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ASCENDING CHAIN CONDITIONS AND STAR OPERATIONS

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One aim of this note is to study ACC on the integral $*$-ideals for a given star operation $*$ on an integral domain D. We show that if $*$ is a star operation on an integral domain D then D satisfies ACC on integral $*$-ideals if and only if for every non-zero fractional ideal A of D, there exists a finite set of elements $x_1,\ldots,x_n \in A$ such that $(x_1,\ldots,x_n)^* = A^*$. This result is then used to offer some characterizations of Krull domains.

Throughout this note we shall use D to denote a commutative integral domain with quotient field K. Let F(D) be the set of non-zero fractional ideals of D. A function $*: F(D) \longrightarrow F(D)$, described as $A \longrightarrow A^*$, is called a <u>star operation</u> if for all $A,B \in F(D)$ and for all $a \in K - \{0\}$

(i) $(a)^* = (a)$ and $(aA)^* = aA^*$,

(ii) $A \subseteq A^*$ and $A \subseteq B$ implies $A^* \subseteq B^*$, and (iii) $(A^*)^* = A^*$.

For details on star operations the reader may consult Sections 32 and 34 of [5], to which we shall refer for specific results.

Yet to make the reading smoother we include some introduction in the following.

Let $*$ be a star operation on D. An ideal $A \in F(D)$ is called a <u>$*$-ideal</u> if $A = A^*$ and A is called a <u>$*$-ideal of finite type</u> if there is a finitely generated $B \in F(D)$ such that $A = B^*$. We call $A \in F(D)$, <u>strictly $*$-finite</u> if there exists a finitely generated $B \in F(D)$ with $B \subseteq A$ and $B^* = A^*$. The function $A \longrightarrow A_v = (A^{-1})^{-1}$, on F(D), is for instance a star operation. We note that a v-ideal is a $*$-ideal for every star operation and that the inverse of each $A \in F(D)$ is a v-ideal. Associated to each star operation $*$, is another star operation $*_s$ defined by $A^{*_s} = \cup F^*$ where F ranges over all finitely generated D-submodules of A. The v_s-operation is called the <u>t-operation</u>. Clearly for each star operation $*$ we have $A^{*_s} \subseteq A^*$; for all $A \in F(D)$. If $*_s = *$, $*$ is said to be of <u>finite character</u>. Further if $*$ is a star operation on D and $A, B \in F(D)$ then $(AB)^* = (A^*B)^* = (A^*B^*)^*$. Moreover if $*_1, *_2$ are two star operations on D such that , for all $A \in F(D)$, $A^{*_1} \subseteq A^{*_2}$ then $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. We also note that $A^* \subseteq A_v$ for any star operation $*$ on D and for any $A \in F(D)$. An ideal $A \in F(D)$ is said to be <u>$*$-invertible</u> if for some $B \in F(D)$, $(AB)^* = D$. In this case it can be easily established that $B^* = A^{-1}$ and if $*$ is of finite character then A and B are strictly $*$-finite. Indeed it is obvious that if B is $*$ invertible then $B^* = B_v$. Given a star operation $*$ of finite character, every proper integral $*$-ideal is contained in a proper integral $*$-ideal maximal w.r.t. being a $*$-ideal. A maximal $*$-ideal is a prime ideal. An integral domain D is called a Mori domain if D satisfies ACC on integral v-ideals. Let us call D, $*$-Noetherian if D satisfies ACC on integral $*$-ideals for a given star operation $*$. Then, because a v-ideal is a $*$-ideal for every star operation $*$, a $*$-Noetherian domain is a Mori domain. We show that (1) D is $*$-Noetherian for some star operation $*$ if, and only if, each $A \in F(D)$ is strictly $*$-finite. Recall that D is a completely integrally closed domain if and only if for all $A \in F(D)$, $(AA^{-1})_v = D$ [5, Th. 34.3]. In order to keep introductions to a minimum we call D a v-domain if for all finitely generated $A \in F(D)$, $(AA^{-1})_v = D$ [5, Th. 34.6]. Obviously a completely integrally closed integral domain is a v-domain. It is well known that a completely integrally closed integral domain is a Krull domain if and only if it is Mori (see for example [4, pp. 12-16]) or equivalently, a Mori domain is Krull if, and only if, it is completely integrally closed. Using (1) we indicate (2) some other conditions under which a Mori domain becomes Krull. We do this by showing that (3) if D has the property that, for all $A \in F(D)$, A_v is of finite type then the following are equivalent for D: (a) D is completely integrally closed (b) D is a v-domain (c) D is an essential domain (to be defined later) and (d) D is a PVMD (to be defined later). We also prove that (4) D is a Krull domain if, and only if, for some star operation $*$, of finite character every $A \in F(D)$ is $*$-invertible. This improves on [6, Theorem 0] which says that D is Krull if, and only if, each $A \in F(D)$ is t-invertible.

This note is split into four small sections. In the first section we introduce strictly $*$-finite ideals; with reference to the ascending chain conditions. In the second section we indicate conditions (other than D being completely integrally closed) which make a Mori domain into Krull. In the third section we characterize Krull domains using star operations. Finally in the fourth section we characterize integral domains D such that for all $A \in F(D)$, A_v is t-invertible, and indicate examples.

1. Strictly *-finite ideals.

From the introduction it is apparent that strictly *-finite ideals are going to play a major role in this work. So it is important (and I thank the referee for pointing this out) that we should have an idea of where the strictly * finite ideals stand. It is easy to see that a *-ideal of finite type is strictly *-finite and that if $A \in F(D)$ is strictly *-finite then A^* is a *-ideal of finite type but as the following example indicates there may be ideals $A \in F(D)$ for some *-operation * such that A is not strictly *-finite while A^* is a *-ideal of finite type. This happens quite often in the case of $* = v$. To indicate this we use the example suggested by the referee. Let (V,M) be a non-discrete rank one valuation domain. Then $M_v = V$, so M_v is a v-ideal of finite type but M, being the only non-zero prime ideal of V, is a maximal t-ideal and so for all finitely generated $A \subseteq M$; $A_v \subseteq M \neq V$.

To establish our result on *-Noetherian domains we prove the following improved form of Theorem 1.1 of [2].

THEOREM 1.1. Let D be an integral domain and let * be a star operation on D. Then the following statements are equivalent.

(1) Each integral $*_s$-ideal is (a $*_s$-ideal) of finite type.

(2) D satisfies ACC on $*_s$-ideals.

(3) D satisfies ACC on integral star ideals.

(4) Every $A \in F(D)$ is strictly $*$-finite.

Moreover if any of the above equivalent statements hold then $* = *_s$.

PROOF. The equivalence of (1),(2) and (3) is Theorem 1.1 of [2]. To complete the proof we show (1) $\Longleftrightarrow$ (4).

(1) $\Longrightarrow$ (4). Let $I^{*_s} = J^{*_s}$ where J is finitely generated. Then, obviously, $I^* = J^*$. Now because J is finitely generated, $J^*=J^{*_s}$ and so $I^* = I^{*_s}$. All that remains is to show that there exists a finitely generated $K \subseteq I$ such that $K^* = I^*$. For this let $J=(b_1,b_2,\ldots,b_n)$. Then $J \subseteq J^{*_s} = I^{*_s} = \cup F^*$ where F ranges over finitely generated D-submodules of I. Obviously for each i there exists $F_i \subseteq I$, F_i finitely generated such that $b_i \in F_i^*$. Consequently $J \subseteq (F_1^*+ F_2^*+\ldots+ F_n^*)$ and so $J^* \subseteq (F_1^* + F_2^* + \ldots + F_n^*)^*$ $= (F_1+ \ldots +F_n)^*$ ([5, Prop. 32.2]) $\subseteq I^*=J^*$. Thus we have $K = F_1+F_2+\ldots+F_n \subseteq I$ such that $K^* = I^*$.

(4) $\Longrightarrow$ (1). Let $J \subseteq I$, with J finitely generated, be such that $J^*=I^*$. Then $I^*=J^*=J^{*_s} \subseteq I^{*_s} \subseteq I^*$, so $I^{*_s} = I^*$.

COROLLARY 1.2. (See [7, Th. 1]). An integral domain D is a Mori domain if and only if $A \in F(D)$ is strictly v-finite.

REMARK 1.3. Querre uses the term <u>quasi fini</u> for strictly v-finite fractional ideals. But obviously, this term is inadequate for our purposes.

2. From Mori to Krull.

In this section we present other conditions which are necessary and sufficient for a Mori domain to be Krull. For this we recall that D is a Prufer v-multiplication domain (PVMD) if for all finitely generated $A \in F(D)$, A is v-invertible and A^{-1} is a v-ideal of finite type [5, p. 427]. Obviously a PVMD is a v-domain. We also recall that D is essential if D has a family $\{P_i\}_{i \in I}$ of prime ideals such that $D = \cap D_{P_i}$ and for each $i \in I$, D_{P_i} is a valuation domain. Slightly less obviously, an essential domain is also a v-domain. To provide an easy and straightforward proof of this fact we note that the operation $A \longrightarrow A^w = \cap AD_{P_i}$ on the essential domain described above is a star operation [5, Th. 32.5]. Now if $A \in F(D)$ is finitely generated then $(AA^{-1})^w = \cap(AA^{-1})D_{P_i} = \cap AD_{P_i}A^{-1}D_{P_i} = \cap AD_{P_i}(AD_{P_i})^{-1}$ (since A is finitely generated) $= \cap D_{P_i} = D$.

It is well known that a Krull domain is a PVMD and that a PVMD is essential. Now we are ready for our next result.

PROPOSITION 2.1. Let D be such that for all $A \in F(D)$, A_v is of finite type. Then the following are equivalent.

(1) D is completely integrally closed.

(2) D is a v-domain.

(3) D is a PVMD.

(4) D is an essential domain.

PROOF. We have already noticed that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4). (2) $\Rightarrow$ (3). Let $A \in F(D)$ be finitely generated. Then, because A^{-1} is a v-ideal, by the condition on D, A^{-1} is of finite

type. Now by (2) for all finitely generated $A \in F(D)$, $(AA^{-1})_v = D$.
(4) $\Rightarrow$ (1). Let $A \in F(D)$ and consider $W = (AA^{-1})_v$. Then $W = (A_vA^{-1})_v$. But by the main condition on D, $A_v = B_v$ where B is finitely generated. Using the fact that A^{-1} is a v-ideal we have $A^{-1} = (A_v)^{-1} = (B_v)^{-1} = B^{-1}$. So $W = (B_vB^{-1})_v$. But as, by (4), D is essential and as B is finitely generated we have $W = (BB^{-1})_v = D$. Thus for all $A \in F(D)$, $(AA^{-1})_v = D$ and this is equivalent to saying that D is completely integrally closed.

Noting that a Mori domain meets the main condition of Proposition 2.1, and recalling that a Mori domain is Krull if and only if it is completely integrally closed we state the following result.

COROLLARY 2.2. Let D be a Mori domain. Then the following are equivalent.

(1) D is completely integrally closed.

(2) D is a Krull domain.

(3) D is a v-domain.

(4) D is a PVMD.

(5) D is essential.

An integral domain D is called a <u>finite conductor domain</u> if for all $a,b \in D$, $aD \cap bD$ is finitely generated. Part (b) of Theorem 4.2 of [3] states that a finite conductor domain is an integrally closed Mori domain if and only if it is Krull. The only if part of this result can, in view of Corollary 2.2, be proved as follows: It is well known that an integrally closed finite conductor domain is a PVMD [8] and by Corollary 2.2, a PVMD which is also Mori is Krull.

3. Characterizations of Krull domains.

THEOREM 3.1. The following are equivalent for D.

(1) There exists a star operation $*$ on D such that (a) every $A \in F(D)$ is $*$-invertible and (b) D satisfies ACC on integral $*$-ideals.

(2) There exists a star operation $*$ of finite character such that every $A \in F(D)$ is $*$-invertible.

(3) D is a Krull domain.

PROOF. (3) $\Longleftrightarrow$ (4) is Theorem 0 of [6]. To complete the proof we prove the equivalence of (1), (2) and (3).

(1) $\Longrightarrow$ (2). According to Theorem 1.1, (1)(b) implies that $*=*_s$. If, in addition, we invoke (1)(a) we get (2).

(2) $\Longrightarrow$ (3). It is easy to see that if $*$ is of finite character then for all $A \in F(D)$ $A^* \subseteq A_t$. Now, by (2), for all $A \in F(D)$, $D = (AA^{-1})^* \subseteq (AA^{-1})_t \subseteq D$.

4. Pre-Krull domains.

It is natural to ask, "What is the difference between Proposition 2.1 and Corollary 2.2?" The difference is indeed that of a v-ideal of finite type and a strictly v-finite ideal. Indeed there do exist integral domains which satisfy all the requirements of Proposition 2.1, along with the equivalent conditions, and which are not Krull. The known examples of these integral domains are the generalized Dedekind domains of [9]. These are integral domains D such that for all $A \in F(D)$, A_v is invertible. The ring of entire functions is an example of a generalized Dedekind domain.

But these are all what were called generalized GCD-domains in [1] i.e. integral domains D such that for each finitely generated $A \in F(D)$, A_v is invertible. On the other hand Proposition 2.1 points to completely integrally closed PVMD's, D such that for each $A \in F(D)$, A_v is t-invertible and not necessarily invertible. Let us call an integral domain D a pre-Krull domain if for each $A \in F(D)$, A_v is t-invertible. Then obviously a Krull domain is pre-Krull. The pre-Krull domains may be characterized as follows.

PROPOSITION 4.1. The following are equivalent for D.

(1) For all $A,B \in F(D)$, $(AB)^{-1} = (A^{-1}B^{-1})_t$.

(2) For all $A \in F(D)$, A^{-1} is t-invertible.

(3) For all $A \in F(D)$, A_v is t-invertible.

(4) D is a v-domain and for all $A \in F(D)$, A_v is of finite type.

(5) D is completely integrally closed and for all $A,B \in F(D)$ $(AB)_v = (A_vB_v)_t$.

PROOF. (1) $\Longrightarrow$ (2). Obviously for all $A \in F(D)$, $A_vA^{-1} \subseteq D$ and so $D \subseteq (A_vA^{-1})^{-1}$. Using (1) we have $D \subseteq (A_vA^{-1})^{-1} = (A_v^{-1}(A^{-1})^{-1})_t = (A^{-1}A_v)_t \subseteq D$.

(2) $\Longrightarrow$ (3). Because $A_v = (A^{-1})^{-1}$, A^{-1} t-invertible implies A_v is t-invertible.

(3) $\Longrightarrow$ (4). Obviously as t-invertible v-ideals are of finite type, we have A_v of finite type for all $A \in F(D)$. Further as $(A_vA^{-1})_t = D$ for all $A \in F(D)$ we have $(A_vA^{-1})_v = D$ and from this it follows that D is completely integrally closed. Obviously a completely integrally closed integral domain is a v-domain.

(4) $\Longrightarrow$ (5). Let $A \in F(D)$. Then as A_v is of finite type and as D is a v-domain we have $D = (A_vA^{-1})_v = (AA^{-1})_v$. Now A being

arbitrary we conclude that D is completely integrally closed. To show that for all $A,B \in F(D)$; $(AB)_v = (A_vB_v)_t$ we note that $A_v = \underline{A}_v$ and $B_v = \underline{B}_v$ where $\underline{A},\underline{B}$ are finitely generated. So $(AB)_v = (A_vB_v)_v = (\underline{A}_v\underline{B}_v)_v = (\underline{AB})_v = (\underline{AB})_t$ (since $\underline{A}$ and $\underline{B}$ are finitely generated) = $(\underline{A}_t\underline{B}_t)_t = (\underline{A}_v\underline{B}_v)_t$.

(5) $\Rightarrow$ (1). Suppose that D is completely integrally closed and that for all $A,B \in F(D)$, $(AB)_v = (A_vB_v)_t$. Then
$D = ((AB)(AB)^{-1})_v = ((AB)_v(AB)^{-1})_t = ((A_vB_v)_t(AB)^{-1})_t$
$= (A_vB_v(AB)^{-1})_t$. Now multiplying both sides by $A^{-1}B^{-1}$ and applying the t-operation we have
$(A^{-1}B^{-1})_t = (A^{-1}B^{-1}A_vB_v(AB)^{-1})_t = ((A^{-1}A_v)(B^{-1}B_v)(AB)^{-1})_t$
$= ((A^{-1}A_v)(B^{-1}B_v)(AB)^{-1})_t$ (since A^{-1} and A_v are both of finite type) $= ((AB)^{-1})_t = (AB)^{-1}$.

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REFERENCES

[1] D.D. Anderson and D.F. Anderson, Generalized GCD-domains, Comment. Math. Univ. St. Pauli, 23(1979), 213-221.

[2] D.D. Anderson and D.F. Anderson, Some remarks on star operations and the class group, (to appear in J. Pure and Appl. Algebra).

[3] V. Barucci, D.F. Anderson and D. Dobbs, Coherent Mori domains and the principal ideal theorem, Comm. Algebra 15(6)(1987), 1119-1156.

[4] R. Fossum, The Divisor Class Group of a Krull Domain, Springer-Verlag, New York(1973).

[5] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York(1972).

[6] J. Mott and M. Zafrullah, On Krull domains, (pre-print).

[7] J. Querre, Sur une propriete des anneaux de Krull, Bull. Sc. Math. 2^c serie 95(1971), 341-354.

[8] M. Zafrullah, On finite conductor domains, Manuscripta Math., 24(1978), 191-203.

[9] M. Zafrullah, Generalized Dedekind domains, Mathematika, 33(1986), 285-295.

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