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ON RIESZ GROUPS

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Let G be a directed p.o. group. An element $x \in G$ is primal in G if $x \in G^+$ and $x \le a_1+a_2$, where $a_i \in G^+$, implies that $x = x_1+x_2$ such that $0 \le x_i \le a_i$. We show that if H is an o-ideal of G such that $x \in H^+$ implies x is primal in G and if G/H is a Riesz group then so is G. Calling $x \in G^+$ an extractor if for all $g \in G^+$ $x \vee g$ exists, we show that if $x \in H^+$ implies that x is an extractor in G and G/H is a l.o. group it is not necessary that G is l.o. unless G satisfies some extra conditions. Connection of these results with a ring-theoretic result of Paul Cohn is indicated.

A directed p.o. group G is called a Riesz group if for $a_i, b_j \in G$ (i,j=1,2) with $a_i \le b_j$ there exists $c \in G$ such that $a_i \le c \le b_j$. The letter G denotes a directed p.o. group throughout. We use + to denote the group operation, which may not be commutative, and use 0 for the identity of G. According to Fuchs [5, (4) Theorem 2.2] G is a Riesz group if and only if every element of G^+ is primal in G. The aim of this paper is to prove the following result and to show some of its consequences and some of its connections with the current literature.

THEOREM 1. Let G be a directed p.o. group and let H be an o-ideal of G such that every member of H^+ is primal in G. If G/H is a Riesz group then so is G.

This result, as we shall explain later, is a general group theoretic version of Cohn's analogue of Nagata's UFD's Theorem for Schreier domains [2, Theorem 2.6]. The other point of interest is the following corollary.

COROLLARY 2. Let G be a p.o. Schreier extension of a directed p.o. group H by another p.o. group F. Then G is a Riesz group if and only if the following hold:

(1) Every member of H^+ is primal in G,

(2) F is a Riesz group.

This note is split into three sections. In the first section we prove THEOREM 1, and use its proof to isolate the largest Riesz subgroup of a given directed p.o. group. We will call such a subgroup the largest interpolating subgroup of the given group. In the second section we show the connection of THEOREM 1, with the above indicated theorem of Cohn's. Using the ring theoretic connection we construct an example to establish that if H is an o-ideal of G such that $x \in H^+$ implies that x is an extractor in G and if G/H is a l.o. group then all we can expect is that G is a Riesz group (of course we show that an extractor is primal). In section 3, we prove a modified version of THEOREM 1 for l.o. groups.

1. THE MAIN RESULT

To prepare for the proof of THEOREM 1, we need two simple lemmas. As these lemmas can be proved for a notion slightly stronger than that of a primal element we introduce it. We shall call $x \in G^+$ completely primal if every element in [0,x] is primal. In fact in THEOREM 1, the fact that H is an o-ideal and the requirement that every member of H^+ is primal in G entail that the elements of H^+ be completely primal. Obviously if x is completely primal in G and $y \in [0,x]$ then y is also completely primal. Later, in the next section, we shall give an example to establish that a primal element may not be completely primal.

LEMMA 3. Let p be a (completely) primal element of G. Then every conjugate of p is (completely) primal.

PROOF. Obvious.

LEMMA 4. In G any sum of (completely) primal elements is (completely) primal.

PROOF. Let p and q be primal and assume that $p+q \leq a_1+a_2$ where $a_i \in G^+$. Then $p \leq a_1+a_2$ implies that $p = p_1+p_2$ such that $0 \leq p_i \leq a_i$. So if $a_i = p_i+x_i$ then $p_1+p_2+q \leq p_1+x_1+p_2+x_2$ or $q \leq [-p_2+x_1+p_2] + [x_2]$. But as q is primal we conclude that $q = q_1+q_2$ where $q_1 \leq -p_2+x_1+p_2$ and $q_2 \leq x_2$. This gives $0 \leq p_2+q_1-p_2 \leq x_1$ and $q_2 \leq x_2$ or $p_1+p_2+q_1-p_2 \leq p_1+x_1 = a_1$ and $p_2+q_2 \leq p_2+x_2 = a_2$. Clearly $p+q = [p_1+p_2+q_1-p_2] + [p_2+q_2]$ and p+q is primal.

Now suppose that p and q are completely primal and let $a_1 \in G^+$ such that $a_1 \leq p+q$. Then there exists $a_2 \in G^+$ such that $a_1+a_2 = p+q$. Using the fact that p is primal we have $p = p_1+p_2$, $0 \leq p_i \leq a_i$ with $a_i = p_i+x_i$, which gives $p_1+x_1+p_2+x_2 = p_1+p_2+q$ or $[-p_2+x_1+p_2] + [x_2] = q$. But as q is completely primal so is $-p_2+x_1+p_2$ and so is its conjugate x_1. But then $a_1 = p_1+x_1$ is a sum of two primal elements and hence a primal element.

The next two lemmas do slightly more than prove the theorem. Before we state and prove these lemmas let us recall that H is an <u>o-ideal</u> of G means that H is a normal, convex and directed p.o. subgroup of G. The order in G/H is defined by $x+H \leq y+H$ if for some $h_1, h_2 \in H$, $x+h_1 \leq y+h_2$ in G.

<u>LEMMA 5</u>. <u>Let</u> H <u>be an o-ideal of</u> G <u>and suppose that</u> H^+ <u>consists of primal elements of</u> G. <u>Let</u> $x,a,b \in G^+$ <u>with</u> $x \leq a+b$. <u>If there exist</u> $x_1, x_2 \in G$ <u>such that</u> $x = x_1+x_2$ <u>and</u>

$$H \leq x_1+H \leq a+H \text{ and } H \leq x_2+H \leq b+H \qquad \text{(i)},$$

<u>then there exist</u> $z_1, z_2 \in G^+$ <u>such that</u> $x = z_1+z_2$ $z_1+H = x_1+H$ <u>and</u> $z_2+H = x_2+H$.

<u>PROOF</u>. By (i) and by the directedness of H, we can find $h_i \in H^+$ such that $y_i = x_i+h_i \geq 0$ and obviously $y_i+H = x_i+H$. Now $y_1+y_2 = x_1+h_1+x_2+h_2 = x_1+x_2+h$ (since H is normal $h \in H$). As x, $h \geq 0$ and as h is primal we have $h = k_1+k_2$ where $k_i \in H$ such that $0 \leq k_i \leq y_i$. But then $x_1+x_2 = [y_1-k_1] + [k_1 + (y_2-k_2)-k_1]$ where each element in the brackets is positive. Putting $z_1 = y_1-k_1$, $z_2 = k_1+(y_2-k_2)-k_1$ we have $x = z_1+z_2$, $z_1+H = y_1+H = x_1+H$ and $z_2+H = y_2+H = x_2+H$ (since k_1, $k_2 \in H^+$ and H is normal).

<u>LEMMA 6</u>. <u>Let</u> H <u>be an o-ideal of</u> G <u>and suppose that every element of</u> H^+ <u>is primal in</u> G. <u>If for</u> $x \geq 0$, x+H <u>is a primal element of</u> G/H <u>then</u> x <u>is a primal element of</u> G. <u>Moreover if for</u> $x \geq 0$, x+H <u>is completely primal then</u> x <u>is completely primal</u>.

PROOF. Let $0 \leq x \leq a+b$ where a, $b \in G^+$ and suppose that $x+H = x_1+x_2+H$ such that $H \leq x_1+H \leq a+H$, $H \leq x_2+H \leq b+H$ (i).

Then as we can assume that $x = x_1+x_2$ where $x_i \in G$, by LEMMA 5 we can assume that $x_i \in G^+$. By (i) and by the directedness (and normality) of H we can assume that there exist k_1, $k_2 \in H^+$ such that $0 \leq x_1 \leq a+k_1$;

$0 \leq x_2 \leq b+k_2$ (ii).

Now we show that for $0 \leq x \leq a+b$, a, $b \in G^+$ we can select x_1, $x_2 \geq 0$ such that $x = x_1+x_2$, $x_1 \leq a$ and $x_2 \leq b$. By (ii) there exist u_1, $u_2 \geq 0$ such that

$x_1+u_1 = a+k_1$ (iii)

and $x_2+u_2 = b+k_2$ (iv).

This leads to $x_1+u_1+x_2+u_2 = a+k_1+b+k_2$ or to $x_1+x_2+u_1'+u_2 = a+b+k_1'+k_2$ where $u_1' = -x_2+u_1+x_2 \geq 0$ and $k_1' = -b+k_1+b \in H^+$. Now as $x \leq a+b$ we have $a+b = x+w$; $w \geq 0$, so that $x+u_1'+u_2 = x+w+k_1'+k_2$ or $u_1'+u_2 = w+k_1'+k_2$. But $k_2 \in H^+$ gives $k_2 = r+s$ where $0 \leq r \leq u_1'$; $0 \leq s \leq u_2$ and r, $s \in H^+$. Substituting for k_2 in (iv) we get $x_2+u_2 = b+r+s$ or $x_2+u_2-s = b+r$. Again as $r \in H^+$ and x_2, $u_2-s \geq 0$ we have $r = r_1+r_2$, such that $0 \leq r_1 \leq x_2$, $0 \leq r_2 \leq u_2-s$. Consequently, $x_2 + (u_2-s-r_2) = b+r_1$, or $[x_2-r_1] + [r_1 + (u_2-s-r_2)-r_1] = b$. Giving $x_2-r_1 \leq b$. Let us write $x_2' = x_2-r_1 \leq b$ and $x_1' = x_1+x_2+r_1-x_2$. Then $x_i' \geq 0$, $x_1'+x_2' = x_1+x_2$, $x_2' \leq b$ and as $x_1 \leq a+k_1$, $x_1' = x_1 + x_2 + r_1 - x_2 \leq a + k_1 + x_2 + r_1 - x_2$.

Thus $0 \leq x \leq a+b$, a, $b \in G^+$ and $x+H$ primal imply that we can find x_1, $x_2 \in G^+$ such that $x = x_1+x_2$, $0 \leq x_1 \leq a+\ell$ and $0 \leq x_2 \leq b$ where $\ell \in H^+$. So, we can find z_1, $z_2 \in G^+$ such that

$z_1 + x_1 = g+a$ $(g = a+\ell - a \in H^+)$ (v)

$z_2 + x_2 = b$ (vi).

Consequently $z_1+z_2'+x = g+m+x$ where $z_2' = x_1+z_2-x_1$ and $m = a+b-x \geq 0$. This gives $z_1+z_2' = g+m$. As $g \in H^+$, $z_i \geq 0$ we have $g = g_1+g_2$ where $g_i \in H^+$, $g_1 \leq z_1$ and $g_2 \leq z_2'$ (vii).

Substituting for g in (v), we get $z_1+x_1 = g_1+g_2+a$ and so $-g_1+z_1+x_1 = g_2+a$.

Again as g_2 is primal we can write $g_2 = g_{21}+g_{22}$ such that $g_{2i} \in H$, $0 \le g_{21} \le -g_1+z_1$ and $0 \le g_{22} \le x_1$. So $-g_{21}-g_1+z_1+x_1 = g_{22}+a$ or $[-g_{22}+(-g_{21}-g_1+z_1)+g_{22}]+[-g_{22}+x_1] = a$.

Writing $x_1' = -g_{22}+x_1$, $x_2' = -x_1+g_{22}+x_1+x_2$ we have $x_i' \ge 0$, $x_1'+x_2' = x_1+x_2$ with $x_1' \le a$ and it remains to show that $x_2' \le b$. For this, recall that $z_2+x_2 = b$... by (vi). Or $[z_2-(-x_1+g_{22}+x_1)]+[-x_1+g_{22}+x_1+x_2] = b$.

But $z_2 - (-x_1+g_{22}+x_1) \ge 0$ because $g_{22} \le g_2 \le z_2' = x_1+z_2-x_1$ (by (vii)). So $z_2-(-x_1+g_{22}+x_1)+x_2' = b$ which establishes that $x_2' \le b$. Finally if $x \ge 0$ and $x+H$ is completely primal, then for $y \in G$ such that $0 \le y \le x$ we have $H \le y+H \le x+H$ which, by the earlier part of the lemma, makes y a primal element.

The proof of COROLLARY 2 depends upon a simple use of the definition of the p.o. extension of a p.o. group for which, if necessary, [4] may be consulted.

Let us call an o-ideal H of G an <u>interpolating o-ideal</u> of G if every element of H^+ is a primal element of G(and hence, due to convexity of H, a completely primal element of G). We now proceed to show that every directed p.o. group G contains an interpolating o-ideal M(G) that is the largest i.e. contains every other interpolating o-ideal. Though of course M(G) may be trivial... in which case G is void of strictly positive (i.e. non-trivial) completely primal elements.

<u>THEOREM 7</u>. <u>Every directed p.o. group G contains an interpolating o-ideal $M = M(G)$ that is the largest that is if M_1 is an interpolating o-ideal of G then $M_1 \subseteq M(G)$. Consequently G/M contains no non-trivial completely primal elements.</u>

<u>PROOF</u>. Let S be the set of all completely primal elements of G. Then by <u>LEMMA 4</u>, S is a monoid and by <u>LEMMA 3</u>, $g+S = S+g$ for all $g \in G$. The convexity of S follows from the fact that every positive element preceding a completely primal element is completely primal. For $s_1, s_2 \in S$, $s_1+s_2 = 0$ implies that $s_1 = s_2 = 0$, because $S \subseteq G^+$ (or...if you like because 0 is completely primal). Finally as $S \subseteq G$, S satisfies the cancellation property. Thus, according to [1, Theorem 1, p.321], $M = \langle S \rangle$ is a directed group and hence a directed subgroup of G. The convexity and normality follow

by the convexity and normality of S and directedness of M. That M is the largest interpolating o-ideal is obvious. Finally the last part of LEMMA 6 ensures that G/M contains no non-trivial completely primal elements.

If Riesz groups (i.e. groups with interpolation property) are of any use then Cohn's completely primal elements may be seen as a tool for isolating a useful part of a general p.o. group.

2. SOME RELATED QUESTIONS

We devote this section to (i) indicating the connection of THEOREM 1 with Cohn's results (ii) giving an example of a primal element that is not completely primal and to (iii) answering the question, why choose Riesz groups for THEOREM 1? Why not l.o. groups?

We first prepare to indicate the connection of THEOREM 1 with Cohn's Theorem. Let D be an integral domain with quotient field K. The set $G(D) = \{xD \mid x \in K\backslash\{0\}$ is a directed p.o. group under $xD \cdot yD = xyD$ and $xD \leq yD$ if and only if $yD \subseteq xD$. Indeed D is the identity of this group and $(G(D))^+ = \{xD \mid x \in D\backslash\{0\}\}$. According to Mott [8, Theorem 2.1] if S is a saturated multiplicative set of D and if $\underline{S} = \{sD \mid s \in S\}$ then $<\underline{S}>$, the group generated by $\underline{S}$ is a convex directed subgroup of G(D) and, conversely, if T is a convex directed group of G(D), then $\underline{T} = \{x \in D \mid xD \in T^+\}$ is a saturated multiplicative set. Moreover for S saturated and multiplicative, $G(D_S) \cong_o G(D)/<\underline{S}>$.

Cohn calls an element x of an integral domain D primal if for a, b ∈ D, x|ab implies that $x = x_1x_2$ such that $x_1|a$ and $x_2|b$, and completely primal if every factor of x is primal. Then he calls an integral domain D Schreier if D is integrally closed and each non-zero element of D is primal. He then proves that every quotient ring of a Schreier domain is Schreier and that for D integrally closed if S is generated by completely primal elements of D and if D_S is Schreier then so is D [2, Theorem 2.6]. Let us call D pre-Schreier if every non-zero element of D is primal. Clearly, D is pre-Schreier if and only if G(D) is a Reisz group. In view of the above discussion and in view of our theorem we make the following statement.

COROLLARY 8. Let D be an integral domain and let S be a multiplicative set generated by a set of completely primal elements of D. If D_S is a pre- Schreier domain then so is D.

For the proof we only need to know that the saturation S′ of S consists of primal elements of D and that $<\underline{S}'>$ is then an o-ideal of G(D), with positive elements primal in G(D).

Although COROLLARY 8 appears to be so, it is not an improvement on Cohn's result, because in his proof Cohn does not use the extra condition that D is integrally closed.

Having indicated the connection we attend to the following natural question: "Is there a primal element that is not completely primal?" From the above discussion it follows, among other things, that "x is (completely) primal in D" translates to, "xD is (completely) primal in G(D)" and vice versa. So it would be sufficient to find an element x in an integral domain D such that x is primal but not completely primal.

Example 9. Let R be an integral domain that is not pre-Schreier, K the quotient field of R and x an indeterminate over K. Then, in the ring $D = \{a_0 + \sum_1^n a_i x^i \mid a_0 \in R, a_i \in K\}$ $= R + xK[x]$ the element x is primal but not completely primal.

Illustration. Note that if $f \in D$ then $f = ax^r(1 + xf_1(x))$ where $f_1(x) \in K[x]$, $a \in K$, $r \geq 0$ and if $r=0$, $a \in R$. Now let $x|fg$, f, $g \in D$. We can write

$$f = ax^r(1 + xf_1(x))$$
$$g = bx^s(1 + xg_1(x)).$$

If both r, $s \geq 1$ we can write $x = d(x/d)$ selecting d so that $db \in R$. Then clearly $d|f$ and $x/d|g$. (The choice of d so that $db \in R$ is to cover the possibility of s being 1... as in that case $g/(x/d) = db(1 + xg_1(x))$.) Further if any of r, s say $s \geq 2$ we can write $x = 1 \cdot x$ where $1|f$ and $x|g$. This leaves the case when one of r, s is 0 and the other is 1. Let $r=0$ and $s=1$. Then $r=0$ gives $f = a(1 + xf_1(x))$ where $a \in R$ and $g = bx(1 + xg_1(x))$.

Now as $x|fg = abx(1 + xf_1(x))\ (1 + xg_1(x))$ we conclude that $ab \in R$. But then we can write $x = a(x/a)$ where $a|f$ and $(x/a)|g$. Having exhausted all the possibilities we conclude that x is primal. Now as R is not pre- Schreier it must have a non-zero element d that is not primal and as every non-zero element of R divides x we have the conclusion that x is not completely primal. (Indeed a use of COROLLARY 8 will establish the sufficiency in the following statement: Given that D, R, K and x are as in EXAMPLE 9, D is a pre-Schreier domain if and only if x is completely primal in D. The necessity is of course obvious.)

It is natural to ask, "Why were Riesz groups chosen for THEOREM 1, and not the l.o. groups?" Obviously we raise this question because we have an answer. As it stands, the question is not properly posed; it does not prepare the grounds for an answer. A properly posed question should first give us elements like the completely primal elements so that if G^+ consists of these elements then G is a lattice ordered group.

DEFINITION 10. Call $g \in G$ an extractor if $g \in G^+$ and for all $x \in G^+$, $g \vee x$ exists.

So the proper question is, "Let H be an o-ideal of G such that every element of H^+ is an extractor in G. Is it true that if G/H is an l.o. group then so is G?" The answer is, "All you can expect is that G is a Riesz group." To justify this answer we need first to establish that an extractor is indeed primal.

PROPOSITION 11. In a directed p.o. group G, any extractor is a completely primal element.

PROOF. We first establish that if $r \in G$ is an extractor then for all $h \in [0,r]$, h is an extractor. We need to establish that for all $d \in G^+$, $h \vee d$ exists. Since $r = h+k$ for some $k \in G^+$, and since $r \vee (d+k)$ exists we conclude that $h \vee d = r \vee (d+k) - k$ also exists. Moreover if r is an extractor then for all $d \in G^+$, $r \wedge d = -(-r \vee -d)$ exists in G and if $r \wedge d = k$ then $r = r_1+k$, $d = d_1+k$ such that $r_1 \wedge d_1 = 0$. Now let r be an extractor and suppose that $r \leq a+b$, $a, b \in G^+$. Then as $r \wedge a = k'$ exists we have $r = k'+r_1$ and $a = k'+a_1$ where $r_1 \wedge a_1 = 0$ and so $r_1 \leq a_1+b$ which implies that

$r_1 \leq b$ (cf. [5, p.10]). Thus if $r \leq a+b$ where a, $b \in G^+$ then $r = k+r_1$ such that $0 \leq k \leq a$ and $0 \leq r_1 \leq b$. Thus an extractor is primal. Combining this result with the first part of the proof we have the proposition.

So, we have established that if H is an o-ideal of G such that every member of H^+ is an extractor in G then the event of G/H being lattice ordered would cause G to be at least a Riesz group. To show that this Riesz group may not be a l.o. group we need to give a counter example... and of course we need to see what extra conditions on H or G will make G a l.o. group. For this we translate the problem into a ring theoretic one... for in ring theory the corresponding problems have received attention.

Let D be an integral domain and let G(D) be the group of divisibility of D. It is easy to see that for a, $b \in D\backslash\{0\}$,

$$aD \cap bD = U(aD, bD) = \{rD \in G(D) \mid aD, bD \leq rD\}$$

and obviously $aD \vee bD$ exists $\Leftrightarrow$ for some $x \in K\backslash\{0\}$ $U(aD, bD) = U(xD)$ $\Leftrightarrow$ $aD \cap bD$ is principal $\Leftrightarrow$ $(a,b)_v = ((a,b)^{-1})^{-1}$ is principal. Further if $(a,b)_v = (aD + bD)_v$ is principal its generator is the GCD of a and b (up to associates). (Indeed, for a, $b \in D\backslash\{0\}$, $(a,b)_v = (aD + bD)_v$ may stand for $L(aD, bD) = \{xD \in G(D) \mid xD \leq aD, bD\}$.) Thus D is a GCD-domain if, and only if, G(D) is a l.o. group. An extractor $rD \in G(D)$ obviously translates to what may be called a <u>lcm extractor</u> i.e. $r \in D\backslash\{0\}$ such that for all $x \in D\backslash\{0\}$ $rD \cap xD$ is principal (or equivalently $(r,x)_v$ is principal). So if we produce a Schreier domain R that is not a GCD-domain but contains a saturated multiplicative set S consisting of extractors such that R_S is a GCD-domain we have our counter-example. For then via Mott's result [8] $G(R)/\langle \underline{S} \rangle \cong_o G(R_S)$ will be a l.o. group where $\langle \underline{S} \rangle$ is an o-ideal of the Riesz group G(R) and $\langle \underline{S} \rangle^+$ consists of extractors from G(R). Most of the examples constructed in [10] will meet the requirements but we shall use example 2.6 of that paper for its directness and accessibility. Besides it affords a less technical illustration.

<u>Example 11</u>. Let E be the ring of entire functions. It is well known that E is a Bezout domain (every finitely generated ideal is principal) [7], and hence a GCD-domain. It is

also well known that an entire function is uniquely expressible as a countable product $\Pi(z-\alpha_i)^{n_i}e_i$ of associates of powers of distinct linear polynomials in a single variable over $\mathbb{C}$ (here $n_i \in N$, obviously $z-\alpha_i$ are primes of E and e_i the units of E). Moreover a non-zero principal prime of E is a maximal ideal of height 1. Now let S be the multiplicative set of E generated by the non-zero principal primes of E and let x be an indeterminate over E. Then the integral domain

$$E^{(S)} = \{a_0 + \sum_{i=1}^{n} a_i x^i \mid a_0 \in E, a_i \in E_S\}\} = E + xE_S[x]$$

is the required Schreier domain.

Illustration. Because E is integrally closed, so is $E^{(S)}$ [3] and it is easy to see that every principal prime of E is a prime of $E^{(S)}$ and that in an integral domain any finite product of non-zero principal primes will have an lcm with each non-zero element of the integral domain. Moreover any factor of a product of powers of these primes is again a product of powers of primes. (Indeed we entertain empty products and so S contains the units of E... and hence of $E^{(S)}$.) Thus S is a saturated multiplicative subset of $E^{(S)}$ such that every member of S is an lcm extractor of $E^{(S)}$ (and of E). But $(E + xE_S[x])_S = E_S[x]$ a GCD-domain (for E_S is a Bezout domain [2] and hence GCD and a ring of polynomials over a GCD-domain is again a GCD-domain). So by Cohn's Theorem or (via Mott's Theorem) by THEOREM 1, $E^{(S)}$ is Schreier. The reason why $E^{(S)}$ is not a GCD-domain is more or less obvious: Let e be an infinite product of (non-zero) principal prime of E. Then e and x do not have a GCD. This is because even though every prime power dividing e divides x only a finite product of prime powers can divide both... as x is divisible only by members of S. So, in this case, we will keep on getting greater and greater common divisors but never the greatest.

The polynomial ring construction used in Example 11 is known as the $D + xD_S[x]$ construction. Introduced in [3], this construction has since been used as an efficient tool for constructing examples. In fact in [10] this author presented this construction as a device for constructing Abelian Riesz groups from Abelian l.o. groups. It would be interesting to see if a parallel or similar group theoretic construction is possible.

3. MODIFIED THEOREM 1 FOR L.O. GROUPS

Finally, in view of Example 11 we ask, "How much more should we modify THEOREM 1 to get a l.o. group conclusion?" On seeing Cohn's Theorem the ring-theorists asked the corresponding question for GCD-domains. An answer was provided in [6]. Then Mott and Schexnayder [9] showed in essence that the conditions imposed on S in [6] made $<\underline{S}>$ into a cardinal summand of G(D). But in group theoretic terms it means the following statement.

Let H be a cardinal summand of G. If H^+ consists of extractors and G/H is a l.o. group then so is G. In group theory this result is stated slightly differently but its triviality needs no proof.

But as the quotient of a l.o. group by any of its o-ideals is again a l.o. group there do exist more general l.o. group extensions of l.o. groups by l.o. groups. So there may be a more general simple answer. (Considering the literature on this topic any answer in the spirit of this paper will indeed be simpler.) The counter example i.e. Example 11 suggests, to this author, the following statement.

THEOREM 12. *Suppose that H is an o-ideal of G such that:*

(1) *every member of H^+ is an extractor of G*

(2) *if for $a, b \in G^+$ there is $h \in H^+ \backslash \{0\}$ such that $h \leq a,b$ then there is $d \leq a,b$ such that for $k \in H^+$ with $k \leq a,b$, $k \leq d$.*

If G/H is a l.o. group then so is G. Moreover if G is a l.o. group then every o-ideal H of G satisfies (1) and (2) and G/H is a l.o. group.

PROOF. First let us note that by the observations in section 2 and by THEOREM 1, G is a Riesz group by (1). The next observation will be referred to as the "Riesz group property". We note that in a Riesz group G if $a, b \in G^+$ are non-disjoint then there exists a strictly positive element h of G such that $h \leq a,b$. For if a and b are non-disjoint and positive then there exists x such that $x \leq a,b$ and x is not less than or equal to 0. But then $0, x \leq a,b$ and by the interpolation property of Riesz groups there exists h such that $0, x \leq h \leq a,b$ and obviously this h must be strictly positive.

It is our task now to show that for each pair $a, b \in G^+$, $a \vee b$ and $a \wedge b$ exist. We choose t o show the existence of $a \wedge b$ leaving $a \vee b$ to the reader as it will fall to similar techniques. Let $a, b \in G^+$. Since G/H is a l.o. group. $a+H \wedge b+H = c+H$ giving $a+H = a_1+c+H$ and $b+H = b_1+c+H$ where $a_1, b_1 \geq 0$, and $a_1+H \wedge b_1+H = H$. Using the fact that H is an o-ideal we can find $h_1, k_1, h_2, k_2 \in H^+$ such that

$$a+h_1 = a_1+c+k_1 \qquad \text{(i)}$$

$$b+h_2 = b_1+c+k_2 \qquad \text{(ii)}$$

since h_1, h_2 are extractors we can find $h_1 \vee h_2 = h_1+d_1 = h_2+d_2$, $d_i \in H^+$. Adding d_1 to the right of (i) and d_2 to the right of (ii) we get

$$a+h_1 \vee h_2 = a_1+c+m_1 \qquad \text{(iii)}$$

$$b+h_1 \vee h_2 = b_1+c+m_2 \qquad \text{(vi)}$$

where $m_i \in H^+$.

Since H is an o-ideal we can find $m_i' \in H^+$ so that

$$a+h_1 \vee h_2 = a_1+m_1'+c \qquad \text{(v)}$$

$$b+h_1 \vee h_2 = b_1+m_2'+c. \qquad \text{(vi)}$$

Subtracting c from the right of both equations we get

$$a+h_1 \vee h_2 - c = a_1+m_1' \qquad \text{(vii)}$$

$$b+h_1 \vee h_2 - c = b_1+m_2'. \qquad \text{(viii)}$$

It is sufficient to show that $a_1+m_1' \wedge b_1+m_2'$ exists. For this we first note that if $0 \leq x \leq a_1+m_1', b_1+m_2'$ then $x \in H$. (For in G/H, we have $H \leq x+H \leq a_1+H, b_1+H$ and $a_1+H \wedge b_1+H = H$.) Now suppose that $a_1+m_1' \wedge b_1+m_2'$ does not exist. Then a_1+m_1' and b_1+m_2' are non-disjoint and so by the Riesz group property there exists a strictly positive t such that $t \leq a_1+m_1', b_1+m_2'$ and as we have noted above $t \in H^+$. But then by (2) there exists $d \in G^+$ with $d \leq a_1m_1', b_1+m_2'$ such that $0 \leq h \leq a_1+m_1', b_1+m_2'$ implies that $h \leq d$. Clearly, by the above observation, $d \in H^+$ and by the definition of the infimum $d = a_1+m_1' \wedge b_1+m_2'$. The moreover part follows from that fact that for all $a, b \in G$, $a \wedge b$ exists. So $a \wedge b$ can be d of part (2), and the rest is either obvious or well-known.

Indeed we can make a statement like COROLLARY 2 in this case as well but we leave it to the reader. We also leave to the reader the proof of the following corollary.

COROLLARY 13. Let S be a saturated multiplicative set of an integral domain D such that

(1) every $s \in S$ is an lcm extractor,

(2) If for a, $b \in D\backslash\{0\}$ there is a non-unit $s \in S$ with $s|a,b$ then there is $d|a,b$ such that for all $t \in S$, $t|a,b$ implies $t|d$.

If D_S is a GCD-domain then so is D. Moreover if D is a GCD-domain then D_S is a GCD-domain and (1) and (2) hold.

REMARK 14. COROLLARY 13 obviously covers the case where $<\underline{S}>$ is a cardinal summand of G(D). For in that case every element of $D\backslash\{0\}$ is expressible as $d = rs$ where r is such that for $t \in S$, $t|r$ implies t is a unit. So given $d_1 = r_1s_1$, $d_2 = r_2s_2$ in $D\backslash\{0\}$ we can determine GCD $(s_1,s_2)=d$ and this obviously will suffice for the d in (2) of the above corollary. Indeed we could let S be generated by extractors...as the product of two extractors is an extractor.

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