INTEGRAL DOMAINS IN WHICH ANY TWO v-COPRIME ELEMENTS ARE COMAXIMAL

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Abstract. Domains in which the star operations d (the trivial star operation) and w coincide have received a good deal of attention recently. These are exactly the domains D in which $I = D$ whenever I is a finitely generated ideal of D with $I^v = D$. In this work, we study what happens when "finitely generated" is replaced by "two-generated." It turns out that these are precisely the domains in which $d = F$, where F is a certain star operation closely connected to, but more complicated than, the w-operation.

Introduction

Throughout this work, D denotes a domain, and K denotes its quotient field. We recall the v-operation: For a nonzero fractional ideal I of D, we set $I^{-1} = (D : I) = \{u \in K \mid uI \subseteq D\}$ and then $I^v = (I^{-1})^{-1}$. (The map $I \mapsto I^v$ is an example of a *star operation*; we review pertinent definitions below as needed.) We say that nonzero elements $a, b \in D$ are *v-coprime* if $(a, b)^v = D$ and *comaximal* if $(a, b) = D$. It is easy to see that a and b are v-coprime if and only if $(a, b)^{-1} = D$ if and only if $(a) \cap (b) = (ab)$. The primary purpose of this work is to study *DF-domains*, domains D in which $a, b \in D$ are comaximal whenever a, b are v-coprime. The terminology arises as follows. In [3] H. Adams studied F-prime (shortened from factorization-prime) ideals. These are primes that contain no pair of v-coprime elements. She called an ideal I of D an F-ideal if whenever $a, b, x \in D$ with $(a, b)^v = D$ and $x(a, b) \subseteq I$ we have $x \in I$. As is pointed out in [16], an F-ideal is precisely an ideal I satisfying $I^F = I$ for a certain star operation F on D, and we shall show that DF-domains are precisely those domains for which the d-operation (the identity star operation) is identical to the F-operation.

Examples of DF-domains include Prüfer domains and one-dimensional domains. If fact, these are examples of *DW-domains*, that is, domains in which the two star operations d and w (reviewed below) coincide. DW-domains were introduced (but not named) in [7] and further studied in [8] (where they were called *t-linkative domains*), [26], [28], and [29]. It is easy to see that D is a DW-domain if and only if I is principal for each finitely generated ideal I of D such that I^v is principal (see [28, Proposition 2.1]). Hence DW-domains are DF-domains, but we shall show (Proposition 5.2) that DF-domains form a properly larger class.

Recall that GCD-domains may be characterized as those domains D in which $(a, b)^v$ is principal for all nonzero $a, b \in D$. Now, it is well known that if $(a, b)^v = (d)$ for a *given* pair of elements a, b in a domain D, then $\gcd(a, b)$ exists and is equal to d, but the converse is false. Thus domains D in which (a, b) is principal whenever

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a, b are elements of D such that $\gcd(a,b)$ exists might be expected to form a strictly smaller class that the class of DF-domains. This is indeed the case. In fact the property just mentioned is easily seen to be equivalent to $(a,b) = D$ whenever a, b are elements of D for which $\gcd(a,b) = 1$, and domains with this property were called *pre-Bézout* domains by Cohn [6]. Interestingly, the "finitely generated version" of this property has recently been studied by Park and Tartarone: they call a domain D *GCD-Bézout* if $(a_1, \ldots, a_n) = (d)$ whenever $a_1, \ldots, a_n \in D$ and $\gcd(a_1, \ldots, a_n) = d$.

In Section 1 we review terminology of star operations and study two particular star operations, the F- and t_2-operations, both defined in [16]. In Section 2 we give several characterizations of DF-domains, study their properties, compare and contrast the class of DF-domains with the other classes mentioned above, and explore what happens when we combine the DF-property with other well-studied properties (such as GCD, Krull). Section 3 is devoted to studying localization. We prove that a domain D for which D_M is a DF-domain for each maximal ideal M of D is a DF-domain, but we also give an example of a DF- (in fact, a DW-) domain D with a maximal ideal M such that D_M is not DF, thus answering a question left open in [28]. We also consider other properties locally, proving, for example, that a domain D is a Prüfer domain if and only if it is a DF-domain that is locally a GCD-domain and is such that F-primes localize (to F-primes). We devote a brief Section 4 to connections with regular sequences. Our main result here is a generalization of the fact that in a Noetherian domain D, an ideal I has (classical) grade at least 2 if and only if $I^{-1} = D$ [25, Exercise 2, page 102]. In Section 5 we analyze an example of Uda [30] to show that the DF-property is weaker than the DW-property. We also study the behavior of the DF-property in pullbacks, yielding many more examples of DF-domains (that are not DW-domains). Finally, in Section 6, we consider polynomial and Nagata rings. We show, for example, that $D[X]$ is a DF-domain if and only if D is a field.

1. The F- and t_2-operations

We begin by recalling some basic facts about star operations. Denote by $\mathbf{F(D)}$ (resp., $\mathbf{f(D)}$) the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of D. A *star operation* on D is then a mapping $I \mapsto I^*$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ such that for all nonzero $a \in K$ and $I, J \in \mathbf{F}(D)$,

(1) $(aD)^* = aD$ and $aI^* = (aI)^*$;
(2) $I \subseteq I^*$, and $I \subseteq J$ implies $I^* \subseteq J^*$; and
(3) $(I^*)^* = I^*$.

For any star operation $*$ on D, two new star operations $*_f$ and $*_w$ can be constructed by setting, for $I \in \mathbf{F}(D)$, $I^{*_f} = \bigcup\{J^* \mid J \subseteq I \text{ and } J \in \mathbf{f}(D)\}$ and $I^{*_w} = \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathbf{f}(D) \text{ with } J^* = D\}$. A star operation $*$ on D is said to be of *finite type* if $* = *_f$; hence $*_f$ and $*_w$ are of finite type. An ideal $I \in \mathbf{F}(D)$ is said to be a $*$-*ideal* if $I^* = I$, and a $*$-ideal is called a *maximal* $*$-*ideal* if it is maximal among proper integral $*$-ideals. We denote by $*$-Max(D) the set of maximal $*$-ideals of D. Assuming D is not a field, it is known that each maximal $*$-ideal is prime, that $*_f$-maximal ideals are plentiful in the sense that each nonzero $*_f$-ideal (and hence each nonzero element) of D is contained in a maximal $*_f$-ideal, that a prime ideal minimal over a $*_f$-ideal is itself a $*_f$-ideal, and that $*_f$-Max$(D) = *_w$-Max(D) [5, Theorem 2.16]. Also, if $I \in \mathbf{F}(D)$, then

$I^{*_w} = \bigcap_{P \in *_f\text{-Max}(D)} ID_P$ [5, Corollary 2.10], and hence $I^{*_w} D_P = ID_P$ for each $P \in *_f$-Max(D). The best-known star operations are the d-, v- (defined above), t-, and w-operations. The d-operation is just the identity function on $\mathbf{F}(D)$, so that $d = d_f = d_w$. The t-operation (resp., w-operation) is given by $t = v_f$ (resp., $w = v_w$). For two star operations $*_1$ and $*_2$ on D, we write $*_1 \leq *_2$ when $I^{*_1} \subseteq I^{*_2}$ for all $I \in \mathbf{F}(D)$ (and $*_1 < *_2$ when $*_1 \leq *_2$ but $*_1 \neq *_2$). It is known that $d \leq *_w \leq *_f \leq * \leq v$, $*_w \leq w$, and $*_f \leq t$ for any star operation $*$ on D.

We next recall the definitions of the t_2- and F-operations.

Definition 1.1. Let $J \subseteq K$ and $I \in \mathbf{F}(\mathbf{D})$.

(1) For the t_2-operation: Set $J' = \bigcup\{(a,b)^v \mid a, b \in J\}$. Then set $I_0 = I$, $I_n = (I_{n-1})'$ for $n > 0$, and $I^{t_2} = \bigcup_{k=0}^{\infty} I_k$. The t_2-operation was shown in [16] to be a finite-type star operation.
(2) For the F-operation: Set $J' = \{x \in K \mid x(a,b) \subseteq J$ for some $a, b \in J$ with $(a,b)^v = D\}$. Then let $I_0 = I$, $I_n = (I_{n-1})'$ for $n > 0$, and $I^F = \bigcup_{k=0}^{\infty} I_k$. It was observed in [16] that this defines a finite-type star operation on D (but most of the necessary details were already present in [3]).

Observe that the t_2- and F-operations are similar to the t- and w-operations, the differences being that finite subsets are replaced by two-element subsets and iteration is required. Clearly, we have $F \leq t_2$, $F \leq w$, and $t_2 \leq t$. In [16], an example was given showing that it is possible to have $F < t_2$; in fact, in that example, it is easy to see that we have $d = F = w < t_2$. In Example 5.1 below, we show that it is possible to have $F < w$ and $t_2 < t$, answering questions posed in [16].

Although the t_2- and F-operations are defined inductively, only one step is needed to determine whether a given ideal is a t_2- or F-ideal:

Lemma 1.2. *Let I be a nonzero ideal of a domain D. Then the following statements hold.*

(1) *I is a t_2-ideal if $(a,b)^v \subseteq I$ whenever $a, b \in I$.*
(2) *I is an F-ideal if $x \in I$ whenever $x(a,b) \subseteq I$ with $x, a, b \in D$ and $(a,b)^v = D$.*
(3) *I is a prime F-ideal (F-prime) if I does not contain any pair of v-coprime elements.*

Proof. Statements (1) and (2) follow easily from the definitions. For (3), suppose that I is as hypothesized and that $x(a,b) \subseteq I$ with $(a,b)^v = D$. Then, $(a,b) \nsubseteq I$, so that we must have $x \in I$. Hence I is an F-ideal by (2). $\square$

As has already been mentioned, for any star operation $*$ on D, we may define $*_w$ by $I^{*_w} = \bigcup\{(I : J) \mid J$ is a finitely generated subideal of I and $J^* = D\}$, and we have $v_w = t_w = w$.

Proposition 1.3. *For any domain D, the F- and F_w-operations on D are identical.*

Proof. Since $F_w \leq F$ by definition, it suffices to show that each F_w-ideal is also an F-ideal. Accordingly, let I be an F_w-ideal of D, and suppose that $x, a, b \in D$ are such that $(a,b)^v = D$ and $x(a,b) \subseteq I$. Since $1(a,b) \subseteq (a,b)$ and $(a,b)^v = D$, we have $(a,b)^F = D$ and hence $x \in I^{F_w} = I$. The result now follows from Lemma 1.2. $\square$

For any $*$-operation on D, it is known that if P is a $*_w$-prime of D, then every prime ideal contained in P is also a $*_w$-prime. Hence we have the following:

Corollary 1.4. *If P is an F-prime of D, then so is every nonzero prime of D contained in P.* □

Questions 1.5. Let D be a domain.

1. Must we have F-Max$(D) \subseteq t_2$-Max(D)?
2. Must we have F-Max$(D) = t_2$-Max(D)?
3. If I is an ideal of D with $I^{t_2} = D$, do we necessarily have $I^F = D$?
4. Do we have t_2-Max$(D) \subseteq F$-Max(D)?
5. What conditions on D ensure $t_2 = t$?
6. In general, we have $F = F_w \leq (t_2)_w \leq w$. When do we have $F = (t_2)_w$ or $(t_2)_w = w$?

It is not difficult to show that Questions (1)-(3) are equivalent:

Lemma 1.6. *Suppose that $*_1 \leq *_2$ are finite-type star operations on D. Then the following statements are equivalent.*

1. *$*_1$-Max$(D) \subseteq *_2$-Max(D).*
2. *$*_1$-Max$(D) = *_2$-Max(D).*
3. *If I is an ideal of D with $I^{*_2} = D$, then $I^{*_1} = D$.*

Proof. Assume (1), and let $M \in *_2$-Max(D). Since $*_1 \leq *_2$, we have $M^{*_1} \neq D$. Hence M is contained in a maximal $*_1$-ideal N of D. However, by assumption, this yields $N \in *_2$-Max(D), and we must therefore have $M = N$, that is, $M \in *_1$-Max(D). Thus (1) $\Rightarrow$ (2). Assume (2), and let I satisfy $I^{*_1} \neq D$. Then $I \subseteq M$ for some $M \in *_1$-Max$(D) = *_2$-Max(D), and we have $I^{*_2} \subseteq M \subsetneq D$. Hence (2) $\Rightarrow$ (3). Finally, assume (3), and let $M \in *_1$-Max(D). Then $M^{*_1} \neq D$, whence, by assumption, $M^{*_2} \neq D$. Since M^{*_2} is a $*_1$-ideal and $M \subseteq M^{*_2}$, this yields $M = M^{*_2}$. Thus M is a $*_2$-ideal. Since every $*_2$-ideal is also a $*_1$-ideal, M cannot be contained in a larger $*_2$-ideal, i.e., $M \in *_2$-Max(D). □

Recall that if $*$ is a star operation on D, then we say that D has *finite $*$-character* if each nonzero element of D is contained in only finitely many maximal $*$-ideals of D. (When $* = d$, one says that D has finite character.)

Proposition 1.7. *If D has finite t_2-character, then t_2-Max$(D) = F$-Max(D).*

Proof. Suppose that D has finite t_2-character, and let $M \in F$-Max(D). If M is not a t_2-ideal, then, since every t_2-ideal is a F-ideal, we have $M^{t_2} = D$. Choose a nonzero element $a \in M$. Then a is in only finitely many maximal t_2-ideals, and, since $M^{t_2} = D$, we may use prime avoidance to find $b \in M$ with (a, b) in no maximal t_2-ideal, that is, $(a, b)^{t_2} = D$. However, this yields $(a, b)^v = D$, contradicting that M is a maximal F-ideal. Thus M must be a t_2-ideal and hence a maximal t_2-ideal. The result now follows from Lemma 1.6. □

Proposition 1.8. *If D has finite t-character, then t-Max$(D) = t_2$-Max$(D) = F$-Max$(D) = w$-Max(D). In particular, finite t-character implies both finite t_2- and finite F-character.*

Proof. Assume that D has finite t-character, and let M be a maximal t_2-ideal of D. If M is not a t-ideal, then $M^t = D$, and, as in the proof of Proposition 1.7, we can find $a, b \in M$ with (a, b) in no maximal t-ideal of D. But then $(a, b)^v = D$, a contradiction. Hence t-Max$(D) = t_2$-Max(D), and D also has finite t_2-character. A similar conclusion for maximal F-ideals now follows from Proposition 1.7. Finally, it is well know that t-Max$(D) = w$-Max(D) in general ([5, Theorem 2.16]). $\square$

It follows from Proposition 1.7 that finite t_2-character implies finite F-character. However, it does not imply finite t-character–see Proposition 5.2 below.

In [22] the authors introduced the class of *TV-domains*, domains in which the t-operation coincides with the v-operation. By [22, Theorem 1.3], TV-domains have finite t-character, so that Proposition 1.8 applies to this class of domains. Now recall that a domain is a *Mori domain* if it satisfies the ascending chain condition on divisorial ideals. It was observed in [22] that the class of TV-domains includes (but is properly larger than) the class of Mori domains. In particular, Proposition 1.8 applies to Noetherian domains. Actually, for Mori domains, we can say a good deal more:

Proposition 1.9. *Let D be a Mori domain. Then every t_2-prime of D is a t-prime.*

Proof. Let P be a t_2-prime of D, and let a be a nonzero element of P. By [19, Theorem 2.1], a is contained in only finitely many t-primes of D. Use prime avoidance to choose $b \in P$ with b in no t-prime Q of D for which $a \in Q$ and $Q \subsetneq P$. Since P is a t_2-prime, $(a, b)^v \subseteq P$. Shrink P to a prime P_0 minimal over $(a, b)^v$. Then P_0 is a t-prime, and by construction we must have $P = P_0$. $\square$

We suspect that Questions (1) - (4) above have negative answers in general. With respect to Question 5, we do not even know whether $t_2 = t$ in a one-dimensional local Noetherian domain. (We do know from Proposition 5.2 below that $t_2 < t$ can occur (albeit in a domain that is far from being Noetherian).)

2. DF-domains

We begin this section with several characterizations of DF-domains. We recall the definition: The domain D is a DF-domain if for $a, b \in D$ with $(a, b)^v = D$, we have $(a, b) = D$. Now recall from [7] that an overring E of a domain D is *t-linked* over D if $(E : IE) = E$ whenever I is a finitely generated ideal of D with $I^{-1} = D$, equivalently, if $(JE)^{t_E} = E$ whenever J is an ideal of D with $J^t = D$. It was shown that every overring of D is t-linked over D if and only if every maximal ideal of D is a t-ideal, i.e., if and only if D is a DW-domain. In [9] the notion of t-linkedness was extended as follows. Given D and an overring E and star operations $*$ on D and $*_1$ on E, E is *$(*, *_1)$-linked over D* if $(JE)^{*_1} = E$ whenever J is an ideal of D with $J^* = D$.

Theorem 2.1. *The following statements are equivalent for a domain D.*

(1) *D is a DF-domain.*
(2) *$a, b \in D$ with $(a, b)^v = (d)$ implies $(a, b) = (d)$.*
(3) *$a, b \in D$ with $(a, b)^v$ principal implies (a, b) principal.*
(4) *Each nonzero ideal of D is an F-ideal; equivalently, the d- and F-operations on D are identical.*
(5) *Each maximal ideal of D is an F-prime.*
(6) *For every overring E of D, E is (F, F_E)-linked over D.*

Proof. (1) $\Rightarrow$ (2): Let D be a DF-domain, and let $a,b \in D$ with $(a,b)^v = dD$ for some $d \in D$. Then $(a/d, b/d)^v = (1/d)(a,b)^v = D$. Since D is a DF-domain, this yields $(a/d, b/d) = D$ and, therefore, $(a,b) = dD$.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): Assume (3). Let I be a nonzero ideal of D, and suppose that $x(a,b) \subseteq I$ with $(a,b)^v = D$. By (3) $(a,b) = (c)$ for some $c \in D$. Hence $D = (a,b)^v = (c) = (a,b)$, and we have $x \in I$. Therefore, $I^F = I$.

(4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (6): Assume (5), let E be an overring of D, and let I be an ideal of D with $I^F = D$. If $(IE)^{F_E} \neq E$, then IE is contained in a maximal F-ideal Q of E. Let M be a maximal ideal of D containing $Q \cap D$. Then M is an F-prime. However, $I^F = D$ and $I \subseteq M$, a contradiction.

(6) $\Rightarrow$ (1): Assume (6), and let $a,b \in D$ with $(a,b)^v = D$. Then $(a,b)^F = D$ also. Suppose, by way of contradiction, that (a,b) is a proper ideal of D, and let M be a maximal ideal containing (a,b). Then there is a valuation overring V of D whose maximal ideal N satisfies $N \cap D = M$. By assumption, we have $((a,b)V)^{F_V} = V$. However, every ideal of V is a t-ideal and hence also an F_V-ideal, and this yields $((a,b)V)^{F_V} = (a,b)V \subseteq N$, a contradiction. □

Since $F \leq w$ for all domains, the following is immediate.

Corollary 2.2. *A DW-domain is a DF-domain.* □

We consider another property stronger than DF. Recall that in [6] Cohn defined a *pre-Bézout* domain to be a domain D satisfying the following property: $a,b \in D$ with $\gcd(a,b) = 1$ implies $(a,b) = D$. We list a few equivalent conditions:

Lemma 2.3. *The following statements are equivalent for a domain D.*

1. *$a,b \in D$ with $\gcd(a,b) = d$ implies (a,b) principal.*
2. *$a,b \in D$ with $\gcd(a,b) = d$ implies $(a,b) = (d)$.*
3. *D is a pre-Bézout domain.*
4. *Each proper 2-generated ideal of D is contained in a proper principal ideal.*

Proof. Assume (1), and let $a,b \in D$ with $\gcd(a,b) = d$. Then (a,b) is principal, say $(a,b) = (c)$. Since $c \mid a$ and $c \mid b$, we have $(d) \subseteq (c)$. On the other hand, $(c) = (a,b) \subseteq (d)$. Hence (1) $\Rightarrow$ (2). That (2) $\Rightarrow$ (3) is trivial. Assume (3), and let $a,b \in D$ be such that (a,b) is contained in no proper principal ideal. Then $\gcd(a,b) = 1$, and we have $(a,b) = D$ (i.e., (a,b) is not a proper ideal) by (3). Thus (3) $\Rightarrow$ (4). Finally, assume (4), and let $a,b \in D$ with $\gcd(a,b) = d$. A standard argument yields $\gcd(a/d, b/d) = 1$, so that $(a/d, b/d)$ is not contained in a proper principal ideal. Thus $(a/d, b/d) = D$ by (4) and hence $(a,b) = (d)$. □

Now suppose that D is pre-Bézout, and let $a,b \in D$ with $(a,b)^v = D$. Then, as we have already observed, $\gcd(a,b)$ exists and is equal to 1 and hence $(a,b) = D$. This yields:

Corollary 2.4. *A pre-Bézout domain is a DF-domain.* □

The converse of Corollary 2.4 is false. Let $L \subsetneq k$ be fields, $\mathbf{X}$ a set of indeterminates over k with $|\mathbf{X}| \geq 2$, M the maximal ideal of $k[\mathbf{X}]$ generated by $\mathbf{X}$, and $D = L + Mk[\mathbf{X}]_{\mathbf{M}}$. It is well known that D is then a local domain whose maximal ideal is divisorial and hence a t-ideal. Since DW-domains are characterized as domains each of whose maximal ideals is a t-ideal ([26, Proposition 2.2] and [7, Lemma 2.1]), D is a DW-domain and hence a DF-domain. However, for $x \neq y \in \mathbf{X}$, we have $\gcd(x, y) = 1$, but $(x, y) \subsetneq D$.

In fact, we can characterize pre-Bézout domains among DF-domains. In [28] the authors call a domain D a GCD-Bézout domain if $(a_1, a_2, \ldots, a_n)$ is principal whenever $a_1, \ldots, a_n$ are elements of D with a greatest common divisor, and they show that a GCD-Bézout domain is a DW-domain. Indeed, in [28, Corollary 2.12], they characterize GCD-Bézout domains as DW-domains that satisfy the PSP-property of Arnold-Sheldon [2]. A domain D satisfies *PSP* (for primitive implies superprimitive) if for each finitely generated ideal I that is not contained in a proper principal ideal we have $I^v = D$. (This terminology arises as follows: an element $f \in D[X]$ is called *primitive* (resp. *superprimitive*) if $c(f)$, the ideal of D generated by the coefficients of f, is not contained in a proper principal ideal of D (resp., satisfies $c(f)^v = D$).) Following [27], let us say that a domain satisfies *LPSP*–for linear PSP– if each two-generated ideal I not contained in a proper principal ideal satisfies $I^v = D$ (that is, if each primitive *linear* polynomial is superprimitive). Then we have the following:

Proposition 2.5. *A domain D is pre-Bézout if and only if it is a DF-domain with LPSP.*

Proof. Suppose that D is pre-Bézout. Then D is a DF-domain by Corollary 2.4. Also, for $a, b \in D$ with (a, b) not contained in a proper principal ideal, we have $\gcd(a, b) = 1$, whence $(a, b) = D$ and then $(a, b)^v = D$. Therefore, D also satisfies LPSP. Now suppose that D is DF and satisfies LPSP, and let $a, b \in D$ with $\gcd(a, b) = 1$. Then (a, b) cannot be contained in a proper principal ideal, whence $(a, b)^v = D$ by LPSP. Since D is a DF-domain, we then have $(a, b) = D$. Therefore, D is pre-Bézout. $\square$

We don't know whether a pre-Bézout domain must be GCD-Bézout (but we doubt it). However:

Proposition 2.6. *A local pre-Bézout domain is GCD-Bézout.*

Proof. Let (D, M) be a local pre-Bézout domain. Then each proper 2-generated ideal of D is contained in a proper principal ideal by Lemma 2.3. We show that (in the local case) this extends to all finitely generated ideals. Thus let $(a_1, \ldots, a_n)$, $n > 2$, be a proper finitely generated ideal. By induction, we may assume that $(a_1, \ldots, a_{n-1}) \subseteq (b)$ for some $b \in M$. We also have $(a_n, b) \subseteq (c)$ for some $c \in M$, and hence $(a_1, \ldots, a_n) \subseteq (c) \subseteq M$, as desired. By [28, Proposition 2.6], D is a GCD-Bézout domain. $\square$

We next consider what happens when the DF-property is combined with other commonly considered properties.

Proposition 2.7. *A DF-domain of finite t-character is a DW-domain.*

Proof. Let D be a DF-domain of finite t-character, and let M be a maximal ideal of D. Then M is a maximal F-ideal and hence a maximal t-ideal by Proposition 1.8. Thus each maximal ideal of D is a maximal t-ideal, whence D is a DW-domain. $\square$

In particular, the DF- and DW-properties coincide for Noetherian domains. Noetherian DW-domains of arbitrary dimension (including ∞) exist–see [21, Examples 2.1 and 2.7].

Let $*$ be a star operation on a domain D. Then the *$*$-dimension* of D is the length of a longest chain of $*$-primes in D (where, for the purposes of this definition, (0) is counted as a $*$-prime). The next two results strengthen [26, Corollary 2.3].

Proposition 2.8. *Let D have F-dimension one. Then the following statements are equivalent.*

(1) $\dim(D) = 1$.
(2) *D is a DW-domain.*
(3) *D is a DF-domain.*

Proof. Since height-one primes are t-primes, we obtain (1) ⇒ (2) immediately, and (2) ⇒ (3) is easy (Corollary 2.2). Assume (3). Then, since each maximal ideal of D is an F-prime and primes within an F-prime are F-primes by Corollary 1.4, D must have dimension one. □

Observe that if D has finite t-character, then the t- and F-dimensions are the same by Proposition 1.8. It is well known that a Krull domain is a Dedekind domain if and only if it has dimension one. Then, since a Krull domain has finite t-character and has t-dimension one, we obtain:

Corollary 2.9. *Let D be a Krull domain. Then the following statements are equivalent.*

(1) *D is a DW-domain.*
(2) *D is a DF-domain.*
(3) *D is a Dedekind domain.* □

Since a Dedekind domain is a PID if and only if it is a UFD, we have:

Corollary 2.10. *Let D be a UFD. The following statements are equivalent.*

(1) *D is a DW-domain.*
(2) *D is a DF-domain.*
(3) *D is a PID.* □

We next give a direct proof of a result of Mott and the second author [27, Corollary 6.6]. Recall that a domain D is *atomic* if each nonzero, nonunit of D factors as a product of atoms (irreducible elements).

Corollary 2.11. *An atomic pre-Bézout domain is a PID.*

Proof. Let D be an atomic pre-Bézout domain. By Corollary 2.10 it suffices to show that D is a UFD, and for this it suffices to show that each atom is prime. Thus let a be an atom, and suppose that $a \mid bc$ for some $b, c \in D$. If (a, b) is not contained in a proper principal ideal, then by assumption, we may write $1 = ar + bs$ with $r, s \in D$; multiplication by c then yields that $a \mid c$. Suppose that $(a, b) \subseteq (d)$ for some nonunit d. Then $a = dt$, $t \in D$. Since a is an atom, and d is not a unit, t must be a unit. Therefore, since $d \mid b$, we have that $a \mid b$, as desired. □

Recall (see [1]) that a domain D is an *almost GCD-domain (AGCD-domain)* (resp., *almost Bézout domain* (ABD), almost Prüfer domain (APD), almost valuation domain (AVD)) if for all nonzero $a, b \in D$ there is a positive integer n for which $(a^n, b^n)^v$ is principal (resp., (a^n, b^n)) is principal, (a^n, b^n) is invertible, $a^n \mid b^n$ or $b^n|a^n$).

We have GCD and AGCD versions of Corollary 2.10; the latter strengthens [26, Corollary 2.6]. The other properties will be considered later.

Proposition 2.12. *Let D be a GCD-domain (resp., AGCD-domain). Then the following statements are equivalent.*

(1) *D is a DW-domain.*
(2) *D is a DF-domain.*
(3) *D is a Bézout domain (resp., AB-domain).*

Proof. We give the proof for the AGCD case; the proof for the GCD case is similar (and easier). That statement (1) implies statement (2) is trivial. Assume statement (2), and let $a, b \in D$. Since D is an AGCD-domain, we have $(a^n, b^n)^v$ principal for some positive integer n. The DF-assumption then yields that (a^n, b^n) is principal. Hence D is an AB-domain. This gives (2) $\Rightarrow$ (3). Finally, if D is an AB-domain, then D is a DW-domain by [26, Corollary 2.6]. □

It is clear that a domain D is local if and only if no two nonunits of D are comaximal. Since a local Bézout domain is a valuation domain, we have the following.

Corollary 2.13. *A domain D is a valuation domain if and only if it is simultaneously a GCD-domain and a DF-domain in which no two nonunits of D are comaximal.* □

Recall that a Prüfer v-multiplication domain (PVMD) may be characterized as a domain D for which D_M is a valuation domain for each maximal t-ideal M of D. Examples of PVMDs include Prüfer, Krull, and GCD-domains. It is easy to see that a PVMD that is also a DW-domain is a Prüfer domain (and this was observed in [29, page 1967]), but we do not know whether a domain that is both a PVMD and a DF-domain must be Prüfer. However, recall that a domain is said to be a *ring of Krull type* if it is a PVMD of finite t-character [15]. Then by Proposition 2.7 (and the fact that $d = t$ in a Prüfer domain):

Corollary 2.14. *The following statements are equivalent for a domain D.*

(1) *D is of Krull-type and is also a DF-domain*
(2) *D is of Krull-type and is also a DW-domain.*
(3) *D is a Prüfer domain of finite character.* □

3. Localization

In this section, we discuss localization in connection with the DF-property. We begin with some facts about the relation between the F-operation on a domain D and the F-operation on a ring of quotients of D.

Lemma 3.1. *Let D be a domain with overring E. Let $*$ (resp., $*_1$) be a star operation on D (resp. E). For each nonzero fractional ideal I of D, set $I^{\delta(*,*_1)} = (IE)^{*_1} \cap I^*$. Then:*

(1) *$\delta(*, *_1)$ is a star operation on D, and $\delta(*, *_1) \leq *$.*
(2) *If $I^* \subseteq (IE)^{*_1}$ for each fractional ideal I of D, then $\delta(*, *_1) = *$; in this case each $*_1$-ideal of E contracts to a $*$-ideal of D.*
(3) *If S is a multiplicatively closed subset of D, then $\delta(F, F_{D_S}) = F$ and hence F-ideals of D_S contract to F-ideals of D.*

Proof. (1) That $\delta(*, *_1)$ is a star operation on D follows immediately from [4, Theorem 2]. It is clear that $\delta(*, *_1) \leq *$.

(2) If $I^* \subseteq (IE)^{*_1}$, then $I^{\delta(*,*_1)} = (IE)^{*_1} \cap I^* = I^*$. Now let A be a $*_1$-ideal of E. Then, by what was just proved, $(A \cap D)^* = ((A \cap D)E)^{*_1} \cap (A \cap D)^* \subseteq A \cap (A \cap D)^* \subseteq A \cap D$.

(3) By (2) we need show only that $I^F \subseteq (ID_S)^{F_{D_M}}$ for each nonzero fractional ideal I of D. Let I be a nonzero fractional ideal of D, and let $x \in D$ be such that $x(a,b) \subseteq I$ for $a, b \in D$ with $(a,b)^v = D$. By [31, Lemma 4], $((a,b)D_S)^v = D_S$, whence $x \in (ID_S)^{F_{D_S}}$. It follows that $I^F \subseteq (ID_S)^{F_{D_S}}$, as desired. □

In [26, Theorem 2.9] Mimouni showed that a domain D for which D_M is a DW-domain for each maximal ideal M of D is itself DW. We have a similar result for the DF-property.

Proposition 3.2. *For a domain D, if D_M is a DF-domain for each maximal ideal M of D, then D is a DF-domain.*

Proof. Let M be a maximal ideal of D. Under the assumption that D_M is a DF-domain, we have that MD_M is an F-prime of D_M. By Lemma 3.1 M is an F-prime of D. Therefore, D is a DF-domain if each localization at a maximal ideal is DF. □

The converse of Proposition 3.2 is false. In fact, we next give an example of a DW-domain D with a maximal ideal M such that D_M is not a DF-domain. Note that this answers a question left open by Park and Tartarone [28, page 60].

Example 3.3. In [17] W. Heinzer and J. Ohm present an example of a domain D which is essential ($D = \bigcap D_{P_\alpha}$, where each P_α is a prime ideal of D and D_{P_α} is a valuation domain) but is not a PVMD. As further analyzed in [27] and [12], D has one height-two maximal ideal M, with M being a t-prime and D_M a regular local ring, and all other maximal ideals of D have height one (and are therefore t-primes). (Moreover, D_P is a rank-one discrete valuation domain for each height-one maximal ideal P; we use this fact below.) Thus D is a DW-domain, and hence a DF-domain, but, since MD_M is not an F-prime (since MD_M is 2-generated and satisfies $(MD_M)^{v_{D_M}} = D_M$), D_M is not a DF-domain.

As usual, we say that a domain has a given property locally if each localization at a maximal ideal has the property. Thus the example above is locally a PVMD. In fact, it is also locally a UFD (by the "moreover" statement in the example) and hence locally a GCD-domain and locally a Krull domain. The example "works" because MD_M is not an F-ideal. Recall from [32] that a domain D is *(conditionally) well behaved* if for each prime (maximal) t-ideal P of D, PD_P is a t-prime of D_P. Let us now call D *(conditionally) F-well behaved* if for each prime (maximal) F-ideal of D, PD_P is an F-prime of D_P. Then the D of the example is neither conditionally well behaved nor conditionally F-well behaved.

It is clear that a Prüfer (resp., almost Prüfer) domain is locally GCD (resp., AGCD). We next find conditions that yield a converse.

Lemma 3.4. *Let D be a local domain. Then the following statements are equivalent.*

(1) *D is an APD.*
(2) *D is an AVD.*
(3) *D is an ABD.*
(4) *D is both an AGCD-domain and a DW-domain.*
(5) *D is both an AGCD-domain and a DF-domain.*

Proof. The equivalence of (1), (2), and (3) follows from [1, Theorem 5.8]. Statements (3), (4), and (5) are equivalent by Proposition 2.12 (since an ABD is clearly an AGCD-domain). □

Proposition 3.5. *The following statements are equivalent for a domain D.*

(1) *D is an APD (resp., Prüfer domain).*
(2) *D is a well-behaved DW-domain that is locally AGCD (resp., locally GCD).*
(3) *D is an F-well-behaved DF-domain that is locally AGCD (resp., locally GCD).*

Proof. We give the proof for the "non-parenthetical" result. Let D be an APD, and let M be a maximal ideal of D. By [1, Theorem 5.8], D_M is an AVD and hence an AGCD-domain. In addition, PD_P is a t-prime of D_P for each t-prime P of D by [1, Lemma 5.2], i.e., D is well behaved. Finally, D is DW by [26, Corollary 2.11]. This gives (1) ⇒ (2). Now let D satisfy the conditions in (2), let P be a prime ideal (automatically an F-prime) of D, and let M be a maximal ideal of D containing P. By hypothesis D_M is an AGCD DW-domain and hence an AVD by the lemma. Therefore, D_P, as an overring of D_M, is an AVD, whence PD_P is (a t- and hence) an F-prime of D_P, as desired. This proves (2) ⇒ (3). Finally, suppose that D is an F-well behaved DF-domain that is also locally an AGCD-domain. If M is a maximal ideal of D, then D_M, being AGCD and DF, is an AVD-domain by the lemma. Hence D is an APD, again by [1, Theorem 5.8]. □

Recall that an *almost Dedekind domain* is a domain for which each localization at a maximal ideal is a rank-one discrete valuation domain.

Proposition 3.6. *The following statements are equivalent for a domain D.*

(1) *D is an almost Dedekind domain.*
(2) *D is a well behaved DW-domain that is also locally a Krull domain.*
(3) *D is an F-well behaved DF-domain that is also a locally a Krull domain.*

Proof. It is clear that (1) ⇒ (2). Let D be as in (2). Then for each maximal ideal M of D, D_M is DW and Krull and hence, by Corollary 2.9, a Dedekind domain. Thus D is in fact one dimensional, and (3) follows easily. Now let D be as in (3), and let M be a maximal ideal of D. Then D_M is both a DF-domain and a Krull domain and hence a (local) Dedekind domain by Corollary 2.9. Hence D_M is a rank-one discrete valuation domain. Therefore, (3) ⇒ (1). □

Similar arguments (using Corollary 2.14) yield the following result.

Proposition 3.7. *A domain D is a Prüfer domain if and only if D is an F-well behaved DF-domain that is locally a ring of Krull type.* □

4. Connections with classical grade

As in [25] we call a sequence $a_1, \dots, a_n$ of D of elements of D an R-sequence if $(a_1, \dots, a_n) \neq D$ and a_i is not a zero divisor on the module $D/(a_1, \dots, a_{i-1})$ for $i = 1, \dots, n$. The classical grade of an ideal I of D, denoted by $G(I)$, is then the length of a longest R-sequence of elements of I. We note that this is "delicate" in the non-Noetherian setting (Kaplansky refrains from defining it there), as Hochster [17] has shown that it is possible for an ideal in a domain to have maximal R-sequences of different lengths.

Now recall Exercises 1 and 2 on page 102 of [25]. According to Exercise 1, if an ideal I of D satisfies $G(I) \geq 2$, then $I^{-1} = D$. Exercise 2 then provides a converse in case D is Noetherian. Note that it follows immediately from Exercise 1 that the first two elements of any R-sequence in D are v-coprime. Now suppose that an ideal I not only satisfies $I^{-1} = D$ but actually contains two v-coprime elements a, b. If $bc \in (a)$ for some $c \in D$, then one sees immediately that $c/a \in (a, b)^{-1} = D$ and hence $c \in (a)$. Therefore, a, b is an R-sequence. We state this formally:

Proposition 4.1. *Let I be a nonzero proper ideal in a domain D. Then $G(I) \geq 2$ if and only if I contains a pair of v-coprime elements (and this pair is then an R-sequence). Thus $G(I) < 2$ for every ideal I of an DF-domain.* □

Corollary 4.2. *Let I be a proper finitely generated ideal of an integral domain D, and suppose that I contains an element a which belongs to only finitely many maximal t-ideals of D. Then $G(I) \geq 2$ if and only if $I^{-1} = D$.*

Proof. That $G(I) \geq 2$ implies $I^{-1} = D$ has already been discussed. Assume $I^{-1} = D$. Pick $a \in I$ with a contained in only finitely many maximal t-ideals of D. Since $I^{-1} = D$, I is contained in no maximal t-ideals of D, and we may use prime avoidance to pick $b \in I$ with (a, b) contained in no maximal t-ideal. We then have $(a, b)^{-1} = (a, b)^v = D$, whence a, b is an R-sequence by Proposition 4.1. □

Corollary 4.3. *Let D be a domain with finite t-character, and let I be a proper finitely generated ideal of D. Then $G(I) \geq 2$ if and only if $I^{-1} = D$.*

We have the following result, which both generalizes, and provides an easier path to a solution of, Exercise 2 of [25].

Corollary 4.4. *If I is an ideal of a TV-domain D, then $G(I) \geq 2$ if and only if $I^{-1} = D$.*

Proof. Let I be an ideal in the TV-domain D, and assume that $I^{-1} = D$. Then $I^t = I^v = D$, and hence $J^{-1} = J^v = D$ for some finitely generated subideal J of I. By Corollary 4.3, we then have $G(I) \geq G(J) \geq 2$. □

We note that the conclusion of Corollary 4.4 is not valid if D is only assumed to have finite t-character, for if D is a valuation domain with nonprincipal maximal ideal M, then D has finite (t-) character, but $M^{-1} = D$ and $G(M) = 1$.

In Proposition 1.8, we saw that in a domain of finite t-character, we have F-Max$(D) = w$-Max(D). In fact, by applying the ideas of this section, we can obtain a stronger conclusion (and thereby generalize [16, Proposition 3.3]):

Corollary 4.5. *In a domain D of finite t-character, we have $F = w$.*

Proof. Let D have finite t-character, and let I be an F-ideal of D. Suppose that $xJ \subseteq I$ for some $x \in D$ and finitely generated ideal J with $J^v = D$. By Corollary 4.3 (and Proposition 4.1), there are elements $a, b \in J$ with $(a, b)^v = D$. Since $x(a, b) \subseteq I$ and I is an F-ideal, this yields $x \in I$. Therefore I is also a w-ideal, as desired. □

5. Examples

In [30, Section 7], H. Uda presents an example showing that classical grade and polynomial grade can differ. We begin with a review of his example and then proceed to adapt it for our purposes. Specifically, we show that an appropriate localization satisfies $t_2 < t$ and $F < w$ and is a DF-domain but not a DW-domain.

Except for a slight change in notation, here is Uda's example:

Example 5.1. Let k be a field and s, t, u indeterminates over k. Then set $A = k[s,t,u]_{(s,t,u)}$, and let P denote the maximal ideal of A. For each $\alpha, \beta \in P$, let $X_{\alpha\beta}$ be an indeterminate, and let $T = A[\{X_{\alpha\beta}\}]$. Let B denote the ideal of T generated by the $X_{\alpha\beta}$, and let $J = B^2$. Let $N = PT + B$, so that N is a maximal ideal of T, generated by s, t, u and the $X_{\alpha\beta}$. Now for each $\alpha, \beta \in P$, let $P_{\alpha\beta} = (\alpha, \beta)A$, and let $R = A + \sum P_{\alpha\beta}X_{\alpha\beta} + J$. Let $M = N \cap R$. Each $f \in R$ has a unique representation $f = f_0 + \sum f_{\alpha\beta}X_{\alpha\beta} + f_1$ with $f_0 \in A$, $f_{\alpha\beta} \in P_{\alpha\beta}$, and $f_1 \in J$.

Proposition 5.2. *In Example 5.1:*

(1) *T is integral over R.*
(2) *M is a maximal ideal of R and a maximal t_2-ideal.*
(3) *$(PR)^t = R$, hence M is not a t-ideal.*
(4) *$T_{R\setminus M} = T_N$.*
(5) *R_M is not integrally closed.*
(6) *MR_M is a t_2-ideal but not a w-ideal of R_M. Hence in $D := R_M$, $t_2 < t$ and $F < w$.*
(7) *D is a DF-domain but not a DW-domain.*
(8) *D is not a pre-Bézout domain.*
(9) *D does not have finite t-character. (Of course, since D is local with maximal ideal a t_2-ideal, D does have finite t_2-character.)*

Proof. (1) This follows from the fact that $A \subseteq R$ and $X_{\alpha\beta}^2 \in R$ for each α, β.
(2) By (1) M is a maximal ideal of R. Let $f, g \in M$. Write $f = f_0 + \sum f_{\alpha\beta}X_{\alpha\beta} + f_1$ and $g = g_0 + \sum g_{\alpha\beta}X_{\alpha\beta} + g_1$ with $f_0, g_0 \in P$, $f_{\alpha\beta} \in P_{\alpha\beta}$, and $f_1, g_1 \in J$. Then $X_{f_0g_0}(f,g)R \subseteq R$, and we have $(f,g)^v \subseteq (R :_R X_{f_0g_0}) \subseteq M$ ([30, Lemma 7.1]).
(3) It follows from [30, Proposition 7.3] that $((s,t,u)R)^v = R$.
(4) Let $f \in T \setminus N$, and write $f = a + g$ with $a \in A$ and $g \in (\{X_{\alpha\beta}\})T$. Then $f^{-1} = (a-g)/(a^2 - g^2)$ with $a^2 - g^2 \in R \setminus M$.
(5) We have $X_{\alpha\beta} \in T \setminus R_M$ for each $\alpha, \beta \in P$.
(6) For $f, g \in M$, represent f, g as in (2). Then $X_{f_0g_0}(f,g)R \subseteq R$ with $X_{f_0g_0} \notin R_M$. It follows that $((f,g)R_M)^{v_{R_M}} \subseteq MR_M$ and hence that MR_M is a t_2-ideal. On the other hand, M is not a w-ideal by (3) (since every maximal w-ideal is a maximal t-ideal); hence MR_M is not a w-ideal.
(7) Since MR_M is a t_2-ideal, it is also an F-ideal. Therefore, D is a DF-domain. On the other hand, D is not a DW-domain, since MR_M is not a t-ideal.
(8) It is clear that s, t are not contained in a proper principal ideal of T_N. Hence $(s,t)D$ is not contained in a proper principal ideal of D, i.e., $(s,t)D$ is primitive. Of course, $(s,t)D$ is not superprimitive and hence D does not have LPSP. Thus D is not pre-Bézout by Proposition 2.5.
(9) Since D is not a DW-domain, D cannot have finite t-character by Proposition 2.7.

□

Remarks/Questions 5.3. Refer to Proposition 5.2.

(1) By (5), D is not integrally closed. Must an integrally closed DF-domain be DW? We doubt that this is true but have no counterexample.

(2) Picozza and Tartarone [29, Theorem 3.7] prove that a DW-domain that is both integrally closed and satisfies the finite-conductor property must be a Prüfer domain. (A domain E is a finite conductor domain if $(a) \cap (b)$ is finitely generated for all $a, b \in E$.) The proof involves two steps: an integrally closed finite conductor domain is a PVMD, and a PVMD that is also a DW-domain must be a Prüfer domain. As we have already remarked, we do not know whether a DF-PVMD must be Prüfer (but we doubt it).
(3) It is clear that $\dim(D) = \infty$. Every one-dimensional domain is a DW-domain. Are there two-dimensional, or at least finite-dimensional, examples of DF-domains that are not DW?
(4) As mentioned in [16], if $n > 2$ and one substitutes n-generated ideals for two-generated ideals in the definitions of the t_2- and F-operations, one obtains new star operations, dubbed the t_n- and F_n-operations (so that $F_2 = F$). Whether we always have $t_n = t$ or $F_n = w$ were left as open questions. However, by making obvious changes in Example 5.1, one can obtain, for each $n > 1$ a local domain D_n whose maximal ideal is a t_n- (and hence also an F_n-) ideal but is not an F_{n+1}-ideal.

In order to produce more examples of DF-domains that are not DW, we investigate the DF-property in pullback diagrams. Though our results generally parallel those of Mimouni for DW-domains [26], our proofs are somewhat more delicate due to the fact that ideals often must be two-generated. We need several facts about the behavior of v-ideals, etc., in pullbacks. For this we use [11] as a convenient reference, but the ideas actually come from [10].

Let T be a domain, M a maximal ideal of T, $\varphi : T \to k := T/M$ the natural projection, and D an integral domain contained in k. Then let $D = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms.

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \varphi\ } & T/M = k \end{array}$$

We shall refer to this as a diagram of type $\square$.

Proposition 5.5 below allows one to produce many examples of DF-domains.

Lemma 5.4. *In a pullback of type $\square$:*
(1) *If A is a F-ideal of D, then $\varphi^{-1}(A)$ is a F-ideal of R.*
(2) *For each nonzero ideal A of D, $\varphi^{-1}(A^F) = \varphi^{-1}(A)^F$.*
(3) *If Q is a maximal F-ideal of T, then $Q \cap R$ is a maximal F-ideal of R.*

Proof. (1) Let A be a F-ideal of D, and let $I = \varphi^{-1}(A)$. Suppose $r(a, b) \subseteq I$, with $r, a, b \in R$ and $(a, b)^v = R$. By [11, Proposition 2.17(2b)], $(\varphi(a), \varphi(b))^{v_D} = D$. Since A is a F-ideal of D, this yields $\varphi(r) \in A$ and hence $r \in I$. Thus I is a F-ideal of R.

(2) Let A be a nonzero ideal of D. By (1), we have $\varphi^{-1}(A^F) \supseteq \varphi^{-1}(A)^F$. We now recall the notation of Defnition 1.1: For a domain E with quotient field L and a subset J of L, we write $J' = \{y \in L \mid y(e, f) \subseteq J$ for some $e, f \in E$ with $(e, f)^{v_E} = E\}$. To complete the proof, it will suffice to show that $\varphi^{-1}(A') \subseteq \varphi^{-1}(A)'$. To this end, let $x \in \varphi^{-1}(A')$. Then $\varphi(x)(d_1, d_2) \subseteq A$ for elements $d_1, d_2 \in D$ with

$(d_1, d_2)^{v_D} = D$. According to [20, Lemma 7 and its proof], there are elements r_1, r_2 in R for which $\varphi(r_i) = d_i$ for $i = 1, 2$ and $\varphi^{-1}(d_1, d_2) = (r_1, r_2)$. By [11, Proposition 2.17(1b)], we have $R = \varphi^{-1}((d_1, d_2)^{v_D}) = (r_1, r_2)^v$. Since $x(r_1, r_2) \subseteq \varphi^{-1}(A)$, we have $x \in \varphi^{-1}(A)'$, as desired.

(3) Let Q be a maximal F-ideal of T, and let $P = Q \cap R$. Suppose that P is not an F-prime of R. Then there are elements $a, b \in P$ for which $(a, b)^v = R$. Note that we cannot have $(a, b) \subseteq M$ since M is divisorial in R. Hence $((a, b)T)^{v_T} = T$ by [11, Proposition 2.5(2)], contradicting that Q is an F-prime of T. □

Proposition 5.5. *In a pullback of type □:*

(1) *If T, D are DF-domains, then R is a DF-domain.*
(2) *If T is local and D is a DF-domain, then R is a DF-domain.*
(3) *If R is a DF-domain, then D is a DF-domain.*

Proof. (1) Let P be a maximal ideal of R. If $P \supsetneq M$, then $P = \varphi^{-1}(p)$ for a maximal ideal p of D [11, Theorem 1.9]. Since D is a DF-domain, p is an F-prime of D and hence P is an F-prime of R by Lemma 5.4. If $P = M$, then P is divisorial (and therefore an F-prime). If P is incomparable to M, then $P = Q \cap T$ for some maximal ideal Q of T [11, Theorem 1.9]. Since T is a DF-domain, Q is an F-prime and hence so is P by Lemma 5.4.

(2) This follows as in the proof of (1).

(3) Assume that R is DF, and let p be a maximal ideal of D. Then $P := \varphi^{-1}(p)$ is a maximal ideal of R, and, since R is a DF-domain, P is an F-prime of R. By Lemma 5.4 $P = P^F = \varphi^{-1}(p^F)$, whence $p = p^F$, that is, p is an F-prime of D. □

According to [26, Theorem 3.1(1)], in a pullback diagram of type (□), if R is DW, then so is D. Hence if we take D to be a DF-domain that is not DW (e.g., the D of Proposition 5.2) and T is either local or a DF-domain, then R is a DF-domain that is not DW.

6. Polynomial rings

Proposition 6.1. *Let D be a domain, and Q a maximal t_2-ideal of $D[X]$. Then Q is a maximal t-ideal of $D[X]$. Hence Q is either an upper to zero or the extension of a maximal t-ideal of D. Moreover, t-Max$(D[X]) = t_2$-Max$(D[X]) = w$-Max$(D[X]) = F$-Max$(D[X])$.*

Proof. If Q is an upper to zero, then Q is a t-ideal and must therefore be a maximal t-ideal. Hence we assume that $P = Q \cap D \neq (0)$. Suppose, by way of contradiction, that $Q^t = D[X]$. Then we have $f_1, \ldots, f_n \in Q$ with $(f_1, \ldots, f_n)^{-1} = D[X]$, and it is clear that we must then have $(c(f_1) + \cdots + c(f_n))^{-1} = D$. By a standard argument, we can then produce $f \in Q$ with $c(f) = c(f_1) + \cdots + c(f_n)$ (take $f = f_1 + X^{k_2} f_2 + \cdots + X^{k_n} f_n$ for appropriately chosen positive integers $k_2, \ldots, k_n$), so that $(c(f))^v = D$. Pick $a \in P$, $a \neq 0$. We claim that $(a, f)^v = D[X]$. (This is a another standard argument: suppose that $g \in (a, f)^{-1}$. Since $ga \in D[X]$, this puts $g \in K[X]$. We then use the content formula to get $c(f)^{r+1} c(g) = c(f)^r c(fg) \subseteq D$ for appropriately chosen r [13, Theorem 28.1]. Since $c(f)^v = D$, this yields $g \in D[X]$. Hence $(a, f)^v = (a, f)^{-1} = D[X]$, as claimed.) However, this contradicts the fact that Q is a t_2-ideal. Hence $Q^t \neq D[X]$. It follows that Q must be a maximal t-ideal of $D[X]$. The "hence" statement now follows from [22, Proposition 1.1]. As to the "moreover" statement, we have w-Max$(E) = t$-Max(E) for all domains E and $w = F$ on $D[X]$ [16, Theorem 4.5]. □

The following corollary strengthens [26, Proposition 2.12].

Corollary 6.2. *For a domain D, $D[X]$ is a DF-domain if and only if D is a field.*

Proof. Suppose that D is not a field, and let M be a maximal ideal of D. Then (M, X) is a maximal ideal of $D[X]$ that is not a t-ideal, hence not a F-ideal. Thus D is not a DF-domain. The converse is trivial. □

For a prime ideal I of a domain D, it is well known that I is a t-ideal of D if and only if $I[X]$ is a t-ideal of $D[X]$. This does not hold, however, for F or t_2-ideals, as the next example shows.

Example 6.3. In Proposition 5.2, $M[X]$ is not an F-ideal of $D[X]$.

Proof. If $M[X]$ is a F-ideal of $D[X]$, then it must be a maximal F-ideal. (The only primes containing $M[X]$ are of the form (M, f) with f monic. Then for any nonzero $a \in M$, we have $(a, f)^v = D[X]$, so that (M, f) is not an F-ideal.) However, by Proposition 6.1, this means that M is a (maximal) t-ideal of D, a contradiction. □

We remark that, although we always have $F = w$ in $D[X]$ [16, Theorem 4.5], we do not know whether a t_2-prime of $D[X]$ must be a t-ideal. (See [16] for some cases where the answer is yes.)

In [24] Kang extended the notion of the Nagata ring as follows. For a star operation $*$ on D, let $N_* = \{g \in D[X] \mid c(g)^* = D\}$ (where $c(g)$ denotes the content of g, i.e., the ideal of D generated by the coefficients of g). The $*$-Nagata ring is then $D[X]_{N_*}$. When $* = d$, we have the classical Nagata ring, usually denoted by $D(X)$. In [29], the authors observe that $D[X]_{N_v}$ is always a DW-domain, and they prove that a domain D is DW if and only if $D(X)$ is DW if and only if $D(X) = D[X]_{N_v}$. This leads to the question: When is $D[X]_{N_F}$ a DF-domain? We answer this question in the next result. We shall use the fact that the maximal ideals of $D[X]_{N_*}$ are the ideals $MD[X]_{N_*}$, where M is a maximal $*_f$-ideal of D [24, Proposition 2.1].

Proposition 6.4. *The following statements are equivalent for a domain D.*

(1) *$D[X]_{N_F}$ is a DF-domain.*
(2) *$D[X]_{N_F} = D[X]_{N_v}$.*
(3) *$D[X]_{N_F}$ is a DW-domain.*

Proof. Suppose that $D[X]_{N_F}$ is a DF-domain. Then each maximal ideal of $D[X]_{N_F}$ is an F-prime, and, using the fact that the F-operation has finite type and the above-mentioned description of Max$(D[X]_{N_F})$, we have that $MD[X]_{N_F}$ is an F-prime of $D[X]_{N_F}$ for each maximal F-ideal M of D. If follows that $MD[X]$ is an F-prime, and hence, by Proposition 6.1, a t-prime of $D[X]$ for each such M. Therefore, each maximal F-ideal of D is in fact a maximal t-ideal, and this yields that $D[x]_{N_F} = D[X]_{N_v}$. Hence (1) ⇒ (2). It is clear that (2) ⇒ (3) ⇒ (1). □

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