Various Facets of Rings between D[X] and K[X]

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ABSTRACT

Since the circulation, in 1974, of the first draft of "The construction $D + XD_S[X]$, J. Algebra 53 (1978), 423-439" a number of variations of this construction have appeared. Some of these are: The generalized $D+M$ construction, the $A+(X)B[X]$ construction, with X a single variable or a set of variables, and the $D+I$ construction (with I not necessarily prime). These constructions have proved their worth not only in providing numerous examples and counter examples in commutative ring theory, but also in providing statements that often turn out to be forerunners of results on general pullbacks. The aim of this paper will be to discuss these constructions and the remarkable uses they have been put to. I will concentrate more on the $A + XB[X]$ construction, its basic properties and examples arising from it.

0. Introduction

Let A be a subring of an integral domain B, and let X be an indeterminate over B. The set $\{f(X) \in B[X] : f(0) \in A\}$ is a ring denoted by $A + XB[X]$. This article is an attempt at a survey of the polynomial ring constructions of the form $A + XB[X]$ that have come into vogue in recent years. The article consists of a slightly modified version of the talk that I gave at the Fez Conference held in the year 2001 and a number of appendices or supplements. In the talk, I briefly surveyed the history of the $A + XB[X]$ construction and the various constructions that seem to have risen from similar considerations. The talk is the first part of the article. In the appendices, I study topics that are either essential to the understanding of the $A+XB[X]$ construction or are ones that give rise to examples that, in my opinion, are useful. In the first appendix, which is part 2, I study the prime ideal structure of $A + XB[X]$ construction. Then in part 3, I indicate how the study of Spec$(A + XB[X])$ led to the construction of various examples and in part 4, I indicate how useful examples can be constructed from the $D + XD_S[X]$ construction. Part 5 is a wish list, i.e., I briefly go over topics that I hoped to write on but could not because that will make the article a bit too long. The necessary terminology is explained where needed, and any terminology that has not been explained can be found either in Gilmer [Gil72] and/or Kaplansky [Kap70].

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1. Part 1 (the Talk)

Let D be an integral domain with quotient field F and let K be a field extension of F. We know a great deal about $D[X]$ and almost everything about $K[X]$, where X is an indeterminate over K. About rings between $D[X]$ and $K[X]$ we have just begun to learn. Let us call the rings between $D[X]$ and $K[X]$ the intermediate rings. At present we can split these rings into two main types: (1) Intermediate rings that are composite, i.e. are of the form $A+XB[X] = \{a_0+\sum_{i=1}^{n} a_iX^i | \; a_0 \in A$ and $a_i \in B\}$ where $D \subseteq A \subseteq B \subseteq K$ are ring extensions, and (2) The intermediate rings that are not of the form $A + XB[X]$.

The only, very, well known rings of the latter kind are the rings of integer-valued polynomials. These rings are very well known and the best I can do is refer the readers to the wonderful book by Cahen and Chabert [CC 97]. There are other less well known though equally important examples, of such rings due to Eakin and Heinzer [EH73]. Indeed rings that are not of the form $A+XB[X]$ have a composite cover as indicated by D.D. Anderson, D.F. Anderson and myself in [AAZ91]. This composite cover, very often, determines some of the properties of these rings.

If you have not got the drift yet, then let me tell you, I intend to spend more time on the composites, i.e., rings of the form $A+XB[X]$. My reasons for choosing this course of action are the following:

(a) The composites, over the past few years, have provided directly constructible examples of rings which were once very hard to construct.

(b) The composites have given rise to new notions and new constructions which make it easier to bring in new concepts and study them.

(c) The composites are pullbacks and it has become customary to prove a statement for a special kind of a composite, then for a general composite, and then for a general pullback. Consequently the appreciation of pullbacks has increased.

My plan is essentially to give a brief description of what it (i.e. the $A+XB[X]$ construction) is, then beef about what it does, and then indicate several of the variants of this construction that have appeared recently. Recently, Tom Lucas [Luc00] has written a survey on examples of pullbacks using the $A + XB[X]$ construction. I will try not to repeat those examples and will cover the material that Tom left out because of space restrictions.

Basic Properties of the intermediate rings

Given a ring R between $D[X]$ and $K[X]$, we can split R into two parts as follows:

$M_R = \{f(x) \in R| \; f(0) = 0\}$ and $S_R = \{f(X) \in R| \; f(0) \neq 0\}$. (Let $S_R(0) = \{f(0) : f \in S_R$.) Of these M_R is a prime ideal and S_R is a multiplicative set with the property that $S_R(0) \cup \{0\} = \{f(0)| \; f(x) \in R\} = R_0$. Now R_0 is a subring of K, though R_0 may or may not be a subring of R, but there are some direct observations that can be made:

(i) The map $\pi : R \longrightarrow R_0$ defined by $\pi(f) = f(0)$ is a ring epimorphism with $\ker \pi = M_R$. Thus $R_0 \approxeq R/M_R$.

(ii) Every unit of R is a unit of R_0, but the converse may not hold. ($Q[X] \subseteq R = Q[X][\sqrt{2}X + \sqrt{3}] \subset \mathcal{R}[X]$, where Q is the rationals and $\mathcal{R}$ is the reals. Clearly $R_0 = Q[\sqrt{3}]$.)

(iii) If $R_0 \subseteq R$ then $R = R_0 + M_R$. Using the fact that if B is a ring I an ideal of B and A a subring of B then $A + I$ is a subring of B, we can construct for each subring A of $R \cap R_0$ a subring $A + M_R$ of R. Now as $R \supseteq D[X]$, we have $XA[X] \subseteq M_R$, and so $A + M_R$ contains $D[X]$, whenever A contains D.

(iv) If it so happens that $R_0 \subseteq R$ and $M_R = XR_1[X]$ for some R_0-subalgebra R_1 of K, then we have $R = R_0 + XR_1[X]$ a ring of the form $A + XB[X]$. Indeed, $M_R = XR_1[X]$ for some R_1 if and only if $\sum_{i=1}^{n} a_i X^i \in R$ implies $a_i X \in R$ for all i such that $1 \leq i \leq n$. In this case, $R_1 = \{a | \ aX \in R\}$ [AAZ91].

(v) Now we have seen that $R_0 \approxeq R/M_R$. For some of us it is grounds enough to set up, for each subring R' of R_0, the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} S = \pi^{-1}(R') & \hookrightarrow & R \\ \downarrow & & \downarrow_{\pi} \\ R' & \hookrightarrow & R/M_R \end{array} .$$

In the usual terminology S is called a pullback. Indeed, if $R' \subseteq R$, then $S = R' + M_R$. Thus the composites $A + XB[X]$ are pullbacks. Yet, while the pullback would give you $\pi^{-1}(R_0) = R$, the composites will go a step further. By [AAZ91, Cor. 2.4] associated with an intermediate ring R, there is a unique composite $S = R_0 + XS_1[X]$, where S_1 is a subring of K generated by $\bigcup_{f \in R} A_f$ and A_f is the D-submodule of K generated by the coefficients of f. Let us call this unique composite the composite cover of R.

Now, what is so special about the composite cover of an intermediate ring R? For one thing, R is integral over $D[X]$ if and only if the composite cover of R is integral over $D[X]$ ([AAZ91, Prop. 2.6]). On its own, $A + XB[X]$ is integrally closed if and only if B is integrally closed and A is integrally closed in B.

(vi) The name composite fits because $A + XB[X]$ is the composite of A and $B[X]$ over the ideal $XB[X]$. Mott and Schexnayder's paper [MS76] gives a good description of composites of several kinds.

Special types of Composites.

Enough of the basic properties. Let us now see the different types of composites. The current wave of study of composites started with the circulation of an earlier version of my paper with Costa and Mott [CMZ78]. The construction to be studied was given in the title,"The construction $D + XD_S[X]$", where D is an integral domain of your choice, S a multiplicative set in D and X an indeterminate. The immediate special case: the then well known $D + XK[X]$, where K is the quotient field of D, which I was using to produce examples of GCD domains each proper principal ideal of which has finitely many minimal primes. I called these GCD domains the unique representation domains (URD's). Indeed, if D is a URD, then so is $D + XK[X]$. At Paul Cohn's suggestion I started looking into $D^{(S)} = D + XD_S[X]$. It turned out that $D + XD_S[X]$ is a GCD domain if and only if D is a GCD domain and for each $d \in D$, GCD(d, X) exists [CMZ78]. It turned out also that if $D^{(S)}$ is a GCD domain then $D^{(S)}$ is a URD iff D is [Z78]. Of interest in a GCD domain D are the PF-primes, say the primes P such that D_P is a valuation domain. The PF dimension of a GCD domain can be defined in the same way as the Krull dimension or the valuative dimension of a domain is defined. Sheldon [Sh74] had studied the PF primes and had conjectured that a GCD domain D with PF-dim(D) = Krull-dim(D) should be Bézout (f.g. ideals are principal).

Next, as $D^{(S)}$ is a polynomial ring construction, it was natural to ask questions about Krull-dim$(D^{(S)})$ with reference to Krull-dim(D); it meant a study of

Spec($D^{(S)}$). I found out that if P is a prime ideal of $D^{(S)}$ with $P \cap S \neq \phi$, then $P = P \cap D + XD_S[X]$, and if $P \cap S = \phi$, then $P = PD_S[X] \cap D^{(S)}$.

Adding a little bit here and a little bit there, my paper was complete. When I was writing my thesis, Paul (Cohn) told me to send whatever I produced to Robert Gilmer, who was and of course is a leading expert in Multiplicative Ideal Theory. So, out of habit, I mailed a copy to Robert who had indeed been very kind to me. This way I got in contact with Joe Mott and Doug Costa who had done some similar things. Between us, we added results including results such as: If D is Noetherian, then $D^{(S)}$ is a coherent ring; $D + XK[X]$ is a PVMD if and only if D is a PVMD. Now this needs a little bit of introduction. A function $*$ on the set $F(D)$ of nonzero fractional ideals is said to be a star operation if for all $a \in K\backslash\{0\}, A, B \in F(D)$, we have (1) $(a)^* = (a)$, $(aA)^* = aA^*$ (2) $A \subseteq A^*$ and $A \subseteq B \Rightarrow A^* \subseteq B^*$ and (3) $(A^*)^* = A^*$. If $*$ is a star operation, we can also define $*$-multiplication as $A \times_* B = (AB)^* = (A^*B)^* = (A^*B^*)^*$. An ideal $A \in F(D)$ is a $*$-ideal if $A = A^*$. The operation defined, on $F(D)$, by $A \longrightarrow A_v = (A^{-1})^{-1}$ is called the v-operation, and the one defined by $A_t = \bigcup F_v$, where F ranges over nonzero finitely generated subideals of A, is called the t-operation. Now D is a PVMD if for all finitely generated $A \in F(D)$, A is t-invertible, i.e., $(AA^{-1})_t = D$.

The message of [CMZ78], like any other construction involving two rings, was: see how, and under what conditions, some properties of D get transferred to $D^{(S)}$. This brought up the question: If D is a PVMD, for what S should $D^{(S)}$ be a PVMD? That meant knowing all the maximal t-ideals of $D^{(S)}$. An integral ideal P that is maximal w.r.t. being a t-ideal is a prime ideal, and Griffin [Gri67] had shown that D is a PVMD if and only if for each maximal t-ideal P of D, D_P is a valuation domain. The main hurdle was that the prime ideals of $D^{(S)}$ that are disjoint with S are contractions of prime ideals of $D_S[X]$. To see how this problem could be resolved, I decided to study the $D + XD_S[X]$ construction from GCD domains [Z88]. (A GCD domain is a PVMD.) It did not give me what I wanted, but it brought simple examples of locally GCD domains that were not GCD, and the fact that if P is a prime t-ideal of D then PD_P may not be a t-ideal of D_P. I also discovered that if D is GCD, then $D^{(S)}$ is GCD if and only if S is a splitting set, i.e., each $d \in D\backslash\{0\}$ is expressible as $d = sd_1$, where $s \in S$ and $(d_1) \cap (t) = (td_1)$ for all $t \in S$. This rediscovery of the splitting sets of [MS76] led not only to a lot of activity from the factorization point of view, but also to the ultimate solution of my problem. The solution turned out to be: If D is a PVMD, $D^{(S)}$ is a PVMD if and only if S is a t-splitting set of D. Now, a saturated multiplicative set is a t-splitting set if for every element $d \in D\backslash\{0\}$ we have $(d) = (AB)_t$, where A and B are integral t-ideals such that $A \cap S \neq \phi$ and $B \cap (t) = Bt$ for all $t \in S$. This result has appeared in a paper of mine with the Anderson brothers [AAZ01].

Apparently the contents of [CMZ78] had started taking effect before it was published. Brewer and Rutter[BR76] came up with the idea: if $R = k + M$, where k is a field, then for every subring D of k you have a subring $D + M$ of R. They called it the generalized $D + M$ construction, generalized because it generalized the celebrated $D + M$ construction, greatly popularized by Gilmer, which required $R = k + M$ to be a valuation domain. Using this construction they were able to recover all that was proved about $D + XK[X]$ in [CMZ78], here $K = qf(D)$ and, on top of that, their construction allowed results on subrings of $k[[X]]$, where k is a field. While this was happening, Malik and Mott [MM83] had come up with their study of strong S-domains. D is a S(eidenberg) domain if for each height one prime P of D, $PD[X]$ is of height one, and D is strong S if for each prime P, D/P is an S-domain. They showed that the $D + XK[X]$ construction is a strong S-ring if and only if D is; yet they pointed out that if $D[X]$ is strong S, it is not necessary that $(D + M)[X]$ should be strong S. Now, [BR76] had encouraged me to go general. I wrote up a piece on the overrings of $D + XL[X]$, where L is an extension of the quotient field of D. Then Costa and Mott gave it the language of generalized $D + M$, and we had another paper [CMZ 86]. (Later Joe and I [MZ 90] showed that if D is a Noetherian Hilbert domain and L is an extension of the quotient field of D, then $D + XL[X]$ is a non-Noetherian Hilbert domain whose maximal ideals are finitely generated. Constructing such a domain by conventional means was quite difficult, as shown by Gilmer and Heinzer [GH76].)

The beauty of the $D + XD_S[X]$ construction seems to lie in the fact that it is so close to well known examples of pullbacks and composites, yet so open to reinterpretation and so easy to work with. Marco Fontana wrote a longish article in Italian [F79] and sent a copy to me in Libya. (Marco tells me that those were his seminar notes.) He had treated all the constructions and composites that I have talked about above with reference to [MS76], including $D^{(S)}$, showing how the spectral space of a pullback can be shown to be connected with the spectral spaces of the constituents of the pullback. I think that Marco's interest in the $D+XD_S[X]$ construction had a profound effect on the development of polynomial ring constructions.

In trying to get some examples in a completely different context, I had found out that if $k \subseteq K$ is an extension of fields and if X is an indeterminate over K, then $k + XK[X]$ satisfies ACC on principal ideals [C89]. When, in 1986, I went to Lyon, (France) I gave, among other talks, a talk on an earlier version of [AAZ91]. There I met several young men, Salah-Eddine Kabbaj included, who were eager to learn and ready to experiment with new ideas and techniques. From these young men issued forth a barrage of papers containing all sorts of variations of $A + XB[X]$ construction. Strong S-domains and Jaffard domains were in vogue. Jaffard domains are domains D such that valuative $\dim(D) =$ Krull $\dim(D)$ (symbolically $(\dim_v(D) = \text{K-}\dim(D))$. Anderson, Bouvier, Dobbs, Fontana and Kabbaj wrote papers, [ABDFK88] and [BK88]. [ABDFK88], using various pullbacks to construct examples, and [BK88] showing that if D is Jaffard, then so is $D + XD_S[X]$. Then Fontana and Kabbaj [FK90] studied the Krull and valuative dimensions of $D + (X_1, X_2,.., X_n)D_S[X_1, X_2,.., X_n] = D^{(S,n)}$. It turned out that $\dim_v(D^{(S,n)}) = \dim_v(D) + n$, and that D is a Jaffard domain if and only if so is $D^{(S,n)}$. Next, they prove that $D^{(S,n)}$ is a strong S ring if and only if both D and $D_S[X_1, X_2,.., X_n]$ are. To top it all, they showed that $D[X]$ is an S-domain for any D. Later, Fontana, Izelgue, and Kabbaj [FIK92] studied the Krull and valuative dimensions of the $A + XB[X]$ construction and showed that the results are different, especially when the quotient field of B is a proper extension of the quotient field of A. Recently, Anderson and Nour-el-Abidine [AN01] have studied the $A+XB[X]$ and $A+XB[[X]]$ constructions from GCD domains.

Current trends

The sole purpose of studying the $D+XD_S[X]$ construction was to get examples of domains that did not satisfy ACC on principal ideals. But we could not ignore the possibility of ACCP holding for an intermediate ring. In [AAZ91], we came up with: Let R be an intermediate ring, then R has ACCP if and only if any ascending chain of principal ideals generated by polynomials of R, of the same degree, terminates. As a demonstration of this, we proved a proposition for $D[X] \subseteq R \subseteq K[X]$, where K is the quotient field of D. Of course we thrashed the case of $A+XB[X]$ for A and B fields, but Barucci, Izelgue, and Kabbaj [BIK95] came up with the somewhat remarkable discovery that if A is a field then $A+XB[X]$ has ACCP no matter what kind of integral domain B is. (I recall having written a good review of this paper but, apparently, what I sent was hard to understand, for reasons I am trying to explain to myself. Possibly some part of the review got deleted!) This remarkable short note had some other gems that started off a lot of activity in the study of factorization properties of $A+XB[X]$ and $A+XB[[X]]$ constructions. The names to mention in this connection are Nathalie Gonzalez [Go99] and [Gon99] [GPR01] , David Anderson and Nour-el-Abidine [AN99], Dumitrescu, Radu, Salihi and Shah [DSRS00].

Let $T(D)$ denote the set of t-invertible t-ideals of D. Then $T(D)$ is a group under t-product and when we quotient it by its subgroup $P(D)$ of principal fractional ideals we get what I call the t-class group $Cl_t(D) = T(D)/P(D)$. This class group was introduced by Bouvier in [B82]. Anderson and Ryckaert [AR88] studied the t-class group of the generalized $D+M$ construction. The fact that $Cl_t(D+XK[X]) \approxeq Cl_t(D)$ came to the fore in a strange way in a paper of mine with Bouvier [BZ88]. Then as the Fontana factor grew, a lot of the above questions were considered for pullbacks. Fontana and Gabelli's [FG96], and independently of them Khalis and Nour El-Abidine [KN97], considered the t-class group of a pullback. Yet the class group of $A + XB[X]$ has also been studied by Anderson, El-Baghdadi and Kabbaj [ABK99]. In [ABK99] the main question studied is: Under what conditions is $Cl_t(A + XB[X]) \approxeq Cl_t(A)$. The same authors go on to study other forms of the t-class group of $A + XB[X]$; a good source for their work is El-Baghdadi's thesis [B01]. Coming back to the pullbacks, the hot questions these days are something like: when is a pullback...? For example, see Houston, Kabbaj, Lucas, and Mimouni, [HKLM94]. Also, see coherent-like conditions in pullbacks as in [GH97].

2. Part 2 (Krull dimension of A+XB[X])

Being honest to goodness polynomial ring constructions, $A + XB[X]$ domains qualify for a comparative study of their Krull dimensions with the Krull dimensions of A, $A[X]$ and $B[X]$. Of course, so do the general pullbacks, but in the case of the $A + XB[X]$ construction, we can get a somewhat better picture. This picture becomes clearer for some special cases of this construction. Now Fontana, Izelgue, and Kabbaj [FIK], [FIK92] (one of these two is a translation of the other) and [FIK94] took good care of this need both for speakers of English and French. In the following, I will try to give an idea of what they produced in this connection for readers who are interested, but not too interested.

Let us start with the observation that for each prime ideal P of $R = A+XB[X]$, either $X \in P$ or $X \notin P$. Clearly if $X \in P$, then $XR \subseteq P$, and so $X^2B[X] \subseteq XR \subseteq P$. But as $X^2B[X] = (XB[X])^2$, we have $XB[X] \subseteq P$. This means that all the prime ideals that contain X are of the form $p + XB[X]$, where p is a prime ideal of A. Now the prime ideals Q that do not contain X are hidden in places that may be hard to reach. Let us fix some notation to make the task a little lighter. Let $\mathcal{L} = \{P \in Spec(R) : X \in P\}$ and $\mathcal{M} = \{P \in Spec(R) : X \notin P\}$. Then $\mathcal{L} = \{p + XB[X] : p \in Spec(A)\}$. Now let $l = sup\{ht_R(P) : P \in \mathcal{L}\} = sup\{ht_R(p + XB[X] : p \in Spec(A)\} = sup\{ht_A(P) + ht_R(XB[X] : P \in \mathcal{L}) = dim(A) + ht_R(XB[X])$. Next, let m be the supremum of lengths of chains in $\mathcal{M}$. Then, if $S = \{X^n : \text{where } n \in N\}$, there is a one-to-one order-preserving correspondence between the primes in $\mathcal{M}$ and the primes in $R_S = (A+XB[X])_S = B[X]_S = B[X, X^{-1}]$; so $m = dim(B[X, X^{-1}])$. Now it has been established that $dim(B[X, X^{-1}] = dim(B[X])$, see for instance [ABDFK88, Proposition 1.14]. Thus $m = dim(B[X])$. Since both these sets of primes come from $Spec(R)$, we have $dim(R) \geq max\{l, m\} = max\{dim(A) + ht_R(XB[X]),\ dim(B[X])\}$. Let us record this for future reference as an observation.

Observation 2.0. Let $R = A+XB[X]$, where $A \subseteq B$ is an extension of domains and X is an indeterminate over B. Then $dim(R) \geq max\{dim(A) + ht_R(XB[X]),$ $dim(B[X])\}$.

Now we must find out the answers to the obvious questions. That is, what is $ht_R(XB[X])$?, is there an upper bound for $dim(R)$?, etc. Besides, even though the sets $\mathcal{L}$ and $\mathcal{M}$ are disjoint, some members of $\mathcal{M}$ may be contained in some members of $\mathcal{L}$. Since the Krull dimension is nothing but the supremum of lengths of chains of prime ideals, we may have to consider the case when, after taking the longest chain in $\mathcal{L}$, we are faced with the possibility that there is a sizeable chain of prime ideals of R contained in $XB[X]$, and obviously each of those prime ideals is coming from $\mathcal{M}$. So let us find out what kind of prime ideals of R will be contained in $XB[X]$.

Lemma 2.1. Let $R = A+XB[X]$, where $A \subseteq B$ is an extension of domains and X is an indeterminate over B. If P is a prime ideal of R such that $P \subsetneqq XB[X]$, then the following hold.

(1) no power of X is contained in P.

(2) $X^{-1}P \cap A = (0)$. (Consequently, if there is a nonzero prime ideal $P \subsetneqq XB[X]$, then $X^{-1}P \cap A = (0)$).

(3) $X^{-1}P$ does not contain a polynomial $f(X)$ such that $f(0) \in A \backslash \{0\}$.

(4) $X^{-1}P$ is a prime ideal of $B[X]$.

Proof. (1) Since P is a prime, any power of X in P means $X \in P$. But then $XR \subseteq P$, which means that $X^2B[X] \subseteq P$. But as we have already observed, $X^2B[X] = (XB[X])^2$ we have $XB[X] \subseteq P$, and a contradiction. (2) Clearly $X^{-1}P$ is an ideal of $B[X]$. Suppose on the contrary that $X^{-1}P \cap A = \alpha \neq (0)$. Then $X^{-1}P \supseteq \alpha[X]$. Select $f(X) \in \alpha[X]$ such that $f(0) \neq 0$. Now $f(X) \in A[X] \subseteq A + XB[X], X \in A+XB[X], f(X) \notin P$ because $f(X) \notin XB[X]$ and $X \notin P$ by (1). Yet $Xf(X) \in P$, because $X\alpha[X] \subseteq P$, contradicting the primality of P. (3) The proof of (2) can be modified to take care of this. (4) Let $f(X)g(X) \in X^{-1}P$ where $f, g \in B[X]$. Then $Xf(X)g(X) \in P$, and hence $X^2f(X)g(X) = Xf(X)(Xg(X)) \in P$, which forces $Xf(X) \in P$ or $Xg(X) \in P$. That is, $f(X) \in X^{-1}P$ or $g(X) \in X^{-1}P$.

Lemma 2.2. Given that A, B, X and R are as in Lemma 2.1. If p is a prime ideal of B such that $p \cap A = (0)$ then $Xp[X]$ is a prime ideal of R contained in $XB[X]$.

Proof. It is enough to note that $p[X] \cap (A + XB[X]) = \{f(X) \in p[X] : f(0) = 0$ and every coefficient of f comes from $\{Xg(X) : g(X) \in p[X]\}$ $Xp[X]$. That $Xp[X]$ is a prime ideal of R follows from the fact that it is a contraction of a prime ideal.

Lemma 2.3. Let $A \subseteq B$ be an extension of domains, let X be an indeterminate over B, and let $R = A + XB[X]$. Next let P be an ideal of B that is maximal w.r.t. the property that $P \cap A = (0)$. Then P is a prime ideal such that for any prime ideal Q strictly containing $P[X]$, either $X \in Q$ or there is a polynomial $f \in Q$ such that $f(0) \in A$. Consequently, if P has the above stated property, there is no prime ideal strictly between $XB[X]$ and $P[X] \cap R = XP[X]$.

Proof. Note that $A\backslash\{0\}$ is a multiplicatively closed set in B. So, P being maximal w.r.t. $P \cap A = (0)$ means P is maximal w.r.t. being disjoint from $A\backslash(0)$. This makes P a prime ideal. Next, suppose that Q is a prime ideal such that $P[X] \subsetneq Q$ and $X \notin Q$. Then there is a polynomial $f(X) \in Q\backslash P[X]$ such that no coefficient of f is in P. Since $X \notin Q$, we can arrange $f(X)$ so that $f(0) \neq 0$. But then, due to the maximality of P w.r.t. disjointness from $A\backslash(0)$, we have $rf(0) + p \in A$ for some $r \in B$ and $p \in P$. Next note that as $P \cap A = (0)$, we have $P[X] \cap R = XP[X] \subseteq XB[X]$. Now if there were a prime ideal H strictly between $XB[X]$ and $XP[X]$, then by (1) of Lemma 2.1, $X^n \notin H$ for any n. Yet, as $H \subseteq XB[X]$, every element of H is of the form $h = Xg(X)$ where $g(X) \in B[X]$. Let $h = Xg(X) \in H\backslash XP[X]$. Then $g(X) \in X^{-1}H\backslash P[X]$. This means that $X^{-1}H \supsetneq P[X]$ and as $X \notin X^{-1}H$ by the first part there is $f(X) \in X^{-1}H$ such that $f(0) \in A\backslash\{0\}$. But this contradicts (3) of Lemma 2.1. Hence there is no prime ideal H strictly between $XB[X]$ and $P[X] \cap R = XP[X]$.

Before we go any further, a word about $dim(B[X])$. Let $C : P_n \supsetneq P_{n-1} \supsetneq ... \supsetneq P_1 \supsetneq (0)$ be a chain of prime ideals in $B[X]$. Jaffard [J60] calls C a special chain if $P_i \in C$ implies $(P_i \cap B)[X] \in C$. In [J60], it was shown that $dim(B[X])$ can be realized as the length of a special chain. Let B be finite dimensional and let C described above be a chain that realizes $dim(B[X])$. Then $P_n \supsetneq (P_n \cap B)[X]$. For if $P_n = (P_n \cap B)[X]$, then $P_{n+1} = (P_n, X) = (P_n \cap B) + XB[X]$, and so there is a longer (special) chain of prime ideals in $B[X]$, a contradiction. Next we note that $(P_n \cap B)[X] = P_{n-1}$. For if not then, say $(P_n \cap B)[X] = P_{n-i}$ because C is a special chain, which gives $P_n \cap B \supseteq P_{n-1} \cap B \supseteq P_{n-i} \cap B = P_n \cap B$. This forces three distinct prime ideals of $B[X]$ to contract to the same prime ideal of B, which is impossible. Finally we note that $P_n \cap B$ must be a maximal ideal of B, because if not, then say a prime ideal $Q \supsetneq (P_n \cap B)$, and then $Q[X] \supsetneq (P_n \cap B)[X] = P_{n-1}$ and we end up, again, with a longer chain $Q + XB[X] \supsetneq Q[X] \supsetneq P_{n-1}$.... (A reader who is seriously interested in dimension theory of polynomial rings may want to read [Gil72, p.366] and chase the references given there.) Now, having made these notes, we can make the following statement.

Observation 2.4. For an integral domain B, $dim(B[X]) = 1 + max\{ht_{B[X]}(P[X]) : P \in Spec(B)\} = 1 + max\{ht_{B[X]}(P[X]) : P$ ranges over maximal ideals of $B\}$.

Lemma 2.5. Let $R = A + XB[X]$, where A is a field. Then $dim(R) = dim(B[X])$ and so $dim(R) = 1$ if and only if B is a field.

Proof. In this case, $\mathcal{L}$ =$\{XB[X]\}$ and so $l = 1$ and $\mathcal{M} = \{P \in Spec(R) : X \notin P\}$, and as before $m = dim(B[X]) = 1 + max\{ht_{B\{X\}}(P[X]) : P \in Spec(B)\}$. Now for each $P \in Spec(B), P \cap A = (0)$, and so $XB[X]$ properly contains $XP[X]$ for each $P \in Spec(B)$. Thus $ht_R(XB[X]) \geq 1 + max\{ht_{B[X]}(P[X]) : P \in Spec(B)\} = dim(B[X])$. On the other hand, each prime ideal properly contained in $XB[X]$ corresponds to a prime ideal properly contained in some member of $Spec(B[X])$ by Lemma 2.3. Thus $ht_R(XB[X]) \leq dim(B[X])$.

Now we are in a position to find out the height of $XB[X]$ in R.

Lemma 2.6. Let A, B, X, and R be as in Lemma 2.1 and let $S = A^* = A \backslash \{0\}$. Then $ht_R(XB[X]) = 1 + max\{ht_B(p[X]) : p \in Spec(B)$ such that $p \cap A = 0\} = dim(B_S[X]) \leq dim(B[X])$.

Proof. Indeed, if for every nonzero prime ideal p of B we have $p \cap A \neq (0)$, then $qf(A) = qf(B)$, and so $ht_R(XB[X]) \leq dim(A + XB[X])_S = dim(K[X]) = 1$ because $XB[X] \cap A = 0$. Now as $ht_R(XB[X]) \geq 1$, we have $ht_R(XB[X]) = 1$. This establishes the lemma for the case when $\{p \in Spec(B)\backslash\{(0)\} : p \cap A = 0\} = \phi$. Now suppose that $\{p \in Spec(B)\backslash\{(0)\} : p \cap A = 0\} \neq \phi$. Then, as $XB[X] \cap A = (0)$, we have $ht_R(XB[X]) = ht_{R_{A^*}}(XB_S[X]) = dim(B_S[X])$ because $R_S = A_S + XB_S[X]$ meets the requirements of Lemma 2.5. The inequality is self evident.

Now let us take a chain of prime ideals $C = P_n \supsetneq P_{n-1} \supsetneq ... \supsetneq P_r \supsetneq P_{r-1} \supsetneq ... \supsetneq P_1 \supsetneq P_0 = (0)$ in $A + XB[X]$, and let us use our trick of spotting X. If X does not belong to any of the P_i, then all the P_i are in $\mathcal{M}$, and so $n \leq m = dim(B[X])$. If X belongs to some, but not all of the P_i, then we reason as follows. If $X \in P_r$, then $P_i = (P_i \cap A) + XB[X]$ for all i such that $r \leq i \leq n$. Now if $P_r = XB[X]$, the largest value that $n - r$ can take corresponds to the longest chain of prime ideals in A. So $n - r \leq dim(A)$. That is, $n \leq dim(A) + r$. But, by Lemma 2.6, $r \leq dim(B[X])$. So we have $dim(R) \leq dimA + dim(B[X]$. Next, according to Lemma 2.6, $ht_R(XB[X]) = dim(B_S[X])$. Combining this information with Observations 0, we have that $dim(R) \geq max\{dim(A) + dim(B_S[X]), dim(B[X])\}$ This completes the proof of the following theorem.

Theorem 2.7. Let $A \subseteq B$ be an extension of domains, X an indeterminate over B and let $R = A + XB[X]$. Then $max\{dim(A) + dim(B_S[X]), dim(B[X])\} \leq dim(R) \leq dimA + dim(B[X])$.

Now the usual questions. Can these bounds be attained? How do these observations link up with earlier work? First of all, note that if $dim(B_S[X]) = dim(B[X])$, then the inequalities are replaced by equalities, that is $dim(R) = dimA + dim(B[X])$. What are the circumstances under which this can happen? Of course one possibility is when $qf(A) \subseteq B$. That is, if B is a field or B is a $qf(A)$-algebra. It would be interesting to know if there is an example of a domain B, where $qf(A) \nsubseteq B$, and still $dim(R) = dimA + dim(B[X])$. It may be noted however that $qf(A) \subseteq B$ if and only if for each $P \in Spec(B)$, $P \cap A = (0)$. Now for the one-ended (lower) limits. Let A be a one-dimensional domain such that $dim(A[X]) = 3$. Then $R = A + XA[X]$ gets the lower limit and understandably misses the upper limit. All we need now is an example of $R = A + XB[X]$ such that $max\{dim(A) + dim(B_S[X]), dim(B[X])\} < dim(R) < dimA + dim(B[X])$. Such an example was constructed in [FIK92, Example 3.1]. I will mention the example below and what it does, and let an interested reader look for proofs in [FIK92].

Example 2.8. Let K be a field, and let X, X_1, X_2, X_3, X_4 be indeterminates over K. Set $A = K[X_1]_{(X_1)} + X_4K(X_1, X_2, X_3)[X_4]_{(X_4)}$, $B = K(X_1)[X_2]_{(X_2)} +$

$X_3K(X_1, X_2)[X_3]_{(X_3)} + X_4K(X_1, X_2, X_3)[X_4]_{(X_4)}$, $R = A + XB[X]$, and $S = A^*$. Then $max\{dim(A) + dim(B_S[X]), dim(B[X])\} < dim(R) < dimA + dim(B[X])$. In the illustration of this example, it is also shown that in this case $dim(R) > dim(A[X])$. This is important to know because in [CMZ78, Theorem 2.6] it was shown that if S is a multiplicative set in A such that $B = A_S$ and $R = A+XA_S[X]$, then $dim(A_S[X]) \leq dim(R) \leq dim(A[X])$. So, this example also serves to show that there was a need for the study of Krull dimension of $A + XB[X]$, and that the general $A + XB[X]$ construction can behave differently from the $A + XA_S[X]$ construction.

I do not know if it has occurred to anyone, but I feel that the inequalities appearing in Theorem 2.7 can be used as a forcing tool, to draw conclusions about $A + XB[X]$ constructions having a certain properties. However, the construction being too general, the forcing that I suggest may have very limited scope. For instance, Theorem 2.7 provides the following estimates for $A + XA_S[X]$: $max\{dim(A) + 1, dim(A_S[X])\} \leq dim(R) \leq dimA + dim(A_S[X])$, and the upper bound may turn out to be somewhat higher than the corresponding inequalities given in [CMZ78,Corollary 2.9] which says $max\{dim(A) + 1, dim(A_S[X])\} \leq dim(R) \leq S{-}dimA{+}dim(A_S[X])$. Here $S{-}dim(A)$ represents the maximum length of the chain of prime ideals $P_n \supsetneq P_{n-1} \supsetneq ..... \supsetneq P_1 = P$ such that $P_i \cap S \neq \phi$.

I must record here the fact that a study of the Krull dimension, nearly on the same lines as [FIK92], has been carried out by Cahen [Ca90] and Ayache[Ay] for the $A + I$ construction. Indeed, the first application of their work is $A + XB[X]$, though it is more useful in the situation when $A + I$ is a subring of $K[X_1, .., X_n]$. (Rings of this kind were studied by Visweswaran [Vis88].) Moreover, the general study of subrings of the form $D+I$ of a domain R has been carried out in [FIK95]. The study of Krull dimension or of chains of prime ideals has also gone on in some other directions in Dobbs and Khalis's joint work[DK01]. In this paper, they also have a construction of the form $A + XA_S[[X]]$.

3. Part 3 (Constructing interesting examples 1)

It is good to have a proof that something exists, but if there is a simple example to support a claim, we would do well to use it. An example at hand may well pave the way to better understanding. After this "philosophical" statement, I should come up with some really interesting constructions. I hope to do just that, but I have to let some excess material from the previous part flow in.

Let us talk a little about the valuative dimension of $R = A + XB[X]$. Recall that if A is an integral domain, then the supremum of $dim(V)$ for all valuation overrings V of A is called the valuative dimension of A and is denoted by $dim_v(A)$. The notion of the valuative dimension was introduced by Jaffard in [J58]. However, Gilmer [Gil72] has given a good basic treatment to this topic, and the following remarks can be traced back to [Gil72, Section 30]. Indeed, for an integral domain A, $dim(A) \leq dim_v(A)$. Now what is so important about the valuative dimension is the result that $dim_v(A[X]) = dim_v(A) + 1$. Following [ABDFK88], we may call an integral domain A a Jaffard domain if $dim_v(A) = dim(A)$. Pulling out two of the several equivalent conditions of Theorem 30.9 of [Gil72], we have that $dim_v(A) = n$ is the same as $dim(A[X_1, X_2, .., X_n]) = 2n$. So, Noetherian domains and Prüfer domains are Jaffard domains, along with a host of other examples of Jaffard domains mentioned in [ABDFK88]. Coming back to the business at hand, we have the following result, in connection with the $A + XB[X]$ construction, to report from [FIK92]. Here, let us recall that if $A \subseteq B$ is an extension of domains, then the degree of transcendence of $qf(B)$ over $qf(A)$ is called the degree of transcendence of B over A, denoted by $tr.deg(B/A)$.

Theorem 3.1. $dim_v(A + XB[X]) = dim_v(A) + tr.deg(B/A) + 1$.

Now if you are interested in the proof, look up [FIK92]. However, I would be more interested in a proof that is based on the observation that every valuation overring of $R = A + XB[X]$ is the ring of a valuation on $qf(R)$ that is an extension of a valuation on $qf(A)$.

Now let us see what Theorem 3.1 has to offer. Indeed, if B is algebraic over A, then $tr.deg(B/A) = 0$. So, if B is algebraic over A, in particular if B is an overring of A, then $dim_v(A + XB[X]) = dim_v(A) + 1$. This obviously takes care of the case when B is a quotient ring of A. Now what is the use?

Corollary 3.2. Suppose that A is a Jaffard domain. Then $dim_v(A+XB[X]) = dim(A) + tr.deg(B/A) + 1$.

Thus if $tr.deg(L/K) = \infty$, where L is a field extension of K, then $R = K+XL[X]$ is a one-dimensional domain whose valuative dimension is infinite. There is a wealth of results on valuative dimensions of pullbacks and generalized $D+M$ constructions in [ABDFK88]. One may wonder about the need to write [FIK92] if pullbacks are so perfect. My response, as usual, is that $A + XB[X]$ constructions, crude though they may look, do provide valuable information which may be hard to glean from pullbacks.

Recall that an integral domain A is an S-domain (S for Seidenberg) if for each height-one prime ideal P of A we have that $P[X]$ is a height-one prime ideal of $A[X]$. Let us also recall from Kaplansky [Kap70], who is responsible for this terminology, that A is a strong S-ring if A/P is an S-domain for each prime ideal P of A. Clearly if A is a strong S-ring, then so are the homomorphic images of A. The terminology, whose motivation can in part be traced back to Seidenberg [S53, Theorem 3], seemed to provide a useful tool for recognizing integral domains that behaved like Noetherian domains in that they satisfied $dim(A[X]) = dim(A) + 1$.

Theorem 3.3. ([Kap70, Theorem 39]). Let A be a strong S-ring, let X be an indeterminate over A, and let P be a prime ideal of A. Then $ht_{A[X]}(P[X]) = ht_A(P)$. Moreover, if Q is a prime ideal of $A[X]$ such that $Q \cap A = P$ and $Q \supsetneq P[X]$, then $ht_{A[X]}(Q) = ht_A(P) + 1$.

Then in a later section, he shows that a valuation ring is a strong S-ring [Kap70, Theorem 68]. Now this is where Malik and Mott [MM83] picked up the strand and started pulling, of course in a multiplicative sort of way. Their results were of the type:

Proposition 3.4. ([MM83, 2.1 and 2.2]) A domain A is an S-domain if and only if A_T is an S-domain for each multiplicative set T, if and only if A_M is an S-domain for each maximal ideal M.

On the strong S-property, they proved, likewise, the following statement.

Proposition 3.5. (MM83, 2.3,2.4]). A ring A is a strong S-ring if and only if A_T is a strong S-ring for each multiplicative set T, if and only if A_M is a strong S-ring for each maximal ideal M.

Now, coupling Proposition 3.5 with Kaplansky's Theorem 68 mentioned above, and adding some more work they stated the following result.

Proposition 3.6 ([MM83, 2.5]). A Prüfer domain is a strong S-domain.

Using their criteria, they came up with the following scheme.

Proposition 3.7. ([MM83, 3.1, 3.2]) For A an S-domain (a strong S-domain), $A[X]$ is an S-domain (resp., a strong S-domain) if and only if $A_P[X]$ is an S-domain (resp., a strong S-domain) for each prime ideal P of R. (Here X denotes a finite set of indeterminates.)

There were several other interesting statements in section 3 of [MM83]. All this culminated in a beautiful result and that can be stated as follows.

Theorem 3.8. ([MM83,3.5]) Let A be a Prüfer domain and let $X = \{X_1, X_2, .., X_n\}$ be a finite set of indeterminates over A. Then $A[X]$ is a strong S-domain.

Next, using Kaplansky's Theorem 39 (Theorem 3.3 here) it is easy to observe that a strong S-domain is a Jaffard domain. This observation was made in [MM83], along with an example of a Jaffard domain that is not a strong S-ring [MM83, 3.11]. It appears that no one has tried to find a minimal set of conditions under which a Jaffard domain should be a strong S-domain. The paper ([MM83]) goes on to display other goodies, but I must leave the rest for the interested readers and hasten to answer the question that has by now started popping up in every reader's mind,"Where is the $A+XB[X]$?" Let me take you to Salah Kabbaj's earlier work. He picked up where [MM83] had left off. I found in an earlier version of his thesis the following statement which stayed as it was in the final version ([Kab, Théorème 0.8, Chap. II]). Let A be an S-domain. Then $A[X_1, X_2, .., X_n]$ is an S-domain for all $n \geq 1$. Looked like a pretty result, it was a considerable improvement on [MM83, 3.1] (which is a part of Proposition 3.7 here), so I started playing with it. The first thing that came to my mind was,"Where is he using the fact that A is an S-domain?" The answer came out, "Only at one place, and that could be avoided." I made the suggestion, in my report on his thesis, indicating how his proof can be modified to prove the following statement.

Theorem 3.9. If A is an integral domain and X an indeterminate over A, then $A[X]$ is an S-domain.

Now, for some reason, this suggestion was not taken and I was hopping mad, (I so wanted Kabbaj to have this result!). I was working on [AAZ91] and I mentioned the result to Dan. He agreed to include the result, but at a price, as usual, he would write his own proof. He did give a pretty proof though. Possibly simultaneously, Kabbaj and Fontana [KF90] did prove Theorem 3.9 and many more interesting results. Now my trouble is that I like both very much. For this reason, I have decided to give a proof that has the flavor of both.

Lemma 3.10. ([AAZ91]) For an integral domain A the following statements are equivalent.

(1) A is an S-domain.

(2) For each height-one prime ideal P of A, A_P is an S-domain.

(3) For each height-one prime ideal P of A, $\overline{A_P}$ (the integral closure of A_P) is a Prüfer domain.

Proof. (1)⇒ (2) by Proposition 3.4 above. (2)⇒ (3) By (2), $A_P[X]$ is two-dimensional and by [Gil, 30.14], $\overline{A_P}$ is Prüfer. (3)⇒ (1). Suppose that for each height-one prime ideal P, $\overline{A_P}$ is one-dimensional Prüfer. Then $\overline{A_P}[X] = \overline{A_P[X]}$ is two-dimensional, which requires that $A_P[X]$ is two dimensional, which means that $PA_P[X] = P(A[X])_{(A\setminus P)}$ is of height one. This indeed means that $PA[X] = P[X]$ is of height one. Now recall that P is any height-one prime ideal of A.

This lemma provides a neat characterization of S-domains. Now before we start proving the theorem, let me digress a little. Call A a stably strong S-domain if $A[X_1, .., X_n]$ is a strong S-domain for all $n \geq 1$. (A being a homomorphic image of $A[X_1, .., X_n]$, stably strong S implies strong S.) So, a Prüfer domain, by Theorem 3.8 above, is a (stably) strong S-domain. Of the various equivalences on one-dimensional domains in [ABDFK88, Theorem 1.10], we recall that for a one-dimensional domain A the properties: (a) S-domain, (b) strong S-domain, and (c) stably strong S-domain, are all equivalent.

Proof of Theorem 3.9. By Lemma 3.10, all we need is to show that $A[X]_P$ is an S-domain for each height-one prime P of $A[X]$. There are two possibilities: (i) $P \cap A = (0)$, (ii) $P \cap A = p \neq (0)$. In the first case, it is well known that $(A[X])_P$ is a valuation domain and hence an S-domain. In the second case, $P = p[X]$, where p is of height one. Now $PA[X]_P = pA[X]_{p[X]}$ is of height one. But $A[X]_{p[X]} = A_p(X) = (A_p[X])_{pA_p[X]}$. So $pA_p[X]$ is of height one. Whence the one dimensional A_p is an S-domain. But then, by the remarks prior to the proof, A_p is a stably strong S-domain. This means that $A_p[X]$ is an S-domain. But then so is every quotient ring of $A_p[X]$ by Proposition 3.4 above. Whence $A_p(X) = A[X]_{p[X]} = A[X]_P$ is an S-domain, and this completes the proof.

An immediate corollary is the following statement.

Corollary 3.11. ([AAZ91, 3.3]) $D + XD_S[X]$ is an S-domain for every multiplicative set S of D.

As already mentioned in part 1, in [FK90] the authors study a construction defined as

$$D^{(S,r)} = D + (X_1, .., X_r)D_S[X_1, .., X_r]$$

and show that $D^{(S,r)}$ is an S-domain. Now the question is, what form Theorem 3.9 will take for $R = A + XB[X]$? The answer comes from [FIK94]. Yet before we quote from [FIK94], let us see what we can do with what we have established so far. The following statement can be regarded as a corollary to Lemma 3.10 and Theorem 3.9.

Proposition 3.12. Let $A \subseteq B$ be an extension of domains such that B is a subring of the quotient field of A. Then $R = A + XB[X]$ is an S-domain.

Proof. We show that R_P is an S-domain for each height-one prime P of R. Note that $ht_R(XB[X]) = 1$ and that $R_{XB[X]} = (A + XB[X])_{XB[X]} = (K[X])_{XK[X]}$ is a valuation domain, and hence is an S-domain. For height-one primes $P \neq XB[X]$, let $S = \{X^n : n \geq 1\}$ and note that $R_P = (R_S)_{P_S} = B[X, X^{-1}]_{P_S}$. But then R_P is a quotient ring of $B[X]$ which is an S-domain, and we know that every quotient ring of an S-domain is again an S-domain (Proposition 3.4).

Now comes the promised result.

Proposition 3.13. ([FIK94, Theorem 1.1]) $R = A + XB[X]$ is an S-domain if and only if $ht_R(XB[X]) > 1$ or B is algebraic over A.

What is the use of this? For one thing, it takes care of Corollary 3.11 and Proposition 3.12. Moreover, you can construct composite examples of S-domains and non S-domains with interesting properties. For instance, using Proposition 3.13, if $\mathcal{Z}$, $\mathcal{Q}$ and $\mathcal{R}$ represent the integers, rationals, and reals, respectively then $\mathcal{Z}+X\mathcal{R}[X]$ is not an S-domain, while $\mathcal{Z}+X\mathcal{Q}(i)[X]$ is an S-domain. Of course, these are just two very simple examples.

In [FIK94], the reader can find a wealth of examples on strong S-domains. For this survey, we select a number of results that could intrigue the reader enough to want to prove some direct results in this direction. Let us recall first that the extension of domains $A \subseteq B$ is said to be incomparable (INC) if two distinct primes P, Q of B contract to the same prime p of A, then P and Q are incomparable. ([Kap70, Section1-6] is a good source for a study of INC and related notions.) For example, $A \subseteq A[X]$, where X is an indeterminate, does not have INC because $(0) \subseteq (X)$ both contract to (0) in A. So if $A \subseteq B$ is incomparable, then every nonzero prime of B contracts to a nonzero prime of A. Next, $A \subseteq B$ is said to be residually algebraic if for each prime P of B, we have B/P algebraic over $A/(P \cap A)$. For this notion the reader may look up [DF84].

Theorem 3.14. ([FIK94, Théorème 1.7]) Let $A \subseteq B$ be an extension of domains. Then the following are equivalent:

(1) $R = A + XB[X]$ is a strong S-domain and $A \subseteq B$ is an incomparable extension.

(2) A and $B[X]$ are strong S-domains and $A \subseteq B$ is a residually algebraic extension.

As a direct consequence of the above equivalence, we have the following statement.

Corollary 3.15. Let D be an integral domain, S a multiplicative set of D and let K be a field containing D as a subring. Then the following hold.

(1) $D + XK[X]$ is a strong S-domain if and only if D is strong S-domain and K is algebraic over the quotient field of D.

(2) $D + XD_S[X]$ is a strong S-domain if and only if D and $D_S[X]$ are strong S-domains.

Now this little corollary gives us a host of examples of strong S-domains, including ones that have terrible and totally unaccommodating properties elsewhere and ones that serve as examples of beautiful new notions. I would mention only two here.

Example 3.16. Let D be a Prüfer domain and let S be a multiplicative set of D. Then $D^{(S)} = D + XD_S[X]$ is a strong S-domain.

In fact, $D^{(S)}$ is a stably strong S-domain, i.e., for any set $\{Y_1, Y_2, .., Y_m\}$ of indeterminates $D^{(S)}[Y_1, Y_2, .., Y_m]$ is a strong S-domain. The main reason is that, by Theorem 3.8, $D[X]$ and $D_S[X]$ are both strong S-domains for any set X of indeterminates and $D_S[X] = (D[X])_S$.

Now $D^{(S)}$ is Prüfer if and only if $S = D^*$ [CMZ78], so Example 3.16 affords an example of a non Prüfer domain that is a strong S-domain. For reasons of organization, I will not go too deep into this example and will refer to it later.

Example 3.17. Let $K \subseteq L$ be an extension of fields. Then $R = K + XL[X]$ is one-dimensional and according to [ABDFK88], R is an S-domain $\Leftrightarrow$ R is a strong S-domain $\Leftrightarrow$ $\overline{R}$ the integral closure is a Prüfer domain $\Leftrightarrow$ R is a stably strong S-domain. Now $\overline{R}$ is Prüfer if and only if L is algebraic over K. Next, let K be of characteristic $p \neq 0$ and let L be a purely inseparable extension of K such that $L^p \subseteq K$. (See [Kap, Theorem 100] for an example with $p = 2$.) Then, apart from being a stably strong S-domain, $R = K + XL[X]$ has the added property that for each pair $f, g \in R$ we have $f^p, g^p \in K[X]$, which is a PID, and so $(f^p, g^p)K[X] = hK[X]$. Now as $K[X]R = R$, we have $(f^p, g^p)R = hR$. From this it also follows that $f^pR \cap g^pR$ is principal.

This example is Example 2.13 of [Z85]. Let me use this example to introduce an interesting set of concepts.

An integral domain D is called an almost GCD domain (AGCD domain) if for each pair $x, y \in D$ there is a natural number $n = n(x, y)$ such that $x^nD \cap y^nD$ is principal. (Indeed, if for all $x, y \in D$ we have $n(x, y) = 1$, we get a GCD domain.) Apart from the above example, there are other well known examples, such as almost factorial domains of Storch. These are Krull domains with torsion divisor class group (see, for instance, Fossum [Fos73]). (The reader can look up [Fos73] to check that an integral domain D is Krull if for each height-one prime P, D_P is a discrete rank one valuation domain, and D is a locally finite intersection of localizations at height-one primes.) For other examples of AGCD domains, that use the $A + XB[X]$ construction, the reader may consult [DLMZ] when it appears. Next, D is called an almost Bézout domain if for each pair $x, y \in D$ there is a natural number $n = n(x, y)$ such that $x^nD + y^nD$ is principal. Example 3.17 above may serve as an example again. For more examples of almost Bézout domains, you may consult [AZ91]. Now, the integral closure of an almost Bézout domain is a Prüfer domain with torsion ideal class group [AZ91, Corollary 4.8]. Indeed, as we already know from [ABDFK88] that a one-dimensional almost Bézout domain is stably strong S-domain, but the general case of almost Bézout domains is not quite clear.

Now, coming back to our task at hand, a ring A is called a Hilbert ring if every prime ideal of A is expressible as an intersection of maximal ideals containing it. So, a one-dimensional domain A is a Hilbert ring if and only if there is a set $\{M_\alpha\}$ of maximal ideals of A such that $(0) = \cap M_\alpha$. Now, because everything in a Hilbert ring seems to be in terms of maximal ideals, it is fair to ask if Cohen's criterion for a Noetherian ring (R is Noetherian if and only if every prime ideal of R is finitely generated) can be relaxed for Hilbert rings to: A Hilbert ring A is Noetherian if and only if every maximal ideal of A is finitely generated? A.V. Geramita asked Robert Gilmer and/or William Heinzer this question and they came up with a non-Noetherian example of a Hilbert domain whose maximal ideals are all finitely generated [GH77]. Later, Joe Mott and I [MZ90] came up with the following theorem.

Theorem 3.18. Let D be a Hilbert domain and let L be a field containing D. Then $D + XL[X]$ is a Hilbert domain.

The proof is straight-forward and short, and if D is a Hilbert PID with quotient field L then $D + XL[X]$ is a two dimensional Bézout domain, with each maximal ideal principal. This example is uncannily similar to the one constructed in [GH77]. Obviously, taking any Noetherian Hilbert domain for D in Theorem 3.18, you can construct a Hilbert domain whose maximal ideals are finitely generated and which is not Noetherian. Theorem 3.18 also pre-empts the obvious question about a Hilbert domain being an S-domain. The answer of course is,"Not in general". (Strangely, [GH77] is still on my recommended reading list because of its useful auxiliary results.) In [AAZ91], we proved something in a slightly different direction. The result can be stated as follows.

Theorem 3.19. Let D be an integral domain and S a multiplicative set of D such that each prime P of D that intersects S, intersects S in detail, that is for each nonzero prime $Q \subseteq P$, $Q \cap S \neq \phi$. Then $R = D + XD_S[X]$ is a Hilbert domain if and only if both D and D_S are Hilbert domains.

It is remarkable that unruly Hilbert domains of [MZ90] did not seem to have as much effect as Theorem 3.19 had. There was a renewed interest in Hilbert domains and out came a paper by Anderson, Dobbs, and Fontana [ADF92] on Hilbert rings arising as pullbacks. In this paper, they discuss as applications the events of $D+(X_1,..,X_n)D_S[X_1,..X_n]$, $A+XB[X]$, and $D+M$ constructions being Hilbert. In particular, it was shown in [ADF92] that, $A+XB[X]$ is a Hilbert domain if and only if A and B are Hilbert domains. This includes Theorems 3.18 and 3.19 above. It appears that someone else was interested in showing when $A+XB[X]$ is a Hilbert domain. On reading this survey Lahoucine Izelgue sent me some of his old work on this topic. I have decided to include it here because it is simple and it is efficient.

When is A+XB[X] a Hilbert domain?

Lahoucine Izelgue

On 1991, M. Fontana and S. Kabbaj asked me about conditions under which a domain of the form $A + XB[X]$ is a Hilbert domain. When looking for an answer to this question, I learned that D.D. and D.F. Andersons and Mohammad [AAZ91], had conjectured that "$D + XD_S[X]$ is a Hilbert domain if and only if so are D and D_S". So I attempted an answer to their question that recovers the conjecture also. However, when I was getting ready with an answer Kabbaj, my adviser, told me that Anderson, Dobbs, and Fontana were working on the same subject and that they had got interesting reslts. Indeed, in April 1992, at the first Fez coference, Fontana Gave me a reprint of their paper [ADF92] in which they had studied the Hilbert property in pulbacks, and, in fact, it recovers my results. Now, after reading a first version of the present paper of Moahmmad, I sent him my results and he told me that they should be brought to light. In fact, even if my results are the same as that of [ADF92], the proofs that I give here are very classical and do not use pullback techniques. Also It may serve as a good example even for undergraduate mathematics students.

Theorem 3.20. Let $A \subset B$ be an extension of integral domains and set $R = A + XB[X]$. Then, R is a Hilbert domain if and only if A and B are Hilbert domains.

Proof.

$(ii) \Longrightarrow (i)$. Let $Q \in Spec(R)$. There are two possibilities: $X \in Q$ or $X \notin Q$.

Case 1: Suppose $X \in Q$, so $Q = q + XB[X]$, where $q = Q \cap A$ (cf. [FIK 92, Lemma1]). Since $R/XB[X] \simeq A$ and A is a Hilbert domain. Hence $q = \bigcap_{i \in I} m_i$, where $q \subseteq m_i \in \max(A)$ for each $i \in I$. Furthermore, for each $i \in I$, $m_i + XB[X] \in max(R)$. It follows that Q is an itersection of all maxcimal ideals containing it.

Case 2: $X \notin Q$. Let $S = \{X^n, n \geqslant 0\}$, a multiplicative subset of both R and $B[X]$, such that $S^{-1}R = S^{-1}B[X] = B[X, X^{-1}]$. Thus, there exists $Q' \in Spec(B[X])$ such that $X \notin Q'$ and $Q' \cap R = Q$. Now, since B is a Hilbert domain, then so is $B[X]$ (cf. [Kap70, Theorem 3.1]). Therefore, $Q' = \bigcap_{s \in J} M_s$, where $Q' \subseteq M_s \in \max(B[X])$ for each $s \in J$. Moreover, there exists $s \in J$ such that $X \notin M_s$. It follows that $Q = \bigcap_{s \in J} M_s \cap R$.

If $X \in M_s$, then $M_s \cap R$ is an intersection of maximal ideals of R (see Case 1).

If $X \notin M_s$, then $X \notin M_s \cap R$ and $M_s \cap R$ is a maximal ideal of R (cf. [FIK92, Lemma 1.1] and [Ca88, Proposition 4].

$(i) \Rightarrow (ii)$. $A \simeq R/XB[X]$ and hence is a Hilbert domain (cf.[Gil72, Theorem31.18]). To show that $B[X]$ is Hilbert, it is sufficient to show tha t $S^{-1}R$ is Hilbert, where $S = \{X^n, n \geqslant 0\}$, since $S^{-1}R = S^{-1}B[X] = B[X, X^{-1}]$ is integral over $B[X + X^{-1}] \simeq B[X]$ (cf. [Gil72, Exercise1- p. 327]), and by [Gil72, Theorem 31.8], B is a Hilbert domain.

Now, let $P \in Spec(S^{-1}R)$, there exists Q a prime ideal of R such that $X \notin Q$ and $P = S^{-1}Q$. Since R is a Hilbert domain, $Q = \bigcap_{s \in J} M_s$, with $Q \subseteq M_s \in \max(R)$ for each $s \in J$. Moreover, there exists $s \in J$ such that $X \notin M_s$. Let $I_1 = \{s \in J;\ X \in M_s\}$ and $I_2 = \{s \in J;\ X \notin M_s\}$. We claim that $P = \bigcap_{s \in I_2} S^{-1}M_s$: indeed, let $f(X) = \frac{h(X)}{X^n} \in \bigcap_{s \in I_2} S^{-1}M_s$, then $h(X) \in M_s$ for each $s \in I_2$ and $X \in M_s$ for each $s \in I_1$. Therefore, $Xh(X) \in \bigcap_{s \in J} M_s = Q$. As $X \notin Q$, then $h(X) \in Q$, and hence $f(X) = \frac{h(X)}{X^n} \in S^{-1}Q = P$. Thus, $\bigcap_{s \in I_2} S^{-1}M_s \subseteq P$. The reverse inclusion is obvious. It follows that P is an intersection of maximal ideals of $S^{-1}R$ and hence $S^{-1}R$ is a Hilbert domain.

The following corollary confirms the conjecture of [AAZ 91] (see alsoTheorem 3.18 and theorem 3.19 above).

Corollary 3.21. Let D be an integral domain, S a multiplicative subset of D and K a field containing D.

1) $D + XK[X]$ is a hilbert domain iff so is D.

2) $D + XD_S[X]$ is a Hilbert domain iff so are D and D_S.

Corollary 3.22. Let $A \hookrightarrow B$ be an integral extension of domains and set $R = A + XB[X]$. Then R is a Hilbert domain iff A is a Hilbert domains iff B is a Hilbert domain.

Proof.

$(i) \Leftrightarrow (ii)$. It follows from Theorem 3.20.

$(ii) \Leftrightarrow (iii)$. It follows from [Gil72, Exercise 1 - p. 387].

Example 3.23. $R = \mathbb{Z} + X\mathbb{Z}[Y, X]$ is a non-Noetherian Hilbert domain of Krull dimension 3, R is also a stablly strong S-domain (c.f [FIK95, Exemple 3.4]).

4. Part 4 (Constructing interesting examples 2)

The emphasis of this part of the article is on the use of $A+XB[X]$ in constructing examples of rings and notions related to star operations. The best approach here would be to include a brief introduction to the star operations, and to refer a more interested reader to Gilmer [Gil], Jaffard [Ja], Halter-Koch [HK,] and to [Z 00]. (I know that I have already provided some introduction to the star operation in my talk, but I include a more complete introduction so that the reader does not have to go back to re-read the terminology.)

Let A be an integral domain with quotient field K, and let $F(A)$ denote the set of nonzero fractional ideals of A.

A star operation is a function $I \mapsto I^*$ on $F(A)$, with the following properties:
If $I, J \in F(A)$ and $a \in K\backslash\{0\}$, then
(i) $(a)^* = (a)$ and $(aI)^* = aI^*$.
(ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$.
(iii) $(I^*)^* = I^*$.

We shall call I^* the $*$-*image* (or $*$-envelope) of I. An ideal I is said to be a $*$-*ideal* if $I^* = I$. Thus I^* is a $*$-ideal, and of course every principal fractional ideal, including $D = (1)$, is a $*$- ideal for any star operation $*$. An ideal $I \in F(A)$ is called a $*$-ideal of finite type if there is a finitely generated J such that $I^* = J^*$. For all $I, J \in F(A)$, and for each star operation $*$, $(IJ)^* = (I^*J)^* = (I^*J^*)^*$. These equations define what is called $*$-*multiplication (or $*$-product*). If $\{I_\alpha\}$ is a subset of $F(A)$ such that $\cap I_\alpha \neq (0)$, then $\cap(I_\alpha)^* = (\cap(I_\alpha)^*)^*$. That is, an intersection of $*$-ideals, for a certain star operation $*$, is again a $*$-ideal. We may call this property the *intersection property*. Also if $\{I_\alpha\}$ is a subset of $F(A)$ such that $\sum I_\alpha$ is a fractional ideal then $(\sum I_\alpha)^* = (\sum I_\alpha^*)^*$.

Define $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup\{F_v|\ 0 \neq F$ is a finitely generated subideal of $I\}$. The functions $I \mapsto I_v$ and $I \mapsto I_t$ on $F(A)$ are more familiar examples of star operations defined on an integral domain. Indeed, if I is finitely generated, then $I_t = I_v$. A v-ideal is better known as a divisorial ideal. An integral ideal M (of A) that is maximal w.r.t. being a t-ideal is a prime ideal, and maximal t-ideals always exist. Moreover the following holds.

Proposition 4.1. ([Gri67]) $A = \cap A_M$, where M ranges over maximal t-ideals of A.

There are of course many more star operations that can be defined on an integral domain A. But for any star operation $*$, and for any $I \in F(A)$, $I^* \subseteq I_v$. An ideal $I \in F(A)$ is said to be $*$-invertible, for a star operation $*$, if there is a $J \in F(A)$ such that $(IJ)^* = A$. An integral domain A is said to be a Prüfer v-multiplication domain (PVMD) if every finitely generated $I \in F(A)$ is t-invertible. A t-invertible t-ideal is a t-ideal of finite type, and hence a v-ideal of finite type. So in a PVMD, for each finitely generated ideal I, I^{-1} is of finite type. Griffin[Gri67] showed that A is a PVMD if and only if for each maximal t-ideal P of A, A_P is a valuation domain. Prüfer domains (finitely generated ideals in $F(A)$ are invertible and hence t-invertible), Krull domains (all $I \in F(A)$ are t-invertible[MZ91]) GCD- domains (for all finitely generated $I \in F(A), I_t$ is principal, and hence invertible) are all examples of PVMD's. A closely related concept, to PVMD's, is that of a v-domain, i.e., every finitely generated ideal is v-invertible. Indeed, it is easy to establish that a v-domain A is a PVMD if and only if for each (nonzero) finitely generated ideal

I, I^{-1} is of finite type. Now let us call a prime ideal P of A essential if A_P is a valuation domain. A domain A is called essential if there is a family $\{P_i\}$ of essential primes such that $A = \cap A_{P_i}$. Clearly, by Proposition 4.1, a PVMD is essential. But is an essential domain a PVMD? Griffin [Gri67], conjectured that the answer was no. Heinzer and Ohm [HO73] came up with the example of an essential domain that is not a PVMD. (It is well known that an essential domain is a v-domain see e.g. [Z78].) But of course the example in [HO73] was lengthy and involved. Now let us prepare for the first useful and simple example of this section. In [Z78], I came up with the following statement.

Proposition 4.2.([Z78]) Let I be a finitely generated ideal of a domain A and lets S be a multiplicative set in A. Then (a) $(IA_S)^{-1} = I^{-1}A_S$ and (b) $(IA_S)_v = (I^{-1}A_S)^{-1} = (I_vA_S)_v$.

It is an extremely simple result and it could be extracted from an exercise in [Gil72], and part (a) is extremely well known, but it did open up a lot of doors. (An interested reader may consult [Z00] for further detail on this result.) Yet, at the same time, it seemed to give the false impression that if S is a multiplicative set in A, and P is a prime t-ideal of A disjoint from S, then PA_S must be a prime t-ideal of A_S. Clearly if A is such that for each finitely generated ideal I we have I^{-1} of finite type, then Proposition 4.2 can be used to show that $(IA_S)_v = I_vA_S$. So in a PVMD, we have no problems, every prime t-ideal extends to a prime t-ideal in a localization, and so localization at each prime t-ideal is a valuation domain[MZ81]. Also using Proposition 4.2, we can show that if A is a PVMD and S is a multiplicative set in A, then A_S is a PVMD. It turns out that the example given in [HO73] was a locally factorial domain [MZ81]. Now the argument is: If every prime t-ideal of A extends to a prime t-ideal in a localization, then for a locally factorial (or locally GCD or PVMD) domain A, take a maximal t-ideal P, P must be contained in a maximal ideal M. That is, PA_M is a prime t-ideal of A_M. Now, A_M is a GCD domain and hence a PVMD. But then $A_P = (A_M)_{P_M}$ is a valuation domain, leading to the conclusion that a locally GCD domain/PVMD is a PVMD. But this goes against the Heinzer-Ohm example of [HO73]. So the conclusion: There must be at least one prime t-ideal P in the Heinzer-Ohm example A such that PA_M is not a prime t-ideal, and there must be a finitely generated I such that $(IA_S)_v \neq I_vA_S$ for some multiplicative set S. Now, this establishes the existence of a prime t-ideal P and a finitely generated ideal I in the domain A of [HO73] such that PA_S is not a prime t-ideal and $(IA_S)_v \neq I_vA_S$. But I would rather see these ideals in action, and I wanted to be sure that there do exist other simpler examples of locally GCD domains, that are not PVMD's. This quest made me study the $D + XD_S[X]$ construction all over again. To show you what I got as a result, I need to prepare a little.

Call $a \in A \backslash \{0\}$ primal, if for $x, y \in A$, $a \mid xy$ implies that $a = rs$ such that $r \mid x$ and $s \mid y$. A nonzero element z is called completely primal if each factor of z is completely primal. Cohn [C68] was the author of this terminology, and for some very good reasons. An integral domain A whose nonzero elements are all primal (and hence completely primal) is called pre-Schreier in [Z87]. Cohn [C68] calls an integrally closed domain with nonzero elements primal a Schreier domain. He had thrown in the integrally closed condition for what I call organizational reasons; polynomial rings over Schreier domains are Schreier, but polynomial rings over pre-Schreier domains are not (necessarily) pre-Schreier. Now, why do I bring Schreier and pre-Schreier in the middle of PVMD's and GCD domains? Apart from the fact that GCD domains are Schreier [C68], the reason is the following statements that I intend to use.

Proposition 4.3. Polynomial rings over PVMD's are PVMD's, and polynomial rings over GCD domains are GCD domains. (The GCD case is elementary via Gauss' Lemma. and the PVMD case, is well known too, see for example [HMM84].

Proposition 4.4. ([Z87]) A PVMD A is a GCD domain if and only if A is Schreier.

Proposition 4.5. ([CMZ78]) If A is a GCD domain and S a multiplicative set in A, then $A + XA_S[X]$ is a Schreier domain.

The reason behind this statement is the fact that a directed union of Schreier rings is again Schreier, and obviously we have $A + XA_S[X] = \bigcup\{A[X/s] : s \in S\}$, and if A is a GCD domain each of $A[X/s]$ is GCD. Indeed, the Proposition could be stated for Schreier domains as well.

Using the above proposition, the following was proved in [CMZ78].

Proposition 4.6. If A is a GCD domain and S a multiplicative set of A, then $A + XA_S[X]$ is a GCD domain if and only if $\text{GCD}(a, X)$ exists for each $a \in A$.

Now, in view of Proposition 4.4, to build a locally GCD domain that is not a PVMD, our best bet would be to build a locally GCD domain that is Schreier but not a GCD domain. It would also help if the concepts involved are well known and/or easy to grasp. Well, let us recall that the ring of entire functions E is a Bézout domain, i.e., every finitely generated ideal is principal, and that every nonzero nonunit x of E can be written, uniquely, as a product $x = \prod p_i^{a_i}$, where p_i are countably many mutually non-associated prime elements, and a_i are natural numbers. The condition that p_i are mutually non-associated forces each of p_iE to be a height-one prime that is also maximal. The following example is taken from [Z88, Example 2.6]. The illustration however is modified to avoid the jargon involved in the original illustration.

Example 4.7. Let E be the ring of entire functions, and let S be the multiplicative set generated by principal nonzero primes of E. Then $R = E + XE_S[X]$ is a locally GCD domain that is not a PVMD.

Illustration. Take any maximal ideal M in R. Then $M \cap S = \phi$ or $M \cap S \neq \phi$. If $M \cap S = \phi$, then $R_M = (R_S)_{MR_S}$. But as $R_S = E_S[X]$ a GCD domain, we have that R_M is a GCD domain. If, on the other hand, $M \cap S \neq \phi$, then $M = M \cap E + XE_S[X]$. Now $M \cap S \neq \phi$ implies that $M \cap E$ contains a finite product of primes of E and so contains at least one of them and, because in a Bézout domain two non-associated primes are comaximal, at most one belongs to $M \cap E$. Thus $M = (p) + XE_S[X]$, which is a height-two prime. Now $R_M \supseteq R_{(R \setminus (p))} = E_{(p)} + XK[X] \supseteq R$ where K is the quotient field of E. So, R_M is a quotient ring of $E_{(p)} + XK[X]$ which is a GCD domain by Proposition 4.6.

Now that we have shown that R is locally GCD, and from Proposition 4.5 we know that R is Schreier, all we need to show is that R is not a GCD domain. For this, take $a \in E$ such that $a = \prod p_i^{a_i}$ is an infinite product of principal primes. Then as X is divisible by all powers of all principal primes, but never by an infinite product of them, a and X will have higher and higher common divisors, but will not have a greatest common divisor. So R is not a GCD domain, and hence not a PVMD.

Remarks 4.8 (i) The above example may also be used to establish that, in general a directed union of GCD domains is not necessarily a GCD domain, and that a directed union of PVMD's is not necessarily a PVMD. Indeed, if S is a multiplicative set in E generated by a finite number of principal primes, then $E^{(S)} = E+XE_S[X]$ is a GCD domain. Indeed, this kind of directed unions of PVMD's are PVMD's, as are GCD domains, under certain situations. For PVMD's, as I have already mentioned, the reader may consult [AAZ01]. Dumitrescu and Moldovan [DM] have also considered a construction similar to $D + XD_S[X]$ in connection with PVMD's. The paper is worth a read for the various interesting notions, such as a generalization of the notion of primal elements, they bring in.

(ii) The ring of entire functions is one of the rings a multiplicative ideal theorist should know about. It is important for several reasons. (For a quick idea of this ring read Helmer [Hel40], Henriksen [Hen52] and [Hen53], and Gilmer [Gil72, page 146, and references there].) One very important property to remember is that the ring of entire functions is a completely integrally closed Bézout domain which is infinite dimensional. So the ring of entire functions is a completely integrally closed domain whose rings of fractions may not be completely integrally closed.

Now for the concrete examples.

Lemma 4.9. Let a be a nonzero nonunit of the ring E of entire functions, and let S be as in Example 4.7. Then in $R = E+XE_S[X]$ we have $(a, X)_v = (a)+XE_S[X]$, and so every ideal of the form $(a) + XE_S[X]$ is divisorial.

Proof. Let $a \neq 0$. If $a \in S$, then $(a, X)_v = (a)R = (a) + XE_S[X]$. If on the other hand $a \notin S$, then $a = \prod p_i^{a_i}$ is an infinite product of principal primes. In this case, $(a, X)^{-1} = \frac{(a)\cap(X)}{aX} = \frac{1}{aX}\{\frac{aX}{s} : s \mid a\}R = \{\frac{1}{s} : s \in S, \text{ and } s \mid a\}R$. Now, $(a, X)_v = ((a, X)^{-1})^{-1} = \{r \in R : r\{\frac{1}{s} : s \in S, \text{ and } s \mid a\}R \subseteq R\} = (a) + XE_S[X]$. Obviously $((a) + XE_S[X])\{\frac{1}{s} : s \in S, \text{ and } s \mid a\}R \subseteq R$, and so $(a) + XE_S[X] \subseteq (a, X)_v$. Now let $b + Xf(X) \in R$ such that $(b + Xf(X))\{\frac{1}{s} : s \in S, \text{ and } s \mid a\}R \subseteq R$, then b is divisible in E by every distinct $p_i^{a_i}$ that divides a and so is divisible in E, by $a = \prod p_i^{a_i}$. So $b = ca$ in E, and so $b + Xf(X) \in (a) + XE_S[X]$. Finally, if $a = 0$, then $(XE_S[X])^{-1} = E_S[X]$, and so $(XE_S[X])_v = (E_S[X])^{-1} = XE_S[X]$.

Lemma 4.10. With R, E, and S as in Example 4.7, every prime ideal of R of the form $P + XE_S[X]$ is a t-ideal, where P is an ideal of E.

Proof. Now if $P = (p)$ is principal, then we know that $p \in S$, and so $(p) + XE_S[X] = p(E+XE_S[X])$ is principal, hence divisorial, and hence a t-ideal. So, let P be non principal. But then $P \cap S = \phi$, because in this case every nonzero element of P is an infinite product of primes. Now let $(a_1 + Xf_1(X), .., a_n + Xf_n(X))$ be a finitely generated ideal of R contained in $P + XE_S[X]$. Then $(a_1 + Xf_1(X), .., a_n + Xf_n(X)) \subseteq ((a_1, .., a_n) + XE_S[X]) = (a) + XE_S[X] \subseteq P + XE_S[X]$. Now $(a) + XE_S[X]$ is divisorial by Lemma 4.9. (Here $(a_1, .., a_n) = (a)$ because E is a Bézout domain.)

Now, let us call a finitely generated ideal I of an integral domain A bad if $(IA_S)_v \neq I_vA_S$, where S is some multiplicative set of A disjoint from I. Also, let us call a prime ideal P of A badly behaved if for some multiplicative set S of A, disjoint from P, PA_S is not a t-ideal. It is clear that any prime t-ideal that contains a finitely generated bad ideal is badly behaved.

Example 4.11. Let R, E, and S be as in Example 4.7, and let $a \in E$ be an infinite product of powers of principal primes. Then (a, X) is a bad ideal in

$R = E + XE_S[X]$ and every prime ideal of the form $P + XE_S[X]$ with $P \cap S = \phi$ is a badly behaved prime t-ideal.

Illustration. Note that $(a, X) \cap S = \phi$ and that $((a, X)R_S)_v = ((a, X)E_S[X])_v = E_S[X]$, yet $(a, X)_v R_S = ((a) + XE_S[X])E_S[X] = (a, X)E_S[X] \neq E_S[X]$. Next, each ideal of the form $P + XE_S[X]$ with $P \cap S = \phi$ contains bad ideals of the form (a, X), where a is an infinite product of principal primes in E, and hence is badly behaved.

Obviously a prime t-ideal P would be well behaved if for each multiplicative set S disjoint from P, PA_S is a t-ideal. Well behaved prime t-ideals were studied in [Z90]. Domains in which every prime t-ideal is well behaved may be called well behaved domains. These domains were characterized in [Z90] as domains A in which no finitely generated nonzero ideal is bad, i.e., for each finitely generated nonzero ideal I and for each multiplicative set S disjoint from I, we have $(IA_S)_v = I_v A_S$. So, most of the well known domains, Noetherian, Krull, PVMD, Prüfer, coherent are well behaved. Among the badly behaved domains are essential domains that are not PVMD's. If you come across a badly behaved domain, it is public service to report it; reporting it is just like putting a warning light close to submerged rocks at the sea.

The question of when $A + XB[X]$ is a GCD domain has been studied (also in the case of $A + XB[[X]]$) by Anderson and Nour-El-Abidine [AN 01] and by Dumitrescu, Radu, Salihi and Shah [DSRS00]. In both papers it is decided that $A + XB[X]$ is a GCD domain if and only if A is a GCD domain and $B = A_S$, where S is a splitting multiplicative set of A. This result extends: Given that A is a GCD domain and S a multiplicative set in A, $A + XA_S[X]$ is a GCD domain if and only if S is a splitting set, i.e., each $a \in A^*$ can be written as $a = bs$, where $s \in S$ and b is such that $(b) \cap (t) = (bt)$ for all $t \in S$ [Z88]. Later, in [AAZa91], the notion of splitting sets was strengthened into lcm splitting sets: splitting sets S of A that have the extra property that for each $s \in S$ and for each $a \in A$, $(a) \cap (s)$ is principal. This notion was then used to prove various Nagata type Theorems. Now, as I have already said in my talk, this notion has developed into t-splitting sets and has proved useful.

In [HM82], Heitmann and McAdam introduced an intriguing notion. They call an integral domain A with quotient field K a distinguished domain if for each $z \in K^*$, $(1) : (z) \nsubseteq Z(R/((1) : (z^{-1})))$. A distinguished domain is integrally closed. Prüfer domains, and Krull domains are distinguished domains and, obviously, a finite conductor distinguished domain is a PVMD [Z78]. (Here, A is a finite conductor domain if for all $a, b \in A$ we have $aA \cap bA$ is finitely generated.) From what they did with the distinguished domains, it seemed as if it was another way of looking at PVMD's. When Evan Houston paraded that paper in front of me (in 1987 or 88), I blurted out ... it is just PVMD's. Then I got busy in first writing a few hurried papers and then in falling ill. In 1998, I happened to look at that paper again and realized the mistake I had made. I talked to Dan Anderson, who quite reluctantly agreed to work with me on it. His reluctance was partly because those days I often faltered even in speech and partly because working on the work of two big giants is no joke. In any case we wrote together with Kwak [AKZ]. We found out, for instance, that A is a distinguished domain if and only if for each $z \in K^*$, we can write $z = \frac{a}{b}$, $a, b \in A$, where $b \notin Z(R/((a) : (b)))$ if and only if for each

$z \in K^*$, we can write $z = \frac{a}{b}$, $a, b \in A$ such that $(a) : (b) = (a) : (b^2)$. Using these new criteria, we showed the strong links between the PVMD's and the distinguished domains. For instance, we showed that a two-dimensional distinguished domain is a PVMD and that if A is a PVMD, then $A[X]$ is distinguished, though this does not quite prove that each PVMD is distinguished. However, I feel that a PVMD is distinguished, and I would be very happy if I am proven wrong (or right for that matter). Now to make the long story short, in [AKZ] we constructed an example which fits here nicely.

Example 4.12. Let $(V, (p))$ be a discrete rank $n-1$ valuation domain for $n \geq 3$, and let $Q = \bigcap_{n=1}^{\infty} p^n V$. Then $R = V + XV_Q[X]$ is an n dimensional distinguished domain that is not a PVMD.

The verification that R is a distinguished domain would take a lot more explanation than I can afford to put here. (If you are interested, look up the paper if and when it appears, or ask me for a copy.) So, we settle for a quick proof of the fact that R is not a PVMD. For this, all we have to note is that the prime ideal $Q + XV_Q[X] = \bigcap_{n=1}^{\infty} p^n R$ is a t-ideal. Being an intersection of principal ideals, it is actually divisorial. In any case, if $S = \{p^n\}$, then $(Q + XV_Q[X])R_S$ $= (Q + XV_Q[X])V_Q[X]$, which is not a t-ideal of $V_Q[X]$, as for every nonzero $q \in Q$, $((q, X)V_Q[X])_v = V_Q[X]$. I have found this construction from a valuation domain a useful source of examples. For instance, Example 4.12 can be used to provide examples of Schreier domains that are not GCD domains via Proposition 4.5. Moreover, the construction in Example 4.12 provides examples of non-Prüfer integrally closed stably strong S-domains.

Recall that a prime ideal P of the ring of polynomials $A[X]$ is called an upper to zero if $P \cap A = (0)$. PVMD's can be characterized as integrally closed integral domains A such that every prime upper to zero contains a polynomial $f(X)$ such that $(c(f))_v = A$ ([Z84]). (Here $c(f)$ denotes the ideal generated by the coefficients of $f(X)$). Now a prime upper to zero, being a minimal prime of a principal ideal, is a prime t-ideal [HH80] (see also [Z78]). Moreover, a prime t-ideal that is t-invertible is a maximal t-ideal [HZ89], and any prime ideal of $A[X]$ that contains a polynomial $f(X)$ with $(c(f))_v = A$ is t-invertible, and conversely. Putting this information all together, we conclude that an integrally closed integral domain A is a PVMD if and only if every prime upper to zero of $A[X]$ is a maximal t-ideal. In [HZ89], we studied integral domains in which every (prime) upper (U) to zero is a maximal (M) t-ideal (T) and called them UMT domains. It was mainly Evan Houston's hobby horse and it has stayed that way ever since. (An interested reader may want to read [Z84].) My contribution to the idea was at best an observation here and a question there. In any case, in a recent paper [FGH98] the authors dub UMT domains as,"PVMD's with the integrally closed hypothesis removed". To prove their point, they show that the UMT property survives polynomial ring formation and taking rings of fractions and that a UMT-domain is indeed well behaved. Now, as I have already mentioned, given that A is a PVMD and S a multiplicative set in A, $R = A + XA_S[X]$ is a PVMD if and only if S is a t-splitting set if and only if (a, X) is t-invertible in R for each $a \in A$, I ask the following question.

Question 4.13. Let A be a UMT-domain, let S be a multiplicative set in A, and let X be an indeterminate over A_S. Under what conditions is $R = A + XA_S[X]$ a UMT-domain?

My conjecture is that with all that is given in Question 4.13, $R = A + XA_S[X]$ is a UMT-domain, if and only if, (a, X) is t-invertible in R for each $a \in A$. If you do get interested in this question, look up [AAZ01] where the PVMD case is dealt with, and look up [FGH98], where questions like Question 4.13 are considered for a couple of kinds of pullback constructions. Another thing that might be useful is the fact that A is a UMT-domain if and only if for each maximal t-ideal P of A, the integral closure of A_P is a Prüfer domain.

5. Part 5 (Other links)

In this part, I intend to include topics that I would have liked to treat in detail, but the thought of this survey getting too long keeps me from doing so.

In [CMZ78] it was shown that if D is a Noetherian domain and S any multiplicative set in D, then $D + XD_S[X]$ is coherent (finite intersection of finitely generated ideals is finitely generated). In this connection here is a scheme of examples of coherent rings that might be of some interest. (These examples are more in the spirit of the generalized $D + M$ construction of [BR76].) For this, let us recall that an integral domain A is a an h-local ring if each nonzero prime ideal P of A is contained in a unique maximal ideal of A, and each nonzero ideal of A is contained in at most a finite number of maximal ideals. Clearly, each quasi local domain is h-local. The standard reference for h-local domains is Matlis' book [Mat72]. Looking at the definition then, we have the rather obvious proposition.

Proposition 5.1. If A is a quasi-local domain and $K = qf(A)$, then $A + XK[X]$ is h-local. If in addition A is coherent, then so is $A + XK[X]$.

The proof is straight-forward and so is left to the reader.

In [N91], an integral domain A was said to have property P* if for every finitely generated nonzero ideal I of A, we have that I^{-1} is of finite type. It turns out that the property P* is equivalent to the property that $I_v \cap J_v$ is of finite type for finitely generated I and J [FG, Proposition 3.6]. Following [FG], the P* property was dubbed as the v-coherent property in [GH97]. In [GH97], the authors study the v-coherent property in the pullbacks stemming from the diagram

$$\begin{array}{ccc} S = \pi^{-1}(A) & \hookrightarrow & B \\ \downarrow & & \downarrow_{\pi} \\ A & \hookrightarrow & B/M \end{array}$$

, where M is a maximal ideal of B and A is a subring of B/M. Now, the v-coherent property is interesting in that a number of well known domains, PVMD's, Krull domains, and Mori domains are v-coherent. (Recall that an integral domain A is a Mori domain if A satisfies ACC on integral divisorial ideals. Equivalently, A is Mori if and only if each nonzero ideal I of A contains a finitely generated ideal J such that $I_v = J_v$.) On the other hand, there is no dearth of non v-coherent domains. Indeed, as we have already mentioned in Part 4, a v-coherent v-domain is a PVMD (This is also [GH97, Proposition 3.2]) . So any v-domain that is not a PVMD is not v-coherent. Now an essential domain being a v-domain, whenever we find an essential domain that is not a PVMD we have a non v-coherent domain. This means that Example 4.7 of this article is a simple enough example of a non v-coherent ring.

Let us also note that an h-local ring A is a ring of finite character, in that there is a set $\Omega = \{P\}$ of prime ideals such that $A = \bigcap_{P \in \Omega} R_P$ and each nonzero nonunit of A belongs to at most a finite number of members of Ω. A Noetherian ring A is also a ring of finite character in that $\Omega =$ the set of maximal primes of principal ideals. In fact, as maximal primes of a principal ideal in a Noetherian domain are t-ideals, being of grade one, we can say that a Noetherian ring is of finite t-character. Some examples of rings of finite t-character can be found in [DLMZ]. The most well known rings of finite t-character are the rings of Krull type of Griffin's [Gri68]. Rings of Krull type are PVMD's of finite t-character. There are a number of examples of rings of finite t-character given in [DLMZ], some of them use the $A + XB[X]$ construction, and some of these examples are not rings of Krull type. One example of a ring of finite t-character, that is not integrally closed, is Example 3.17 of this article. It appears that we could do with a few more examples of rings of finite t-character that are not rings of Krull type nor are Mori domains, and I am sure that the $A + XB[X]$ construction would provide plenty of them.

It is odd but true that the $A+XB[X]$ construction can provide examples that are simple enough to be taken to the undergraduate classroom. I recall having written a paper [JZ96], in collaboration with Tess Jackson, when I was at Winthrop College. This paper appeared in a journal called Primus and never got reviewed, so very few people know about it. In this paper, we used the $A + XB[X]$ construction to give examples of non-Noetherian, non UFD rings that satisfy ACC on principal ideals, and hence are atomic, such as $Q + XR[X]$ where Q represents rationals and R represents reals respectively. Of course there were other examples, but of importance, in my opinion, was the appeal to the algebraists to be on the look-out for good and simple examples in their area of research. Simple examples, without a lot of jargon, in my opinion, enhance understanding and help remove the impression that algebra is hard. Giving undergraduates more examples has another advantage, these youngsters will remember them and may later find ways of using them. Now that I have a chance, let me beat the drum for simplicity a bit harder. I will not make a long argument. God Almighty wanted us to learn about the so called Pythagoras Theorem, so He put the examples right under our noses, as close as 3, 4, 5. It seems to me that the Babylonians were so impressed that they even used the product of these digits to devise their, sexagecimal, number system. Of course some problems need sustained effort, and for them we have the deductive system, but that again is a way of breaking up a harder problem into simpler steps. Let us keep it that way. Coming back to [JZ96] and to Tess. Tess died in a car accident a couple of years after the publication of the paper. She was an eager pedagogist and a very pleasant person, may she rest in peace. She died young, in fact her promotion to Associate Professorship came posthumously.

Gabriel Picavet [Pic97] has written an article on "composite rings", meaning the rings of the form $R = A + XB[X]$. He forgot to mention that there is, already, extensive material on these rings. But, what he has produced is a useful article. If you are interested in category theory, it is a must read. He shows for instance, that the symmetric algebra $S_A(B)$ maps onto R. He also handles the direct limit questions very well. What is more impressive from my point of view is that he actually considers rings (which may have zero divisors). I have been hoping to do it once I settle down some place. In any case, now we have [Pic97] and I expect some good papers coming out of it.

G.W. Chang, a student of B.G. Kang, has recently sent me a preprint on a generalization of Krull domains [Cha]. He comes up with the rather interesting notion of a Pinched Krull domain (pinched) at a prime ideal. An integral domain A may be called a pinched Krull domain at prime ideal P, if I is t-invertible for every ideal $I \subsetneq P$. (Recall that A is a Krull domain if and only if every nonzero ideal of A is t-invertible, see [Ja60, p82] or [MZ91, Theorem 2.5].) Indeed, it attaches Krull's name to some very unlikely rings: a one dimensional (quasi) local domain is pinched Krull if pinched at its maximal ideal and so is a discrete rank 2 valuation domain (V, pV) if pinched at $\bigcap p^n V$. On the other hand, A is a Krull domain if A is pinched Krull at (0). What is of interest here is the fact that if A is a Krull domain and K is a field containing A, then $A + XK[X]$ is pinched Krull at $XK[X]$. Chang [Cha] also mentions an old example of an integrally closed non-Mori domain, whose prime t-ideals are all of finite type. The example appeared in [MZ91, page 564], and it was constructed as $D = \overline{Q}[\pi] + XR[X]$, where $\overline{Q}$ represents the algebraic closure of rationals in the reals R and π is your most notorious irrational. In Chang's terminology $\overline{Q}[\pi] + XR[X]$ is also a pinched PID (UFD) at $XR[X]$ in that every ideal (principal ideal) not contained in $XR[X]$ is principal (respectively a product of principal primes). What interests me in this example is the easily verifiable fact that every element of $\overline{Q}[\pi] + XR[X]$ that is expressible as a product of irreducible elements is a product of primes and hence has a unique factorization. This is something that does not hold in $Q + XR[X]$, (or in $\overline{Q} + XR[X]$) because in this ring X is an irreducible element that is not a prime because $X \mid (\frac{X}{\pi})^2$ and $X \nmid \frac{X}{\pi}$. However, as indicated in [AAZ91, page 121], $Q+XR[X]$ is a, so called, half factorial domain, i.e., every nonzero non-unit f of $Q+XR[X]$ has the property that every expression of f as a product of irreducibles has the same length. The notion of HFD was introduced by Zaks [Zak76], [Zaks76]. An interested reader may also want to look up Skula's paper [Sku76] on c-semigroups for a comparative study.

It may be noted that an integral domain A is an HFD, if A is atomic, and every nonzero non-unit f of A has the property that, every expression of f as a product of irreducibles has the same length. This reminds me of an old observation of mine. Abraham Zaks started [Zak76], with the following: "Let R be a commutative domain with 1. We call R a Half Factorial domain (HFD) provided the equality $\prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j$ implies $m = n$, whenever the x's and the y's are non unit and irreducible elements of R." Then he went on to mention a general criterion that required each of the nonzero nonunit elements of his HFD to have a "complete decomposition" as a product of irreducible elements. Now, let us call A an Unrestricted Half Factorial domain (UHFD) provided the equality $\prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j$ implies $m = n$ whenever the x's and the y's are non unit and irreducible elements of A.

For the ease of expression, if $x \in A$ is an element that is expressible as a product $x = \prod_{i=1}^{n} x_i$, where x_i are atoms, we shall say that x has an atomic factorization of length n. So, in simple language, A is an UHFD if for every element x of R that has an atomic factorization of length n, every other atomic factorization of x will be of the same length. Just to establish that the class of proper UHFD's is nonempty, let us see some straight-forward examples. Obviously every HFD is an UHFD. Now we can use the following proposition to make as many examples as we want.

Proposition 5.2. Let A be an UHFD that is not a field, let K be the quotient field of A, and let X be an indeterminate over K. Then $A + XK[X]$ is an UHFD.

Proof. A typical element $f(X)$ of $R = A + XK[X]$, can be written as $f(X) = a_0 + Xg(X)$ where $a_0 \in A$ and $g(X) \in K[X]$. Now if $a_0 = 0$, then as X is divisible by every nonzero element of A, there is no hope of writing $f(X)$ as a product of atoms. If, on the other hand $a_0 \neq 0$, then $f(X) = a_0(1 + Xh(X))$. Now it is easy to show that $(1 + Xh(X))$ is a product of primes of R, so that in this case $f(X)$ is expressible as a product of atoms if and only if a_0 is expressible as a product of atoms. Now as all the atomic factorizations of a_0 are of the same length, we have that all the atomic factorizations of $f(X)$ have the same length.

Of interest here is the fact that in $f(X) = a_0(1+Xh(X))$, the factor $(1+Xh(X))$ is a product of primes, and hence is an inert constituent of the factorization of $f(X) = a_0(1 + Xh(X))$. Thus, if $f(X)$ is a product of atoms, then $f(X)$ would have all the properties of atomic factorization that a_0 has. So the above proposition is a scheme for defining unrestricted UFD's as well.

Clearly, every GCD domain is an unrestricted UFD, because in a GCD domain each atom is a prime. Another example comes from the so called pre-Schreier domains, integral domains whose group of divisibility is a Riesz group. It was shown by Cohn [C68] that in a Schreier domain every atom is a prime. Using the same procedure, we can show that every pre-Schreier domain is an unrestricted UFD. Another example of unrestricted UFD's is the rings A, that satisfy the property that every primitive polynomial of $A[X]$ is super primitive. Here $f \in A[X]$ is primitive if $c(f)$ is not contained in any proper principal ideal of A, and f is super primitive if $(c(f))^{-1} = A$. It was shown by Arnold and Sheldon in [AS75] that in a domain with PSP property every atom is a prime. It would be interesting to find other examples of unrestricted HFD's and UFD's etc. One could address a question such as: Suppose that $A \subseteq B$ is an extension of HFD's such that $B[X]$ is an HFD. Must $A + XB[X]$ be an UHFD? It may not be a very hard question but often the simple questions lead to more interesting situations. In this connection the reader may look up [BIK95, section 3].

These days a new variation of $A + XB[X]$ construction is appearing and it is a welcome new addition. It is more in the tradition of $D+I$, but sufficiently pleasant for me to remember it. It starts with a domain A, picks an ideal I in A, and forms the ring: $A + XI[X]$. A study of $A + XI[X]$ was carried out in [GPR01] assuming that A is a UFD, and recently Sebastian Pellerin has sent me a preprint [PR] that carries out a similar study with A a Dedekind domain. Let me end the article with a brief description of elasticity, the topic of study of these articles. Let A be an atomic domain, and let $\mathcal{A}(A)$ denote the set of all irreducible elements of A. Then $\rho(A) = sup\{m/n : x_1...x_m = y_1...y_n,\ x_i, y_j \in \mathcal{A}(A)\}$ is called the elasticity of A. From the definition it follows that $1 \leq \rho(A) \leq \infty$ and that A is an HFD if and only if $\rho(A) = 1$. In [GPR01], for instance, it was shown that if A is a UFD and I is a non-prime ideal then $\rho(A + XI[X]) \geq 2$ and that for A a UFD, $A + XI[X]$ is an HFD if and only if I is a prime ideal.

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