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Finite Character Representations for Integral Domains.

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Sunto. – *Si formulano alcune condizioni necessarie e sufficienti affinchè un dominio integro D abbia rappresentazione di carattere finito $\cap\{D_P \mid ht P = 1\}$. Si dimostra, per esempio, che D possiede una rappresentazione siffatta se e solo se ogni suo ideale principale proprio è un t-prodotto di ideali primari, ovvero se e solo se per ogni suo ideale primo P minimale sopra un ideale principale (x) risulta che $(x)_P \cap D$ è t-invertibile. Questo risultato è impiegato per caratterizzare alcune classi di domini integri. Infine si studia quando avviene che l'interpretazione $\cap\{D_P \mid P \in t\text{-Max}(D)\}$ ha carattere finito.*

1. – Introduction.

Let D be an integral domain. The title actually refers to two types of finite character representations for D. First, we study when $D = \cap D_{P_\alpha}$ for some collection $\mathcal{S}$ of prime ideals of D where the intersection has finite character, that is, each $0 \neq d \in D$ is in only finitely many P_α from $\mathcal{S}$. Theorem 3.1 gives several characterizations of domains for which $D = \cap D_{P_\alpha}$ has finite character where $\mathcal{S} = X^{(1)} = X^{(1)}(D)$, the set of height-one prime ideals of D, while Theorem 4.3 considers the case where $\mathcal{S} = t\text{-Max}(D)$, the set of maximal t-ideals of D. (For definitions concerning t-ideals, see Section 2.) Second, we study «finite character» product representations for ideals of D; more precisely, we study the situation where every proper principal ideal is a t-product of primary ideals. In fact, according to Theorem 3.1, $D = \bigcap_{P \in X^{(1)}} D_P$ where the intersection has finite character if and only if every proper principal ideal of D is a t-product of primary ideals. We use Theorem 3.1, the main result of this paper, to give finite-character characterizations of several classes of integral domains including various classes of Krull domains and generalizations of factorial domains.

Section 2 reviews some of the basic properties of star-operations, especially the t-operation. Lemma 2.2 and Corollary 2.4 give two useful results concerning ideals in an integral domain D having a finite character representation $\cap D_{P_\alpha}$. Lemma 2.2 states that for a nonzero ideal A, if each AD_{P_α} is principal, then A is t-invertible while Corollary 2.4 states that if AD_M is principal for each maximal ideal M, then A is invertible.

Recall that a domain D is a Krull domain precisely when $D = \bigcap_{P \in X^{(1)}} D_P$ has finite character and each D_P is a DVR. Section 3 investigates integral domains D where $D = \bigcap_{P \in X^{(1)}} D_P$ has finite character (but the hypothesis that each D_P is a DVR has been dropped). Just as properties of a Krull domain are reflected by its divisor class group, we show that properties of a domain D with a finite character representation of the form $D = \bigcap_{P \in X^{(1)}} D_P$ are reflected by its t-class group $\mathrm{Cl}_t(D)$. For example, $\mathrm{Cl}_t(D)$ is torsion if and only if some power of each element of D is a product of primary elements (Theorems 3.4).

In the final section we move from the height-one case $D = \bigcap_{P \in X^{(1)}} D_P$ to the more general case $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. This involves replacing the notion of a P-primary ideal by the more general notion of a P-pure ideal. We define an ideal A to be P-pure, P a prime ideal, if $A_P \cap D = A$. We show (Theorem 4.3) that every proper principal ideal of D is a t-product of P_α-pure ideals where each $P_\alpha \in t\text{-Max}(D)$ if and only if $D = \bigcap_{P \in t\text{-Max}(D)} D_P$ has finite character and for maximal t-ideals P and Q, there is no nonzero prime ideal contained in $P \cap Q$. This result should be viewed as the pure ideal analog of Theorem 3.1.

For terms and results not defined or stated in the paper, see [8] or [13].

2. – Star-operations.

Let D be an integral domain with quotient field K. Let $F(D)$ denote the set of nonzero fractional ideals of D and let $f(D)$ denote the subset of finitely generated members of $F(D)$. A mapping $A \to A^*$ of $F(D)$ into itself is called a *star-operation* on D if the following conditions hold for all $a \in K\text{-}\{0\}$ *and* $A, B \in F(D)$.

(1) $(a)^* = (a)$, $(aA)^* = aA^*$;

(2) $A \subseteq A^*$, if $A \subseteq B$, then $A^* \subseteq B^*$; and

(3) $(A^*)^* = A^*$.

It is easily proved that $(AB)^* = (A^*B)^* = (A^*B^*)^*$ for all $A, B \in F(D)$. A fractional ideal $A \in F(D)$ is called a *$*$-ideal* if $A = A^*$ and A is said to be *$*$-finite* if $A^* = B^*$ for some $B \in f(D)$.

A star-operation $*$ on D is said to be of *finite character* if $A^* = \cup\{J^* \mid J \subseteq A \text{ with } J \in f(D)\}$ for each $A \in F(D)$. If $*$ has finite character and A is $*$-finite, then $A^* = B^*$ for some $B \in f(D)$ with $B \subseteq A$. Recall that the function on $F(D)$ defined by $A \to A_v = (A^{-1})^{-1} = [D: [D: A]]$ is a star-operation on D called the *v-operation*. The function $A \to A_t = \cup\{J_v \mid J \subseteq A \text{ with } J \in f(D)\}$ is a finite character star-operation on D, called the *t-operation*. If $*$ is any star-operation on D, then $(A^*)_v = A_v = (A_v)^*$ for all $A \in F(D)$, while if $*$ has finite character, then $(A^*)_t = A_t = (A_t)^*$ for all $A \in F(D)$. The reader may consult [8, sections 32 and 34] for the basic properties of star-operations and the v-operation. Also, see [9-11].

Let $*$ be a star-operation on D. A fractional ideal A is said to be *$*$-invertible* if there exists a $B \in F(D)$ with $(AB)^* = D$. In this case, we can take $B = A^{-1}$. Observe that if A is $*$-invertible and $A^* = (A_1A_2)^*$, then A_1 and A_2 are also $*$-invertible. If A is $*$-invertible and $*$ has finite character, then A is $*$-finite.

Suppose that $*$ is a finite character star-operation. An easy Zorn's Lemma argument shows that each proper integral $*$-ideal is contained in a maximal proper integral $*$-ideal. Moreover, such a maximal $*$-ideal is prime. Let $*\text{-Max}(D) = \{P \mid P \text{ is a maximal } *\text{-ideal}\}$. For any $A \in F(D)$, $A^* = \cap\{A^*D_P \mid P \in *\text{-Max}(D)\}$ ([9, Proposition 4] or [11, Proposition 2.8]). In particular, $D = \cap\{D_P \mid P \in *\text{-Max}(D)\}$. Note that A is $*$-invertible $\Leftrightarrow AA^{-1} \not\subseteq P$ for each $P \in *\text{-Max}(D)$. Thus if A is $*$-invertible, then A is $*$-finite and for each $P \in *\text{-Max}(D)$, A_P is principal. The converse is also true: if A is $*$-finite and A_P is principal for each $P \in *\text{-Max}(D)$, then A is $*$-invertible [11, Proposition 2.6]. In particular, an ideal A is t-invertible $\Leftrightarrow A$ is t-finite and A_P is principal for each maximal t-ideal P. Also note that if A is t-invertible, then $(A_t)_P = A_P$ for each $P \in t\text{-Max}(D)$. (For $A_P \subseteq (A_t)_P \subseteq (A_P)_t = A_P$ where the containment $(A_t)_P \subseteq (A_P)_t$ follows from [11, Lemma 3.4] and the equality follows since A_P is principal.)

For any integral domain D, the set of t-invertible t-ideals forms a group under the t-product $A*B = (AB)_t$. The *t-class group* of D is $\mathrm{Cl}_t(D)$ – the group of t-invertible t-ideals modulo the subgroup of principal fractional ideals. For D a Krull domain, $\mathrm{Cl}_t(D) = \mathrm{Cl}(D)$, the divisor class group, while for D a Prüfer domain, $\mathrm{Cl}_t(D) = C(D)$ the class group. For result on the t-class group, the reader is referred to [6] and [7].

Suppose that $\mathcal{S} = \{P_\alpha\}$ is a set of nonzero prime ideals of D with $D = \cap\{D_{P_\alpha} | P_\alpha \in \mathcal{S}\}$. (Recall that $D = \cap D_{P_\alpha} \Leftrightarrow$ each proper ideal of the form $(a):(b)$ is contained in some P_α [13, Exercise 20, page 42].) The mapping $A \to A^{*_\mathcal{S}} = \cap AD_{P_\alpha}$ is a star-operation satisfying the following properties: (1) $AD_{P_\alpha} = A^{*_\mathcal{S}} D_{P_\alpha}$, (2) $(A \cap B)^{*_\mathcal{S}} = A^{*_\mathcal{S}} \cap B^{*_\mathcal{S}}$ for all $A, B \in F(D)$, (3) $P_\alpha^{*_\mathcal{S}} = P_\alpha$ for each $P_\alpha \in \mathcal{S}$, and (4) if A is an integral ideal of D, then $A^{*_\mathcal{S}} \neq D \Leftrightarrow A \subseteq P_\alpha$ for some α. Moreover, if the intersection $D = \cap D_{P_\alpha}$ has finite character, then $*_\mathcal{S}$ has finite character. For the proof of this last result and for other properties of such star-operations, the reader is referred to [3].

LEMMA 2.1. – *Let $\mathcal{S}$ be a set of prime t-ideals such that no two distinct elements of $\mathcal{S}$ are comparable. Suppose that $D = \cap\{D_{P_\alpha} | P \in \mathcal{S}\}$ where the intersection has finite character. Then the following statements hold.*

(1) $*_\mathcal{S}$-Max$(D) = \mathcal{S} = t$-Max(D).

(2) *A is $*_\mathcal{S}$-invertible $\Leftrightarrow A$ is t-invertible.*

(3) *If A is t-invertible, then $A_t = A^{*_\mathcal{S}}$.*

PROOF. – (1) Since the prime ideals in $\mathcal{S}$ are incomparable, $*_\mathcal{S}$-Max$(D) = \mathcal{S}$. Let $P \in \mathcal{S}$, so P is a prime t-ideal. If $Q \supsetneq P$ is a prime ideal, then $Q_t \supseteq Q^{*_\mathcal{S}} = D$. Hence $P \in t$-Max(D). Conversely, if $P \in t$-Max(D), then $P^{*_\mathcal{S}} = P \neq D$, so $P \subseteq Q \in \mathcal{S}$. But Q is a t-ideal, so $P = Q \in \mathcal{S}$.

(2) A is $*_\mathcal{S}$-invertible $\Leftrightarrow AA^{-1} \not\subseteq P$ for each $P \in *_\mathcal{S}$-Max$(D) = t$-Max$(D) \Leftrightarrow A$ is t-invertible.

(3) $A_t = \cap\{A_t D_P | P \in t\text{-Max}(D)\} = \cap\{AD_P | P \in t\text{-Max}(D)\} = \cap\{AD_P | P \in \mathcal{S}\} = A^{*_\mathcal{S}}$ where the middle equality follows since $A_t D_P = AD_P$ if A is t-invertible and $P \in t$-Max(D). ■

Note that in Lemma 2.1, we need *not* have $A_t = A^{*_\mathcal{S}}$ if A is not t-invertible. See [3] for a discussion of this point.

We will be especially interested in the case $D = \cap\{D_P | P \in X^{(1)}(D)\}$ with the intersection being of finite character where $X^{(1)} = X^{(1)}(D)$ is the set of height-one prime ideals of D. Since a height-one prime ideal is a t-ideal, the set $X^{(1)}$ satisfies the hypotheses on $\mathcal{S}$ given in Lemma 2.1.

As previously mentioned, an ideal A is $*$-invertible (for $*$ with finite character) if and only if A is $*$-finite and A_P is principal for each $P \in *\text{-Max}(D)$. For star-operators induced by finite character intersections of localizations, we may drop the hypothesis that A is $*$- finite.

LEMMA 2.2. – *Let $\mathcal{S}$ be a collection of nonzero prime ideals of D. Suppose that $D = \cap \{D_{P_\alpha} | P_\alpha \in \mathcal{S}\}$ where the intersection has finite character. If $A \in F(D)$ with AD_{P_α} principal for each P_α, then A is $*_{\mathcal{S}}$-invertible and hence t-invertible.*

PROOF. – We may assume that A is integral. Since the intersection $D = \cap D_{P_\alpha}$ has finite character, $AD_{P_\alpha} = D_{P_\alpha}$ for all P_α, except say for $P_1, \ldots, P_n$. For $i = 1, \ldots, n$, choose $a_i \in A$ with $AD_{P_i} = a_i D_{P_i}$. Let $P_{n+1}, \ldots, P_r$ be the additional primes from $\mathcal{S}$ that contain $(a_1, \ldots, a_n)$. Now $A \subsetneq P_{n+1} \cup \ldots \cup P_r$, so there exists $b \in A - (P_{n+1} \cup \ldots \cup P_r)$. Let $B = (a_1, \ldots, a_n, b)$. Then $BD_{P_\alpha} = AD_{P_\alpha}$ for each $P_\alpha \in \mathcal{S}$. So $B^{*_{\mathcal{S}}} = A^{*_{\mathcal{S}}}$. Now $(BB^{-1})_{P_\alpha} = B_{P_\alpha} B_{P_\alpha}^{-1} = D_{P_\alpha}$ since B_{P_α} is principal. Thus $(BB^{-1})^{*_{\mathcal{S}}} = D$. So $D = (BB^{-1})^{*_{\mathcal{S}}} = (B^{*_{\mathcal{S}}} B^{-1})^{*_{\mathcal{S}}} = (A^{*_{\mathcal{S}}} B^{-1})^{*_{\mathcal{S}}} = (AB^{-1})^{*_{\mathcal{S}}}$. Hence A is $*_{\mathcal{S}}$-invertible. But $*_{\mathcal{S}}$ has finite character, so $D = D_t = ((AB^{-1})^{*_{\mathcal{S}}})_t = (AB^{-1})_t$. Hence A is also t-invertible. ■

COROLLARY 2.3. – *Suppose that $D = \bigcap_{P \in X^{(1)}} D_P$ where the intersection has finite character. Then for $0 \neq x \in P$ where $P \in X^{(1)}$, $xD_P \cap D$ is a t-invertible primary t-ideal.*

PROOF. – Now $xD_P \cap D$ is P-primary since $\operatorname{ht} P = 1$. Thus $(xD_P \cap D)_P = xD_P$ while $(xD_P \cap D)_Q = D_Q$ for $Q \neq P$ with $\operatorname{ht} P = 1$. By Lemma 2.2, $xD_P \cap D$ is t-invertible. Also, note that $xD_P \cap D$ is a $*_{X^{(1)}}$-ideal. By Lemma 2.1(3), $xD_P \cap D$ is a t-ideal. ■

It is well known that an ideal in an integral domain is invertible if and only if it is finitely generated and locally principal. (This is the special case where the star-operation is the *d-operation*, $A_d = A$.) Under certain conditions, a locally principal ideal must be invertible. Our next corollary generalizes the well known result that a locally principal ideal in a semi-quasi-local domain is invertible.

COROLLARY 2.4. – *Suppose that an integral domain D is a finite character intersection of localizations. Then a nonzero fractional ideal A is invertible if and only if A_M is principal for each maximal ideal M of D.*

PROOF. – Suppose that $D = \cap D_{P_\alpha}$ has finite character. Since A_M is principal for each maximal ideal M, each A_{P_α} is principal. By Lemma 2.2, A is t-invertible. Hence A is t-finite. By [2, Theorem 2.1], A is invertible. ■

3. – Finite character representation.

The main result of the paper Theorem 3.1 gives a number of characterizations, involving the t-operation and primary ideals, for an integral domain D to be a finite character intersection of its localizations at height-one prime ideals. Corollary 3.2, an immediate consequence of Theorem 3.1, gives a new proof of several well known equivalent conditions for D to be a Krull odmain. The point of view here is that if in Theorem 3.1 we replace «primary» by «prime» we go from domains having a height-one finite character representation $D = \bigcap_{P \in X^{(1)}} D_P$ to the case where each D_P is a DVR and hence Krull domains. The general result Theorem 3.1 is then used to study several classes of integral domains generalizing certain classes of Krull domains. In each case, the Krull domain assumption is replaced by the weaker condition that $D = \bigcap_{P \in X^{(1)}} D_P$ has finite character, the divisor class group is replaced by the t-class group, and «prime» is replaced by «primary». Then these theorems are combined with Corollary 3.2, we obtain the known results for Krull domains.

THEOREM 3.1. – *For an integral domain D, the following conditions are equivalent.*

(1) $D = \bigcap_{P \in X^{(1)}} D_P$ *where the intersection is of finite character.*

(2) *Every proper principal ideal of D has a primary decomposition where all the associated prime ideals have height-one.*

(3) *Every proper principal ideal of D is a t-product of primary (t-) ideals.*

(4) *Every proper t-ideal of D is a t-product of primary (t-) ideals.*

(5) *Every nonzero prime ideal contains a t-invertible primary (t-) ideal.*

(6) *Every proper principal ideal of D is a finite intersection of t-invertible primary (t-) ideals.*

(7) *If P is a prime ideal minimal over a proper principal ideal (x), then $(x)_P \cap D$ is t-invertible.*

PROOF. – (1) $\Leftrightarrow$ (2) [4, Theorem 13]. (1) $\Rightarrow$ (4). Let A be a proper t-ideal. Let $*$ be the (finite character) star-operation $I \to I^* = \bigcap_{P \in X^{(1)}} ID_P$. Then $A = A^* = \bigcap_{P \in X^{(1)}} AD_P = Q_1 \cap \dots \cap Q_n$ where $Q_i = AD_{P_i} \cap D$ is P_i-primary and $\{P_1, \dots, P_n\}$ is the set of height-one prime ideals containing A. Note that $(Q_1 \dots Q_n)^* = Q_1 \cap \dots \cap Q_n$. Hence $(Q_1 \dots Q_n)_t = ((Q_1 \dots Q_n)^*)_t = (Q_1 \cap \dots \cap Q_n)_t = A_t = A$. So A is a t-product of primary ideals. Observe that also $A = (Q_{1t} \dots Q_{nt})_t$ and Q_{it} is a P_i-primary t-ideal because $P_i \in X^{(1)}$, $\sqrt{Q_{it}} = P_i$, and $Q_{it} = Q_{it} D_{P_i} \cap D$.

(4) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (5). Let P be a nonzero prime ideal and let $0 \neq x \in P$. Then $Dx = (Q_1 \dots Q_n)_t$ where $Q_1, \dots, Q_n$ are primary ideals each of which is necessarily t-invertible being a t-factor of a principal ideal. Then $P \supseteq Dx \supseteq Q_1 \dots Q_n$, so some $Q_i \subseteq P$. Hence P contains a t-invertible primary ideal.

(5) $\Rightarrow$ (1). Let M be a maximal t-ideal of D. Let $0 \subsetneq P \subseteq M$ be a prime ideal. By hypothesis, there exists a t-invertible primary ideal $Q \subseteq P$. Then $Q_M \subseteq P_M$ and Q_M is a principal primary ideal. Thus in D_M every nonzero prime ideal contains a nonzero primary element. Suppose that in D_M, (q) is P-primary where P is a prime ideal of D_M. If $\operatorname{ht} P > 1$, then there exists a prime ideal $Q \subsetneq P$ and hence a Q'-primary ideal for some prime ideal $Q' \subseteq Q \subsetneq P$. But this contradicts [4, Theorem 4] which states that distinct primes having invertible primary ideals must be incomparable. We next show that D_M is the intersection of its localizations at height-one primes. To show this, it suffices to show that every ideal $(aD_M : bD_M) \neq D_M$ is contained in a height-one prime ideal. Now since the saturation of the set of products of primary elements of D_M is $D_M - \{0\}$, there exists an $r \in D_M$ so that ar is a product of primary elements. We can assume that $ar = q_1 \dots q_n$ where (q_i) is P_i-primary and that the P_i's are distinct primes. Then $(ar) = (q_1 \dots q_n) = (q_1) \cap \dots \cap (q_n)$. (See the next to the last paragraph of [4, page 146].) Moreover, each $\operatorname{ht} P_i = 1$. Then

$$(aD_M : bD_M) = (raD_M : rbD_M) = (((q_1) \cap \dots \cap (q_n)) : rbD_M) = ((q_1) : rbD_M) \cap \dots \cap ((q_n) : rbD_M)$$

and each component in this intersection is either P_i-primary or D_M.

Since $(aD_M : bD_M) \neq D_M$, $(aD_M : bD_M)$ is contained in some P_i. Thus D_M is the intersection of its localizations at height-one primes. Since $D = \bigcap_{M \in t\text{-Max}(D)} D_M$, D itself is an intersection of its localizations at height-one primes. It remains to show that this intersection has finite character. Let $\mathcal{S} = \{I$ is a nonzero finitely generated ideal of $D \mid$ there exists a finitely generated ideal J such that $(IJ)_t$ is a t-product of t-invertible primary ideals$\}$. By hypothesis, $\mathcal{S} \neq \emptyset$. Note that $\mathcal{S}$ is multiplicatively closed. Suppose that there exists a proper principal ideal $(x) \notin \mathcal{S}$. Then $T \not\subseteq (x)$ for every $T \in \mathcal{S}$. (For if some $T \subseteq (x)$, then $T = (x)C$, so $T_t = ((x)C)_t$. Moreover, since $C = x^{-1}T$, we can take C to be finitely generated. But then $(x) \in \mathcal{S}$, a contradiction.) By Zorn's Lemma, we can enlarge (x) to an ideal B maximal with respect to the property that $T \not\subseteq B$ for every $T \in \mathcal{S}$. Moreover, it is easily verified that B is prime [1, Theorem 2.2]. But this contradicts the hypothesis that every nonzero prime ideal contains a t-invertible primary ideal. Thus for each proper principal ideal (x), there is a finitely generated ideal J such that $((x)J)_t = (Q_1 \dots Q_n)_t$ where Q_i is P_i-primary. Suppose that $P \supseteq (x)$ where $\operatorname{ht} P = 1$. Then $P \supseteq (x)J$ so $P = P_t \supseteq ((x)J)_t = (Q_1 \dots Q_n)_t \supseteq Q_1 \dots Q_n$. Hence $P \supseteq$ some P_i, so $P = P_i$ for some i since $\operatorname{ht} P = 1$. Thus each proper principal ideal is contained in only a finite number of height-one primes, so the intersection $D = \bigcap_{P \in X^{(1)}} D_P$ has finite character.

So (1)-(5) are equivalent where in each of (3)-(5) the primary ideals in question are not assumed to be t-ideals. Denote by (n'), for $n = 3, 4, 5$, the statement in (n) where each primary ideal is required to be a t-ideal. In the proof of (1) $\Rightarrow$ (4) we actually showed that (1) $\Rightarrow$ (4′). Since clearly (4′) $\Rightarrow$ (3′) $\Rightarrow$ (3), we have that (3′) and (4′) are equivalent to (1)-(5). Since (5′) $\Rightarrow$ (5) and (3′) $\Rightarrow$ (5′), we have that (5′) is equivalent to (1)-(5).

(6) $\Rightarrow$ (5). Let P be a nonzero prime ideal and let $0 \neq x \in P$. Then $P \supseteq (x) = Q_1 \cap \dots \cap Q_n \supseteq Q_1 \dots Q_n$ where each Q_i is a t-invertible primary ideal. Thus $P \supseteq Q_i$ some $i = 1, \dots, n$.

(1) $\Rightarrow$ (6). Let (x) be a proper principal ideal. Let $P_1, \dots, P_n$ be the height-one prime ideals containing (x). Then $(x) = \bigcap_{P \in X^{(1)}} xD_P = (xD_{P_1} \cap D) \cap \dots \cap (xD_{P_n} \cap D)$. By Corollary 2.3, each $xD_{P_n} \cap D$ is a t-invertible primary t-ideal.

(1) $\Rightarrow$ (7). Since P is minimal over (x), P is a t-ideal. By Lemma 2.1, $\operatorname{ht} P = 1$. By Corollary 2.3, $(x)_P \cap D$ is a t-invertible t-ideal.

(7) $\Rightarrow$ (5). Let Q be a nonzero prime ideal. Let $0 \neq x \in Q$. Shrink

Q to a prime ideal P minimal over (x). Then $(x)_P \cap D$ is a t-invertible ideal contained in P and hence in Q. Moreover, $(x)_P \cap D$ is primary since P is minimal over (x). ■

Suppose that D is Noetherian. Then $D = \bigcap_{P \in X^{(1)}} D_P$ if and only if every grade-one prime ideal of D has height one. In this case the intersection is necessarily of finite character.

If in Theorem 3.1 we replace «primary ideal» by «prime ideal» or «power of a prime ideal», we get characterizations of Krull domains. This is stated in Corollary 3.2. While the equivalence of most of the statements in Corollary 3.2 is known (see [12] for another proof of part of Corollary 3.2 and a history of some of these characterizations), our proof is entirely different.

COROLLARY 3.2. – *For an integral domain D, the following conditions are equivalent.*

(1) *D is a Krull domain.*

(2) *Every proper principal ideal has a primary decomposition of the form $(x) = P_1^{(n_1)} \cap \ldots \cap P_s^{(n_s)}$ where $\operatorname{ht} P_i = 1$.*

(3) *Every proper principal ideal is a t-product of prime (t-) ideals.*

(4) *Every proper t-ideal is a t-product of prime t-ideals.*

(5) *Every nonzero prime ideal contains a t-invertible prime (t-) ideal.*

(6) *Every proper principal ideal has the form $(x) = (P_1^{n_1})_t \cap \ldots \cap (P_s^{n_s})_t$ where each P_i is a t-invertible prime ideal and each $n_i \geq 1$.*

(7) *If P is a prime ideal minimal over a proper principal ideal (x), then $(x)_P \cap D = (P^n)_t$ for some $n \geq 1$ and is t-invertible.*

PROOF. – (1) $\Rightarrow$ (2)-(7). Well known. Since (6), (7) $\Rightarrow$ (5) and (4) $\Rightarrow$ (3) $\Rightarrow$ (5), it suffices to show that (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). By Theorem 3.1, each of (2)-(7) implies that $D = \bigcap_{P \in X^{(1)}} D_P$ where the intersection is of finite character. So $X^{(1)} = t\text{-Max}(D)$. It suffices to show that (2) and (5) imply that D_P is a DVR for $P \in X^{(1)}$. Now (2) gives that every principal ideal of D_P has the form P_P^n, so certainly D_P is a DVR. Suppose that (5) holds. Now since P is a maximal t-ideal, P_P must be principal. Hence D_P is a DVR. ■

It is easily seen that to the list in Corollary 3.3 we may add

(8) every t-ideal of D is t-invertible.

Theorem 3.1 and Corollary 3.2 show that a finite character representation $D = \bigcap_{P \in X^{(1)}} D_P$ can be obtained from the definition of a Krull domain by replacing «prime» by «primary». In a similar manner, if we replace «prime» by «primary» in the definition of a factorial domain, we get the notion of a weakly factorial domain. An integral domain D is said to be *weakly factorial* if every nonzero nonunit element of D may be written as a product of primary elements. Weakly factorial domains were introduced in [4]. In [5] it was shown that the following conditions on an integral domain D are equivalent: (1) D is weakly factorial, (2) every convex directed subgroup of $G(D)$, the group of divisibility of D, is a cardinal summand of $G(D)$, (3) if P is a prime ideal of D minimal over a proper principal ideal (x), then $\mathrm{ht}\, P = 1$ and $(x)_P \cap D$ is principal, (4) $D = \bigcap_{P \in X^{(1)}} D_P$ where the intersection has finite character and $\mathrm{Cl}_t(D) = 0$. Note that by Theorem 3.1 (or really [4, Theorem 4]), (3) may be replaced by (3′) if P is a prime ideal of D minimal over a proper principal ideal (x), then $(x)_P \cap D$ is principal.

A domain D is called a *π-domain* if every proper principal ideal of D is a product of prime ideals (necessarily invertible). Many characterizations of π-domains are known, see, for example, [2, Theorem 3.1]. Several of these equivalences include: (1) D is a π-domain, (2) every nonzero prime ideal of D contains an invertible prime ideal, (3) D is a locally factorial Krull domain, (4) D is a Krull domain in which every divisorial ideal is invertible, i.e., $\mathrm{Cl}(D) = \mathrm{Pic}(D)$. Note that in Corollary 3.2 if «t-product» is replaced by «product» and «t-invertible», then we get characterizations of π-domains. This point of view has also been put forth in [12].

In our next theorem we investigate what happens to the notion of a weakly factorial domain when «principal» is replaced by «invertible» or equivalently what happens to the notion of a π-domain when «prime» is replaced by «primary».

THEOREM 3.3. – *For an integral domain D, the following conditions are equivalent.*

(1) *Every principal ideal is a product of primary ideals.*

(2) *Every proper principal ideal has a reduced primary decomposition in which the primary components are invertible.*

(3) *Every invertible ideal is a product of primary ideals.*

(4) *For P a prime ideal minimal over a proper principal ideal (x), $(x)_P \cap D$ is invertible.*

(5) $D = \bigcap_{P \in X^{(1)}} D_P$ *has finite character and every t-invertible t-ideal is invertible, i.e.,* $\mathrm{Cl}_t(D) = \mathrm{Pic}(D)$.

(6) $D = \bigcap_{P \in X^{(1)}} D_P$ *has finite character and for each maximal ideal M of D, D_M is weakly factorial.*

PROOF. – By Theorem 3.1, each of (1)-(6) entails that $D = \bigcap_{P \in X^{(1)}} D_P$ where the intersection has finite character. Moreover, $X^{(1)}(D) = t\text{-Max}(D)$. For each maximal ideal M of D, properties (1)-(4) are inherited by each D_M where now «invertible» may be replaced by «principal». By [5, Theorem] each such D_M is weakly factorial. Thus (1)-(4) $\Rightarrow$ (6).

Suppose that (5) holds. By Theorem 3.1, every proper t-ideal A of D is a t-product of primary t-ideals, say $A = (Q_1 \dots Q_n)_t$ where Q_i is a primary t-ideal. Now if A is either principal or invertible, each Q_i is t-invertible, being a factor of a t-invertible ideal. Thus $\mathrm{Cl}_t(D) = \mathrm{Pic}(D)$ gives that each Q_i is invertible. Thus $Q_1 \dots Q_n$ is invertible, so $A = (Q_1 \dots Q_n)_t = Q_1 \dots Q_n$—a product of primary ideals. Since the product of two invertible P-primary ideals is still P-primary [4, Theorem 1] and since a product of invertible primary ideals with distinct radicals is equal to its intersection [4, Theorem 3], A has a reduced primary decomposition whose components are invertible primary ideals. Thus (5) $\Rightarrow$ (1)-(3). Moreover, (5) $\Rightarrow$ (4) since by Theorem 3.1, $(x)_P \cap D$ is a t-invertible t-ideal and hence is invertible.

It remains to prove (6) $\Rightarrow$ (5). Let A be a t-invertible t-ideal. We must show that A is invertible. We can assume that A is a proper integral ideal of D. Now $A = A^{*_{X^{(1)}}} = ((A_{P_1} \cap D) \dots (A_{P_n} \cap D))^{*_{X^{(1)}}}$ where $P_1, \dots, P_n$ are the height-one prime ideals containing A. It suffices to show that each $A_{P_i} \cap D$ is invertible. Moreover, since each $A_{P_i} \cap D$ is t-invertible and hence of finite type, by Corollary 2.4 it suffices to show that for each maximal ideal M, $(A_{P_i} \cap D)_M$ is principal. Now A is t-invertible, so $(A_{P_i} \cap D)_M = (a_i D_{P_i} \cap D)_M = a_i D_{P_{iM}} \cap D_M$ is principal since D_M is weakly factorial. ■

It is now easy to obtain characterization for π-domains such as those given in [2, Theorem 3.1] by combining Corollary 3.2 and Theorem 3.3. These characterization will either take the form of Corollary 3.2 with the t-operation deleted (or more accurately, replaced by the d-operation) or the form of Theorem 3.3 with «primary» replaced by «prime». We remark that several other equivalences may be added to Theorem 3.3; for example, (a) every proper t-ideal in a product of primary t-ideals, or (b) for each prime ideal P minimal over a proper invertible ideal I, $ID_P \cap D$ is invertible. However, a condition from Theorem 3.1 that can *not* be added to the list is that every prime ideal contains an invertible primary ideal. For example, if D is a local Krull domain with nonzero torsion divisor class group, then every prime ideal contains a principal primary ideal, but D is not weakly factorial.

Storch [14] defined a Krull domain D to be *almost factorial* if $\mathrm{Cl}(D)$ is torsion. Several conditions equivalent to a Krull domain being almost factorial are given in [14] and [2, Theorem 3.2]. One of these is that D is a Krull domain in which some power of each proper principal ideal is a product of principal primary ideals. We define an integral domain D to be *almost weakly factorial* if for each nonzero nonunit $x \in D$, some positive power of x is a product of primary elements. Our next theorem gives some conditions equivalent to D being almost weakly factorial. An interested reader may add to this list.

THEOREM 3.4. – *For an integral domain D, the following conditions are equivalent.*

(1) *D is almost weakly factorial, i.e., for each nonzero nonunit $x \in D$, there exists a positive integer $n(x)$ such that $x^{n(x)}$ is a product of primary elements.*

(2) *For P a prime ideal minimal over a proper principal ideal (x), there exists a natural number $n = n(x, P)$ so that $x^n D_P \cap D$ is principal.*

(3) *$D = \bigcap_{P \in X^{(1)}} D_P$ has finite character and $\mathrm{Cl}_t(D)$ is torsion.*

PROOF. – Note that (1) and (2) imply that each nonzero prime ideal contains a principal primary ideal. Thus by Theorem 3.1, $D = \bigcap_{P \in X^{(1)}} D_P$ where the intersection has finite character and $X^{(1)} = t\text{-Max}(D)$. Let $*$ be the star operation $A^* = \bigcap_{P \in X^{(1)}} D_P$.

(1) $\Rightarrow$ (3). Let A be a t-invertible t-ideal. We must show that $(A^m)_t$ is principal for some $m \geq 1$. We can assume that $A \subsetneq D$. Let $P_1, \ldots, P_n$ be the height-one prime ideals containing A. Now A_{P_i} is

principal (since A is t-invertible), so there exists a natural number m_i with $A_{P_i}^{m_i} = a_i D_{P_i}$ where (a_i) is P_i-primary. Let $m = m_1 \dots m_n$. Then $A_{P_i}^m = a_i^m D_{P_i}$ and (a_i^m) is P_i-primary. Then

$$(A^m)_t = ((A^m)^*)_t = ((A_{P_1}^m \cap D) \cap \dots \cap (A_{P_n}^m \cap D))_t =$$

$$= ((a_i^m) \cap \dots \cap (a_n^m))_t = (a_1^m) \cap \dots \cap (a_n^m) = (a_1^m \dots a_n^m)$$

is principal.

(3) $\Rightarrow$ (2). Since P is minimal over (x), P is a t-ideal and hence $\operatorname{ht} P = 1$. By Corollary 2.3, $xD_P \cap D$ is a t-invertible t-ideal. Hence there exists a natural number n so that $((xD_P \cap D)^n)_t$ is principal. But $((xD_P \cap D)^n)_t = (((x^n D_P \cap D)^n)^*)_t = (x^n D_P \cap D)_t = x^n D_P \cap D$, so $x^n D_P \cap D$ is principal.

(2) $\Rightarrow$ (1). Let $P_1, \dots, P_n$ be the (height-one) prime ideals minimal over (x). For each i, choose $m_i \geqslant 1$ so that $x^{m_i} D_{P_i} \cap D$ is principal. Let $m = m_1 \dots m_n$. Then

$$(x^m) = (x^m)^* = ((x^m D_{P_1} \cap D) \cap \dots \cap (x^n D_{P_n} \cap D))^* =$$

$$= ((x^m D_{P_1} \cap D) \dots (x^m D_{P_n} \cap D))^* = (x^m D_{P_1} \cap D) \dots (x^m D_{P_n} \cap D)$$

is a product of principal primary ideals. ■

Generalizing both of the conditions given in Theorem 3.3 and 3.4, we can also consider integral domains which satisfy the property that for each proper principal ideal there exists a natural number $n(x)$, so that $(x)^{n(x)}$ is a product of (invertible) primary ideals. Using Theorem 3.1 and [2, Theorem 2.3], the reader can readily obtain the next theorem along with some additional characterizations. The case where «primary» has been replaced by «prime» is given in [2, Theorem 3.3].

THEOREM 3.5. – *For an integral domain D, the following conditions are equivalent.*

(1) *For each proper principal ideal (x) of D, there exists a natural number $n(x)$ so that $(x)^{n(x)}$ is a product of primary ideals.*

(2) *For each proper principal ideal (x) and P a prime minimal over (x), there exists a natural number $n = n(x, P)$ so that $x^n D_P \cap D$ is invertible.*

(3) $D = \bigcap_{P \in X^{(1)}} D_P$ *where the intersection has finite character*

and for each maximal ideal M of D, D_M is almost weakly factorial.

(4) $D = \bigcap_{P \in X^{(1)}} D_P$ *where the intersection has finite character and the local t-class group* $\mathrm{Cl}_t(D)/\mathrm{Pic}(D)$ *is torsion.* ■

4. – Pure ideals.

The results of the previous section were restrictive in the sense that the prime ideals involved all have height-one. This restriction is interrelated with the concept of primary ideal. To investigate finite character representations $D = \cap D_P$ where the prime ideals are not assumed to have height-one, we need a generalization of the notion of primary ideal. In this section, we sketch an approach based on what we call pure ideals.

If an ideal A is P-primary, then $A_Q \cap D = A$ for each prime ideal $Q \supseteq P$. Let us call a proper ideal A *P-pure* for a prime ideal P if $A_P \cap D = A$. This is of course equivalent to $A \cap (s) = As$ or $A:(s) = A$ for each $s \in D - P$. Observe that if P' is a prime ideal with $P' \supseteq P$ and A is P-pure, then A is P'-pure and that A is P-primary $\Leftrightarrow A$ is P-pure and P is minimal over A. Hence if $P \in X^{(1)}$, A is P-pure $\Leftrightarrow A$ is P-primary. We call an element $x \in D$ *P-pure* if (x) is P-pure.

If $D = \bigcap_{P \in X^{(1)}} D_P$ has finite character, then each proper principal ideal is a t-product of primary ideals. Our first theorem of this section generalizes this result by replacing primary ideals by pure ideals.

THEOREM 4.1. – *Let $D = \cap\{D_{P_\alpha} | P_\alpha \in S\}$ be of finite character and suppose that no two distinct $P_\alpha \in S$ contain a nonzero prime ideal. Then every proper principal ideal of D may be written as a finite t-product of P_α-pure ideals, each of which is necessarily t-invertible.*

PROOF. – Let $A^{*_S} = \bigcap_\alpha AD_{P_\alpha}$. Let (x) be a proper principal ideal of D. Let $P_1, \ldots, P_n$ be the elements of S containing (x). Put $X_i = xD_{P_i} \cap D$. Then $X_{iP_i} \cap D = xD_{P_i} \cap D = X_i$, so X_i is P_i-pure. Now $X_{iP_i} = xD_{P_i}$. If $P_\alpha \neq P_i$, then $X_i D_{P_\alpha} = D_{P_\alpha}$. For $X_{iP_i} \subsetneq D_{P_\alpha} \Rightarrow X_i \subset P_\alpha D_{P_\alpha} \Rightarrow X_i \cap (D - P_\alpha) = \emptyset$. Hence $X_i \cap (D - P_i)(D - P_\alpha) = \emptyset$. For if $rs \in X_i$ where $r \notin P_i$ and $s \notin P_\alpha$, then $s \in X_{iP_i} \cap D = X_i$, so $s \in X_i \cap (D - P_\alpha)$, a contradiction. So we can enlarge X_i to a prime ideal Q with $Q \cap (D - P_i)(D - P_\alpha) \neq \emptyset$. But then $Q \subseteq P_i \cap P_\alpha$, a contradiction. Thus $(X_1 \dots X_n)^{*_S} = \bigcap_\alpha (X_1 \dots X_n) D_{P_\alpha} = X_1 \cap \dots \cap X_n = (x)$. Since $*_S$ has finite character, $(x) = (X_1 \dots X_n)_t$ where X_i is P_i-pure. ∎

While many of the results from the previous section may be extended to this more general setting, we only give extensions of two results, namely of Theorem 3.1 and of some characterizations of weakly factorial domains.

LEMMA 4.2. – *Let A be a t-invertible P-pure ideal where $P \in t\text{-Max}(D)$. Then A is a t-ideal. If $A \subseteq Q$ where $Q \in t\text{-Max}(D)$, then $P = Q$.*

PROOF. – By hypothesis $A_P \cap D = A$. Since A is t-invertible and $P \in t\text{-Max}(D)$, $A_P = (A_t)_P$. Hence $A = A_P \cap D(A_t)_P \cap D \supseteq A_t$. Hence A is a t-ideal, and therefore a v-ideal since A is t-finite. Suppose that $A \subseteq Q$ where $Q \in t\text{-Max}(D)$, with $P \neq Q$. Let $q \in Q - P$. Then $A \cap (q) = Aq$. Now

$$(A^{-1}, q^{-1})^{-1} = (A^{-1})^{-1} \cap (q) = A_v \cap (q) = A \cap (q) = Aq,$$

so $(A^{-1}, q^{-1})_t = (A^{-1}, q^{-1})_v = (Aq)^{-1} = A^{-1}q^{-1}$. Hence

$$D = (A^{-1}q^{-1}Aq)_t = ((A^{-1}, q^{-1})Aq)_t = (A^{-1}Aq, A)_t = \\ = ((A^{-1}Aq)_t, A)_t = (q, A)_t.$$

But $(q, A)_t \subseteq Q_t = Q$, a contradiction. ∎

THEOREM 4.3. – *For an integral domain D, the following statements are equivalent.*

(1) *$D = \bigcap_{P \in t\text{-Max}(D)} D_P$ has finite character and for P, $Q \in t\text{-Max}(D)$, $P \cap Q$ contains no nonzero prime ideal.*

(2) *Every proper principal ideal (x) is a finite t-product of P_α-pure ideals where $P_\alpha \in t\text{-Max}(D)$.*

(3) *For $P \in t\text{-Max}(D)$ and $0 \neq x \in D$, $xD_P \cap D$ is t-invertible.*

(4) *Every nonzero prime ideal Q contains a t-invertible P-pure ideal for some $P \in t\text{-Max}(D)$.*

PROOF. – (1) ⇒ (2). Theorem 4.1. (1) ⇒ (3). The proof of Theorem 4.1 shows that $(x) = ((xD_{P_1} \cap D) \dots (xD_{P_n} \cap D))_t$ where $P_1, \dots, P_n$ are the maximal t-ideals of D containing (x). Clearly each $xD_{P_i} \cap D$ is P_i-pure. (3) $\Rightarrow$ (4). Let Q be a nonzero prime ideal. Let $0 \neq x \in Q$. Shrink Q a prime ideal Q' minimal over (x). Then Q' is a t-ideal. Then $xD_P \cap D \subseteq xD_{Q'} \cap D' \subseteq Q$ and is by hypothesis t-invertible.

(2) $\Rightarrow$ (4). Let Q be a nonzero prime ideal. Let $0 \neq x \in Q$. Then $(x) = (A_1 \dots A_n)_t$ where A_i is P_i-pure for $P_i \in t\text{-Max}(D)$. Moreover, each A_i is t-invertible. Since Q is prime, $A_j \subseteq Q$ for some $j = 1, \dots, n$.

(4) $\Rightarrow$ (1). We always have $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. Let P, $Q \in t\text{-Max}(D)$ be distinct. Suppose that there exists a prime ideal $0 \neq N \subseteq P \cap Q$. Let $A \subseteq N$ be a P'-pure t-invertible ideal where $P' \in t\text{-Max}(D)$. Then P, $Q \supseteq A$, so by Lemma 4.2, $P = P' = Q$, a contradiction. It remains to show that $D = \bigcap_{P \in t\text{-Max}(D)} D_P$ has finite character. Let $\mathcal{S} = \{I$ is a finitely generated ideal $|$ there exists a finitely generated ideal J so that $(IJ)_t$ is a t-product of P_α-pure ideals where $P_\alpha \in t\text{-Max}(D)\}$. By hypothesis, $\mathcal{S} \neq \emptyset$. Certainly $\mathcal{S}$ is multiplicatively closed. Proceeding as in the proof of (5) $\Rightarrow$ (1) of Theorem 3.1, we see that each proper principal ideal (x) is in $\mathcal{S}$. So let $0 \neq x \in D$ be a nonunit. Then there exists a finitely generated ideal J so that $((x)J)_t = (A_1 \dots A_n)_t$ where A_i is P_i-pure for $P_i \in t\text{-Max}(D)$. Let Q be a maximal t-ideal containing x. Then $Q \supseteq ((x)J)_t \supseteq A_1 \dots A_n$, so $Q \supseteq A_j$ for some j. Then by Lemma 4.2, $Q = P_j$. Hence x is contained in only finitely many maximal t-ideal. ■

Unlike the case of Theorem 3.1 where we need not have $D = \bigcap_{P \in X^{(1)}} D_P$, we always have $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. Moreover, if D is Noetherian, then this intersection has finite character [13, Theorem 123].

Note that if A and B are t-invertible P-pure ideals where $P \in t\text{-Max}(D)$, then $(AB)_t$ is also a t-invertible P-pure ideal. For certainly $(AB)_t$ is t-invertible and $(AB)_t = \bigcap_{Q \in t\text{-Max}(D)} (AB)_t D_Q = (AB)_t D_P \cap D$. Thus in Theorem 4.3 (2) we can write $(x) = (A_1 \dots A_n)_t$ where A_i is a t-invertible P_i-pure ideal (and hence a t-ideal) and the P_i's are distinct maximal t-ideals. Note that A_i is uniquely determined, for $A_i = A_{iP_i} \cap D = A_1 \dots A_n D_{P_i} \cap D = xD_{P_i} \cap D$. We leave it to the reader to apply these comments to (2) of Corollary 4.4.

COROLLARY 4.4. – *For an integral domain D, the following conditions are equivalent.*

(1) *$D = \bigcap_{P \in t\text{-Max}(D)} D_P$ has finite character, for distinct $P, Q \in t\text{-Max}(D)$, $P \cap Q$ does not contain a nonzero prime ideal, and $\mathrm{Cl}_t(D) = 0$.*

(2) *Every nonzero nonunit $x \in D$ may be written in the form $x = x_1 \dots x_n$ where x_i is P_i-pure for $P_i \in t\text{-Max}(D)$.*

(3) *For each $x \in D$ and $P \in t\text{-Max}(D)$, $xD_P \cap D$ is principal.*

(4) *The natural map $G(D) \to \prod_{P \in t\text{-Max}(D)} G(D_P)$ has image $\bigoplus_{P \in t\text{-Max}(D)} G(D_P)$.*

PROOF. – (1) ⇒ (2). By Theorem 4.3, $(x) = (A_1 \dots A_n)_t$ where A_i is P_i-pure. Now each A_i is t-invertible, so by Lemma 4.2, each A_i is a t-ideal. Hence A_i is principal, say $A_i = (x_i)$. By modifying x_1 by a unit factor, if necessary, $x = x_1 \dots x_n$ where x_i is P_i-pure.

(2) ⇒ (3). If x is either zero or a unit, certainly $xD_P \cap D$ is principal. So write $x = x_1 \dots x_n$ where x_i is P_i-pure for $P_i \in t\text{-Max}(D)$. We may assume that the P_i's are distinct. Let $P \in t\text{-Max}(D)$. Note that by Lemma 4.2, $x \in P \Leftrightarrow P = \text{some } P_i$ and that in this case $x_i \in P_i$ while $x_i \notin P_j$ for $j \neq i$. If $x \notin P$, then $xD_P \cap D = D$ is principal. While if $P = P_i$, then $xD_{P_i} \cap D = x_i D_{P_i} \cap D = x_i D$ is principal.

(3) ⇒ (1). By Theorem 4.3 we only need that $\mathrm{Cl}_t(D) = 0$. Let A be a t-invertible t-ideal. We must show that A is principal. We may assume that A is integral. Now as in the proof of Theorem 4.1, $A = ((A_{P_1} \cap D) \dots (A_{P_n} \cap D))_t$ where $P_1, \dots, P_n$ are the maximal t-ideals containing A. But $P_i \in t\text{-Max}(D)$ and A is t-invertible, so A_{P_i} is principal, say $A_{P_i} = a_i D_{P_i}$. But then by hypothesis $a_i D_{P_i} \cap D = A_{P_i} \cap D$ is principal. Hence A is principal.

(4) ⇒ (3). Let $P_0 \in t\text{-Max}(D)$ and $0 \neq x \in D$. There exists a $0 \neq y \in D$ with $yD_{P_0} = xD_{P_0}$ and $yD_Q = 1D_Q$ for $Q \in t\text{-Max}(D) - \{P_0\}$. Then $xD_{P_0} \cap D = \bigcap_{P \in t\text{-Max}(D)} yD_P = yD$.

(1)-(3) ⇒ (4). Since $D = \bigcap_{P \in t\text{-Max}(D)} D_P$ has finite character, $\operatorname{im}\varphi \subseteq \bigoplus_{P \in t\text{-Max}(D)} G(D_P)$ where $\varphi\colon G(D) \to \prod_{P \in t\text{-Max}(D)} G(D_P)$, $xD \to (xD_P)$ is the natural map. To see that $\operatorname{im}\varphi = \bigoplus_{P \in t\text{-Max}(D)} G(D_P)$ it suffices to note that if $xD_{P_0} \cap D = yD$, then $yD_{P_0} = xD_{P_0}$ while $yD_Q = 1D_Q$ for $Q \in t\text{-Max}(D) - \{P_0\}$. ∎

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