D)$ is torsion.

Corollary 2.10. *Let S be an almost lcm splitting set in an integral domain D. If $\mathrm{Cl}_t(D_S)$ is torsion, then so is $\mathrm{Cl}_t(D)$. If D is root closed and D_S is an AGCD domain, then D is an AGCD domain. Hence if D is a Krull domain with D_S almost factorial, then D is almost factorial.*

Proof. By Theorem 2.8, $D_{S^\perp}$ is an AGCD domain and hence $\mathrm{Cl}_t(D_{S^\perp})$ is torsion. Thus $\mathrm{Cl}_t(D_S) \times \mathrm{Cl}_t(D_{S^\perp})$ is torsion. By Theorem 2.9, $\ker\theta$ is torsion. Then $\mathrm{Cl}_t(D)$ itself is torsion.

Suppose that D is root closed. Then D_S is a root closed AGCD domain and hence is integrally closed. By the same reasoning, $D_{S^\perp}$ is integrally closed. Thus $D = D_S \cap D_{S^\perp}$ is integrally closed. By [1, Theorem 3.9], D is an AGCD domain. □

We end this section with a characterization of the integral domains with the property that every saturated multiplicative set is an almost splitting set. Recall that an integral domain D is *weakly Krull* if $D = \bigcap_{ht P=1} D_P$ where the intersection has finite character.

Theorem 2.11. *An integral domain D is weakly Krull with $\mathrm{Cl}_t(D)$ torsion if and only if every saturated multiplicative set of D is an almost splitting set.*

Proof. ($\Leftarrow$) Suppose that every saturated multiplicative set of D is an almost splitting set. By [9, Theorem 3.4], it suffices to show that if P is a prime ideal minimal over a proper principal ideal D, then there is a natural number $n = n(x, P)$ with $x^n D_P \cap D$ principal. But since $S = D - P$ is an almost splitting set, this follows from Proposition 2.7.

($\Rightarrow$) Suppose that D is a weakly Krull domain with $\mathrm{Cl}_t(D)$ torsion and let S be a saturated multiplicative set of D. Let d be a nonzero nonunit of D. Since some power of d is a product of primary elements [9, Theorem 3.4], it suffices to show that each nonzero primary element q not in S is in $S^\perp$. As S is saturated, qD is disjoint from S and hence so is its radical. Since qD is primary, $qD : s = qD$ for each $s \in S$. Thus $q \in S^\perp$. □

3. Polynomial Extensions of AGCD Domains

Let D be an integral domain and S a multiplicative set of D. In this section we consider the question of when $D[X]$ or $D+XD_S[X]$ is an AGCD domain. We show (Theorem 3.4) that $D[X]$ is an AGCD domain if and only if D is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension while (Theorem 3.12) $D + XD_S[X]$ is an AGCD domain if and only if D and $D_S[X]$ are AGCD domains and S is almost splitting.

Let E be an overring of D. According to [2], we say that D is *t-linked under E*, if whenever $x_1, \ldots, x_n \in D^*$ with $((x_1, \ldots, x_n)E)_v = E$, we have $((x_1, \ldots, x_n)D)_v = D$. We shall use the following result from [2].

Lemma 3.1. ([2, Theorem 5.9]) *A domain D is an AGCD domain if and only if*
(i) *D' is an AGCD domain,*
(ii) *$D \subseteq D'$ is a root extension, and*
(iii) *D is t-linked under D'.*

Remark 3.2. By the proof of [2, Theorem 5.9], it follows that in Lemma 3.1 condition (iii) can be replaced by condition
(iii′) whenever $x, y \in D^*$ are v-coprime in D', x, y are v-coprime in D.

We next prove that the t-linked-under property is stable under a polynomial base change.

Proposition 3.3. *Let D be a domain, E an overring of D, and K the quotient field of D. The following assertions are equivalent.*
(a) *D is t-linked under E,*
(b) *$D[X]$ is t-linked under $E[X]$, and*
(c) *whenever $f, g \in D[X]^*$ are v-coprime in $E[X]$, then f, g are v-coprime in $D[X]$.*

Proof. (a) $\Rightarrow$ (c). For this implication, our argument is patterned after the proof of [10, Theorem 3.5]. Let f, g be nonzero elements of $D[X]$ such that f, g are v-coprime in $E[X]$. By [10, Theorem 3.2], f, g are v-coprime in $E[X]$ if and only if f, g are v-coprime in $K[X]$ and $(c(f)E + c(g)E)_v = E$, where $c(f)$ is the content ideal of f in D. As D is t-linked under E, $(c(f) + c(g))_v = D$. A new appeal to [10, Theorem 3.2] shows that f, g are v-coprime in $D[X]$.

(c) $\Rightarrow$ (b). Let I be a nonzero finitely generated ideal of $D[X]$ such that $(IE[X])_v = E[X]$. By [11, Lemma 4.4], there exist $0 \neq a \in IE[X] \cap E$ and $f \in IE[X]$ with $c(f)_v = E$, where $c(f)$ is the content ideal of f; moreover, $((a, f)E[X])_v = E[X]$ for every such a and f. Fix a and f as above. We can write $a = c_1h_1 + \cdots + c_mh_m$ with $c_1, \ldots, c_m \in E$ and $h_1, \ldots, h_m \in I$. As E is an overring of D, there exists $0 \neq s \in D$ such that $sc_1, \ldots, sc_m \in D$. Then $0 \neq sa \in I \cap E = I \cap D$. Therefore, replacing a by sa, we may assume that $0 \neq a \in I \cap D$. Since $f \in IE[X]$, we can write $f = b_0f_0 + \cdots + b_nf_n$ with $b_0, \ldots, b_n \in E$ and $f_0, \ldots, f_n \in I$. Choose k greater than the degree of each f_i and let $g = f_0 + f_1X^k + \cdots + f_nX^{nk} \in I$. Then a, g are v-coprime in $E[X]$. Indeed, if this were not the case, the image of g in $E[X]/aE[X] = (E/aE)[X]$ is a zero divisor. Hence $dg \in aE[X]$ for some $d \in E - aE$. Then $df_i \in aE[X]$ for each i, so $df \in aE[X]$, contradicting the fact that a, f are v-coprime in $E[X]$. So $a, g \in I \subseteq D[X]$ are v-coprime in $E[X]$, hence a, g are v-coprime in $D[X]$ by our assumption. As $a, g \in I$, we get $I_v = D[X]$, thus (b) holds. The implication (b) $\Rightarrow$ (a) is an easy consequence of the definition. □

The following theorem is the main result of this paper.

Theorem 3.4. *Let D be a domain. Then $D[X]$ is an AGCD domain if and only if D is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension.*

Proof. Assume that $D[X]$ is an AGCD domain. By Lemma 3.1, $D[X] \subseteq D'[X] = (D[X])'$ is a root extension. Now, let $a, b \in D^*$. As $D[X]$ is an AGCD domain, there exist a positive integer n and $c \in D[X]$ such that $a^n D[X] \cap b^n D[X] = cD[X]$. Then $c \in D$ and it follows easily that $a^n D \cap b^n D = cD$. So D is an AGCD domain.

Conversely, assume that D is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension. By Lemma 3.1, D is t-linked under D', so $D[X]$ is t-linked under $D'[X]$ by Proposition 3.3. As D is an AGCD domain, so are D' and $D'[X]$, cf. [1, Theorems 3.4 and 5.6]. Hence $D[X]$ is an AGCD domain by Lemma 3.1. □

Example 3.5. By [2, Theorem 4.17], $\mathbb{Z}[2i]$ is an AGCD domain with integral closure $\mathbb{Z}[i]$. An easy computation shows that $f^2 \in \mathbb{Z}[2i][X]$ for each $f \in \mathbb{Z}[i][X]$. By Theorem 3.4, $\mathbb{Z}[2i][X]$ is an AGCD domain.

However, as the following example shows, D an AGCD domain need not always imply that $D[X]$ is an AGCD domain.

Example 3.6. If m is a square-free integer $m \equiv 5 \pmod 8$ and $D = \mathbb{Z}[\sqrt{m}]$, then D is an AGCD domain [2, Theorem 4.17], while $D[X]$ is not AGCD, because $D[X] \subseteq D'[X]$ is not a root extension. Indeed, if it were, then reducing modulo 2, $\mathbb{F}_2[X] \subseteq \mathbb{F}_4[X]$ would be a root extension, where $\mathbb{F}_n$ is the field with n elements (we have used the easy-to-obtain isomorphism $D'/2D' = \mathbb{F}_4$). But if $t \in \mathbb{F}_4 - \mathbb{F}_2$, it is easy to see that no power of $t + X$ lies in $\mathbb{F}_2[X]$.

This example can also serve to establish that if $D \subseteq E$ is a root extension, it is not necessary that $D[X] \subseteq E[X]$ should also be a root extension.

Example 3.7. Let A be a GCD domain of characteristic 2. By the paragraph following [3, Remark 4.1], $D = A[Y^2, Y^3]$ is an AGCD domain and $D' = A[Y]$. As $D'^2 \subseteq D$, it follows that $(D'[X])^2 \subseteq D[X]$, so $D[X]$ is an AGCD domain by Theorem 3.4.

When D contains a field, it is easy to describe when $D[X] \subseteq D'[X]$ is a root extension.

Proposition 3.8. *Let $D \subseteq E$ be an extension of domains such that D contains a field. Assume that $D[X] \subseteq E[X]$ is a root extension.*

(a) *If $D \supseteq \mathbb{Q}$, then $D = E$.*

(b) *If $D \supseteq \mathbb{F}_p$ with p a positive prime, then $D \subseteq E$ is a purely inseparable extension.*

Proof. Let $a \in E$. Then $(1 + aX)^n \in D[X]$ for some positive n. Hence $na, a^n \in D$. So (a) holds, because $n \in U(D)$ in this case.

Now assume that $D \supseteq \mathbb{F}_p$ and decompose n as $n = p^e m$ with $(m, p) = 1$. As $(1 + aX)^n = (1 + a^{p^e} X^{p^e})^m$, we get $ma^{p^e} \in D$, so $a^{p^e} \in D$, because $m \in U(D)$. □

The above proposition provides two types of AGCD domains D such that $D[X]$ an AGCD domain implies that $D[X_1, X_2, \ldots, X_n]$ is an AGCD domain for indeterminates $X_1, X_2, \ldots, X_n$. One is when D contains the field of rational numbers. In this case, $D[X]$ being AGCD forces D to be integrally closed and it is well known [1] that if D is an integrally closed AGCD domain, then so is a polynomial ring over D in several variables. The other case is when D has positive characteristic. In this case, the $D[X]$ being an AGCD domain forces the integral closure of D to be purely inseparable over D and thus paves the way for $D[X_1, X_2, \ldots, X_n]$ to be AGCD. This leads us to the following question.

Question 3.9. Does $D[X]$ an AGCD domain imply that $D[X_1, X_2, \ldots, X_n]$ is an AGCD domain for finitely many indeterminates $X_1, X_2, \ldots, X_n$?

Apparently the answer to this question is related to the following question.

Question 3.10. If $D \subseteq E$ is a root extension such that $D[X] \subseteq E[X]$ is a root extension, is $D[X, Y] \subseteq E[X, Y]$ also a root extension?

Let D be a domain and $S \subseteq D^*$ a multiplicative set. We consider the composite ring $D + XD_S[X] = \{f \in D_S[X] \mid f(0) \in D\}$. By [12, Corollary 1.5] or [13, Corollary 2.6] $D + XD_S[X]$ is a GCD domain if and only if D is a GCD domain and $\bar{S}$ is a splitting set. When D is integrally closed, it was shown in [3, Theorem 3.1] that $D + XD_S[X]$ is an AGCD domain if and only if D is an AGCD domain and $\bar{S}$ is an almost splitting set. We extend this result to arbitrary domains. Nevertheless, in our proof we use the result cited above. We need the following lemma (see [14, Theorem 1] for a similar result).

Lemma 3.11. *Let D be an AGCD domain, $S \subseteq D^*$ a multiplicative set, and T the saturation of S in D'. If S is almost splitting in D, then so is T in D'.*

Proof. Let $0 \neq a \in D'$. Since $D \subseteq D'$ is a root extension (Lemma 3.1) and S is almost splitting in D, there exists a positive integer n such that $a^n = bs$ with $b \in D$ v-coprime in D to every element of S and $s \in S$. By the paragraph before [2, Theorem 5.11], b is also v-coprime in D' to every element of S. Thus $b \in T^{\perp}$. Hence T is an almost splitting set. □

Theorem 3.12. *Let D be a domain and $S \subseteq D^*$ a saturated multiplicative set. Then $D + XD_S[X]$ is an AGCD domain if and only if*
(i) *D and $D_S[X]$ are AGCD domains, and*
(ii) *S is almost splitting.*

Proof. Set $E = D + XD_S[X]$. Assume that E is an AGCD domain. As in the proof of Theorem 3.4, we can show that D is an AGCD domain. As shown in [1, Section 5], a ring of quotients of an AGCD domain is still an AGCD domain. Hence $E_S = D_S[X]$ is an AGCD domain. So (i) holds. The fact that S is almost splitting was shown in the proof of [3, Theorem 3.1]. For the convenience of the reader, we repeat the argument here. Let $0 \neq a \in D$. Since E is an AGCD domain, there exists a positive integer n and $s \in E$ such that $((a^n, X^n)E)_v = sE$. Since $s \mid a^n$, $s \in D$ and since $s \mid X^n$, $s \in S$ (because S is saturated). Set $b = a^n/s \in D$ and let $t \in S$. Since $t \mid X^n/s$, we derive successively that $((b, X^n/s)E)_v = E$, $((b, t)E)_v = E$, and $((b, t)D)_v = D$. Hence $a^n = bs$ with $b \in D$ v-coprime to every element of S and $s \in S$, that is, S is almost splitting.

Conversely, assume that (i) and (ii) hold. We prove that E is an AGCD domain using Lemma 3.1 and Remark 3.2. Let T be the saturation of S in D'. By the preceding lemma, T is almost splitting in D'. Since D is an AGCD domain, D' is an integrally closed AGCD domain [1, Theorem 3.4]. The integral closure of E is $E' = D' + XD'_T[X] = D' + XD'_S[X]$, cf. [15, Theorem 2.7]. By [3, Theorem 3.1], E' is an AGCD domain. For proving that $E \subseteq E'$ is a root extension, let $f \in E'$. As D and $D_S[X]$ are AGCD domains, $D \subseteq D'$ and $D_S[X] \subseteq D'_S[X]$ are root extensions, cf. Lemma 3.1. So there exist two positive integers, m and n, such that $f^m \in D_S[X]$ and $f(0)^{mn} \in D$. It follows that $f^{mn} \in E$. Thus $E \subseteq E'$ is a root extension. To complete the proof, it suffices to verify that condition (iii$'$) of Remark 3.2 holds in E. Let $f, g \in E^*$ such that f, g are v-coprime in E'. As noted in [16], E is the directed union (limit) of its subrings $D[X/s]$ for $s \in S$. Similarly, E' is the directed union of $D'[X/s]$ for $s \in S$. Note that $D[X/s]$ is D-isomorphic to $D[X]$. Let $s \in S$ such that $f, g \in D[X/s]$. We claim that f, g are v-coprime in $D[X/s]$. Indeed, $D'[X/s]$ is an AGCD domain [1, Theorem 5.6], so there exist a positive integer k and $h \in D'[X/s]$ such that $((f^k, g^k)D'[X/s])_v = hD'[X/s]$. In particular, h is a common divisor of f and g not only in $D'[X/s]$ but also in E'. As f, g are v-coprime in E', so are f^k, g^k. Then $h \in U(E') = U(D'[X/s]) = U(D')$. So f^k, g^k are v-coprime in $D'[X/s]$. By part (5) of [1, Lemma 1.1], f, g are v-coprime in $D'[X/s]$. By Proposition 3.3, $D[X/s]$ is t-linked under $D'[X/s]$, so f, g are v-coprime in $D[X/s]$. Now since f, g are v-coprime in every $D[X/s]$ that contains them, a direct limit argument shows that f, g are v-coprime in E. □

Corollary 3.13. *Let D be a domain with quotient field K. Then $D + XK[X]$ is an AGCD domain if and only if D is.*

References

[1] Zafrullah, M. A general theory of almost factoriality. Manuscripta Math. **1985**, *51*, 29–62.
[2] Anderson, D.D.; Zafrullah, M. Almost Bézout domains. J. Algebra **1991**, *142*, 285–309.
[3] Dumitrescu, T.; Lequain, Y.; Mott, J.L.; Zafrullah, M. Almost GCD domains of finite t-character. J. Algebra **2001**, *245*, 161–181.
[4] Anderson, D.D.; Anderson, D.F.; Zafrullah, M. Splitting the t-class group. J. Pure Appl. Algebra **1991**, *74*, 17–37.
[5] Anderson, D.D.; Zafrullah, M. Splitting sets in integral domains. Proc. Amer. Math. Soc. **2001**, *129*, 2209–2217.
[6] Mott, J.L. Convex directed subgroups of a group of divisibility. Canad. J. Math. **1974**, *26*, 532–542.
[7] Mott, J.L.; Schexnayder, M. Exact sequences of semi-value groups. J. Reine Angew. Math. **1976**, *283/284*, 388–401.
[8] Anderson, D.D.; Anderson, D.F.; Zafrullah, M. Atomic domains in which almost all atoms are prime. Comm. Algebra **1992**, *20*, 1447–1462.
[9] Anderson, D.D.; Mott, J.L.; Zafrullah, M. Finite character representations for integral domains. Boll. Un. Mat. Ital. B (7) **1992**, *6*, 613–630.
[10] Uda, H. LCM-stableness in ring extensions. Hiroshima Math. J. **1983**, *13*, 357–377.
[11] Hedstrom, J.; Houston, E. Some remarks on star-operations. J. Pure Appl. Algebra **1980**, *18*, 37–44.
[12] Zafrullah, M. The $D + XD_S[X]$ construction from GCD-domains. J. Pure Appl. Algebra **1988**, *50*, 93–107.
[13] Anderson, D.D.; Anderson, D.F.; Zafrullah, M. The ring $D + XD_S[X]$ and t-splitting sets. Arab. J. Sci. Eng. Sect. C Theme Issues **2001**, *26*, 3–16.
[14] Dumitrescu, T.; Zafrullah, M. LCM-splitting sets in some ring extensions. Proc. Amer. Math. Soc. *in press*.

[15] Anderson, D.D.; Anderson, D.F.; Zafrullah, M. Rings between $D[X]$ and $K[X]$. Houston J. Math. **1991**, *17*, 109–129.
[16] Costa, D.; Mott, J.L.; Zafrullah, M. The construction $D + XD_S[X]$. J. Algebra **1978**, *53*, 423–439.

Department of Mathematics, The University of Iowa, Iowa City IA 52242, USA
E-mail address: dan-anderson@uiowa.edu

Facultatea de Matematică, Universitatea Bucureşti, Str. Academiei 14, Bucharest, RO-70109, Romania
E-mail address: tiberiu@al.math.unibuc.ro

Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085, USA
E-mail address: zafrmuha@isu.edu
URL: http://www.isu.edu/~zafrmuha/