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ON SOME CLASS GROUPS OF AN INTEGRAL DOMAIN

BY

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The aim of this article is to study some extreme cases of two notions of class groups based on the **t-operation.** These groups are defined as follows. Let R be a commutative integral domain. The set T(R) of all t-invertible t-ideals of R is a group under the operation of **t-multiplication.** The group T(R) contains as subgroups the set P(R) of all non-zero principal fractional ideals of R and the set Inv(R) of all invertible ideals of R. The quotient groups Cl(R) = T(R)/P(R) and G(R) = T(R)/Inv(R) are respectively called the **class group** and the **local class group** of R. Obviously Cl(R), defined thus, contains as a subgroup the **Picard group:** Pic(R) = Inv(R)/P(R). We study the cases when Cl(R) is trivial or torsion (G(R) is trivial or torsion). Given below are some results which can be stated in general terms. (1) If S is a multiplicative set of R generated by primes p_i such that each prime p_iR is of rank one and the intersection of infinitely many distionct p_iR is zero, and if $Cl(R_s) = 0$ then Cl(R) = 0. Moreover, with the same hypothesis for p_i, if $Pic(R_s) = 0$ then Pic(R) = 0. (2) If $G(R_M) = 0$ for all maximal ideals M of R then G(R) = 0. This result combined with a rather technical result gives: If Krull dim R = 1 then G(R) = 0.

0. Introduction.

Let R be a commutative integral domain with quotient field K and let F(R) be the set of non-zero fractional ideals of R. A function*: F(R) → F(R) is called a **star operation** on R if for A, B ∈ F(R) and for a ∈ K - {0},

(i) $(a)^* = (a)$, $(aA)^* = aA^*$,

(ii) $A \subseteq A^*$ and $A \subseteq B$ implies $A^* \subseteq B^*$

(iii) $(A^*)^* = A^*$.

Given a star operation * on R and given that A, B ∈ F(R) we have $(AB)^* = (A^*B)^* = (A^* B^*)^*$. These equations determine what is called **star multiplication.** A function on F(R) defined by $A \to (A^{-1})^{-1} = A_v$ is another star operation called the v-operation. Based on the v-operation we

define for all $A \in F(R)$, $A_t = \cup F_v$ where F ranges over finitely genera-
ted R-submodules of A. The fuction $F(R) \to F(R)$ defined by $A \to$
A_t is yet another star operation called the **t-operation.** For a detailed
study of star operations the reader may consult [8] and [10]. For our
purposes we include here some basic terminology.

Given a star operation * on R, an ideal $A \in F(R)$ is called a ***-ideal**
if $A = A^*$ and a ***-ideal of finite type** if $A = B^*$ for some finitely genera-
ted $B \in F(R)$. An ideal $A \in F(R)$ is called ***-invertible** if there exists $B \in$
$F(R)$ such that $(AB)^* = R$. In this case $B^* = A^{-1}$ (Jaffard [10, p..23]).
It is well known that if $A \in F(R)$ is a t-invertible t-ideal then A and A^{-1}
are t-ideals of finite type (and hence v-ideals of finite type [9]). Moreover, it
is easy to verify that, if A, $B \in F(R)$ are both t-invertible t-ideals then so
is $(AB)_t$. Based on these observations a notion of a class group, of a gene-
ral integral domain R, was introduced in [3]; as follows.

Let $T(R) = \{A \in F(R) \mid A \text{ is a t-invertible t-ideal}\}$ and

$$P(R) = \{xR \mid x \in K - \{0\}\}.$$

Then T(R) is a group under t-multiplication. This group contains P(R) as its
subgroup; since principal is invertible and hence is a t-invertible t-ideal.

The quotient group $Cl(R) = T(R)/P(R)$ is called the class group of
R relative to the t-operation. This quotient group, as noted in [3], reduces
to the **divisor class group** of R if R is a Krull domain and reduces to the
ideal class group if R is a Prufer domain. Moreover, as demonstrated in
[19], Cl(R) can be of use in the study of some aspects of **Prufer v-multi-
plication domains (PVMD**'s); that is integral domains in which every v-i-
deal of finite type is t-invertible. Following [2], another notion of a class
group called the **local class group** was introduced in [3]. This was done
as follows. Let Inv(R) be the set of all invertible ideals of R. Then Inv(R) is
a group and because every invertible ideal is a t-invertible t-ideal of R,
Inv(R) is a subgroup of T(R). The quotient group $G = T(R)/Inv(R)$ is
called the local class group of R. We note that these class proups are rela-
ted to a well known class group; the Picard group $Pic(R) = Inv(R)/P(R)$.
Now because

$$T(R)/Inv(R) \cong \left(T(R)/P(R)\right) / \left(Inv(R)/P(R)\right)$$

we have the following exact sequence of groups; if we regard them additive:
$0 \to Pic(R) \to Cl(R) \to G(R) \to 0$. Here the homomorphisms are canonical.

In the study of class groups it is often important to know when a class
group is trivial or torsion. We devote this article to answering the questions:

Question 1. Under what conditions on R is Cl(R) trivial (torsion)?

Question 2. Under what conditions on R is G(R) trivial (torsion)?

We provide general answers to the four questions arising from these two. Yet to give an idea we mention that for R a PVMD, Cl(R) = 0 if and only if R is a GCD-domain (compare with: For R Krull, Cl(R) = 0 if and only if R is a UFD (see e.g. Fossum [7])). Further, for R a PVMD, Cl(R) is torsion if and only if R is an almost GCD-domain of [19] i.e. for all f, g $\in$ R, there exists $n(f, g) \in N$ such that $f^nR \cap g^nR$ is principal (compare with: R Krull is almost factorial of Stroch [17] if and only if Cl(R) is torsion).

We allocate a section, however small, to each of the four questions arising from the above two. Yet, before we give away the plan of the paper, it seems necessary to make a few remarks about notation etc. All unexplained notation can be easily found in current literature. For this paper we use the letter R to denote a commuative integral domain with field of fractions K. Apart from this we use $D_f(R)$ to denote the set of v-ideals of finite type (of R).

In the first section we study the case when Cl(R) = 0. In this section we show that Cl(R) = 0 if and only if every t-invertible t-ideal A of R has the property that every finitely generated R-submodule of A is contained in a syclic R-submodule of A. Using this we show that if R is a pre-Schreier domain of [22] then Cl(R) = 0. Consequently if R is a GCD domain Cl(R) = 0. As mentioned already we also show that if R is a PVMD then Cl(R) = 0 if and only if R is a GCD domain. It is also shown that if R is quasi local with its maximal ideal M a t-ideal then Cl(R) = 0. We end this section with the following interesting result. If S is a multiplicative set of R generated by principal primes $\{p_i\}$ such that p_iR are of rank one and any infinite intersection of p_iR is zero and if $Cl(R_S) = 0$ then Cl(R) = 0. Moreover if, with the same hypothesis for S, $Pic(R_S) = 0$ then Pic(R) = 0. We use the above result to prove that the construction $R = K_1 + XK_2[X]$, where K_1 is a subfield of the field K_2, is such that Cl(R) = 0. In section 2, we study the case when G(R) = 0. We show that G(R) = 0 if and only if for all I, J $\in$ T(R) : IJ $\in$ T(R) if and only if for all I, J $\in$ T(P) $(IJ)^{-1} = I^{-1}J^{-1}$. Using this we show that if R is a $*$-domain of [22] i.e. if for all $a_1, \ldots, a_m; b_1, \ldots, b_n \in R - \{0\}$ we have $(\cap(b_j))(\cap(b_j)) = \cap(a_i b_j)$ then G(R) = 0. From this it follows that for a PVMD, R, G(R) = 0 is equivalent to R being a $*$-domain or a G-GCD domain of [1]. Finally we show that if $G(R_M)$

= 0 for every maximal ideal M then G(R) = 0. Consequently if each maximal ideal of R is a t-ideal then G(R) = 0. We use this observation to prove on the one hand that if dim R = 1 then G(R) = 0 and on the other hand that if R is a CP domain of [13] then G(R) = 0. Here R is a CP domain if every family $\{P_j\}$ of prime ideals of R has the property that for every integral ideal $A \subseteq \cup P_j$ we have $A \subseteq P_j$ for some j. In section 3 we study the case when Cl(R) is torsion; we give the general characterization and do little more than mention results on Cl(R) torsion in case R is a PVMD. These results have already been published [19]. In section 4, we characterize R for which G(R) is torsion and give a result treating PVMD's R with G(R) torsion.

1. The case when Cl(R) = 0

Theorem 1.1. Let R be an integral domain. Then the following are equivalent.

(i) Cl(R) = 0,

(ii) Every t-invertible t-ideal is principal,

(iii) If I and J are two finitely generated non-zero fractional ideals with $(IJ)_v$ principal then I_v and J_v are principal.

Proof. (i) ⇔ (ii). This is just the definition.

(ii) ⇒ (iii). If $(IJ)_v = (d)$, then since IJ is finitely generated, we have $(IJ)_v = (IJ)_t = (d)$ and so $(IJd^{-1})_t = R$. But then I and hence I_t is t-invertible and, because $I_t = I_v$, by (ii) I_v is principal.

(iii) ⇒ (ii). Let I be a t-invertible t-ideal of R. Then there exist two finitely generated ideals I′ and J′ such that $I = I'_v = I'_t$ and $R = (I'_t J')_t = (I' J')_t = (I' J')_v$. So, by (iii) $I = I'_v$ is principal.

Example 1.2. If R is a GCD-domain then Cl(R) = 0. This is because, in a GCD-domain R for all finitely generated $A \in F(R)$; A_v is principal.

An R-module M is called locally cyclic if every finitely generated R-submodule of M is contaived in a cyclic R-submodule of M. We note that a v-ideal of finite type is principal if and only if it is locally cyclic and so we can make the following statement.

Proposition 1.3. For R the following two properties are equivalent.

(i) Cl(R) = 0

(ii) Every t-invertible t-ideal of R is locally cyclic.

Recall that if x is a non-zero no-unit of R then x is **primal** if $x \mid ab$ in R implies that $x = a_1 b_1$ such that $a_1 \mid a$ and $b_1 \mid b$ [5]. If every non-zero non-unit of R is primal then R is called **pre-Schreier** [22]. Now D is pre-Schreier if and only if the inverse of every finitely generated fractional ideal is locally cyclic (see [22, Corollary 1.5 of [20] and the reference there]). These observations give rise to the following result.

Proposition 1.4. If R is a pre-Schreier domain then $Cl(R) = 0$. Thus in a pre-Schreier domain the following are equivalent for $A \in F(R)$.

(1) A is t-invertible,

(2) A_t is invertible,

(3) A_t is principal.

The proof is based on the fact that if A is t-invertible then A_t is a v-ideal of finite type; which is the inverse of a finitely generated ideal.

Corollary 1.5. Let R be a PVMD then the following are equivalent for R.

(1) $Cl(R) = 0$,

(2) R is a GCD-domain,

(3) R is a pre-Schreier domain.

Proof. (2) and (3) are equivalent by Theorem 3.6 of [22]. (2) ⇒ (1) follows from Example 1.2 and (1) ⇒ (2) because for every finitely generated $A \in F(R)$ we have A_v t-invertible and hence principal by (1).

We now proceed to indicate some interesting examples of integral domains R which are not GCD-domains, nor are they pre-Schreier, but for which $Cl(R) = 0$. For this we note the following result.

Proposition 1.6. Let R be a quasi local domain in which the maximal ideal is a t-ideal. Then a t-invertible fractional ideal of R is invertible, and hence principal.

Proof. Let M be the maximal ideal of R. Further let $A \in F(R)$. Then $(AA^{-1})_t = R$. Suppose that $AA^{-1} \neq R$. Then as $AA^{-1} \subseteq R$ we conclude that $AA^{-1} \subseteq M$. But then $(AA^{-1})_t \subseteq M$; because M is a t-ideal. But $(AA^{-1})_t = R$ a contradiction. Thus we conclude that $AA^{-1} = R$.

Corollary 1.7. If R is a quasi local domain with its maximal ideal a t-ideal then $Cl(R) = 0$.

It is easy to see from Example 1.2 that if R is a quasi local domain with $Cl(R) = 0$ then it is not necessary that its maximal ideal should be a t-ideal.

Corollary 1.8. If R is a one dimensional quasi local domain then $Cl(R) = 0$.

Proof. Because every integral t-ideal is contained in a maximal t-ideal (which is integral) and because a maximal t-ideal is prime (see Griffin [9] or

Jaffard [10]) we conclude that the maximal ideal of R is a t-ideal.

We now give an example of a quasi local domain R with $Cl(R) = 0$ is which for every $I \in F(R)$ there exists a $J \in F(R)$ such that $(IJ)_v$ is principal without I_v and J_v being principal.

Example 1.9. The one dimensional completely integrally closed non-valuation domain R constructed by Nagata in [11] and [12] is an example of an R with $Cl(R) = 0$, but with the property that for some $I, J \in F(R)$, $(IJ)_v$ is principal without I_v, J_v being principal.

Illustration. Because R is completely integrally closed, according to [8, Theorem 34.3], for all $I \in F(R)$ there exists $J \in F(R)$ such that $(IJ)_v = R$. If I_v and J_v are both of finite type then by Theorem 1.1., both must be principal. But since R is not a valuation domain (and hence is not a GCD domain) I_v is not principal for some finitely generated $I \in F(R)$. According to Theorem 1.1., this is because there is no finitely generated $J \in F(R)$ such that $(IJ)_v$ is principal.

To close this section we give yet another example of an integral domain R for which $Cl(R) = 0$. The importance of this example lies in the fact that to establish it we use a theorem similar to Nagata's theorem for UFD's.

Example 1.10. Let K_1 be a subfield of a field K_2 and let X be an indeterminate over K_2. Then the integral domain

$$R = K_1 + XK_2[X] = \{f(X) = a_0 + \sum_{i=1}^{n} a_i X^i \mid a_0 \in K_1 \text{ and } a_i \in K_2\}$$

has the property that $Cl(R) = 0$.

Lemma 1.11. Let R be an integral domain and let S be a subset of R generated multiplicatively by a family $\{p_i\}$ of primes such that for eah i, p_iR is of rank one and the intersection of an infinity of distinct p_iR is zero. If $Cl(R_S) = 0$ then $Cl(R) = 0$.

Proof. Let A be a t-invertible t-ideal of R. Since every fractional ideal F of R can be written as $F = B/d$ where B is an integral ideal of R and since F being a t-invertible t-ideal is equivalent to B being a t-invertible t-ideal, we can assume that $A \subseteq R$. By Lemma 2.5, to be proved in the next section, AR_S is a t-invertible t-ideal. But then, as $Cl(R_S) = 0$, $AR_S = aR_S$ for some $a \in R$.

Now $AR_S \cap R = aR_S \cap R = \{x \in R \mid xs \in aR\} = \{x \in R \mid a \mid xs$ for some $s \in S\}$. Because of the hypothesis on S we can write $a = a_1a_2$

where $a_2 \in S$ and a_1 is coprime to each member of S. But then $a_1a_2 \mid xs$, for $s \in S$, implies that $a_1 \mid x$. Consequently $A \subseteq AR_S \cap R = aR_S \cap R = a_1R$. Now being a t-invertible t-ideal A is a v-ideal of finite type and so $A = (x_1, \ldots, x_n)_v$. But $x_i = a_1d_i$, where $d_i \in R$, and so $A = a_1(d_1, \ldots, d_n)_v$. Now as $AR_S = a_1R_S$ we have $(d_1, \ldots, d_n)_vR_S = R_S$ which implies that $(d_1, \ldots, d_n)_v \cap S \neq 0$. Further, because of the property of $\{p_i\}$, $d_1, \ldots d_n$ have a greatest common divisor, say s, in S and consequently if $d_i = se_i$ we have $A = a_1s(e_1, \ldots, e_n)_v$. We claim that $1 \in (e_1, \ldots, e_n)_v$. To establish this claim we note that, because $(e_1, \ldots, e_n)_vR_S = (d_1, \ldots, d_n)_vR_S = R_S$, $(e_1, \ldots, e_n)_v \cap S \neq \emptyset$. Now suppose that $1 \notin (e_1, \ldots, e_n)_v$. Then there exists a fraction $x/y \in K$ such that $(e_1, \ldots, e_n)_v \subseteq (x/y)$ and $x \nmid y$. Then $y(e_1, \ldots, e_n)_v \subseteq (x)$ and so $x \mid yk$ for all $k \in (e_1, \ldots, e_n)_v$. In particular $x \mid yt$ for $t \in (e_1, \ldots, e_n)_v \cap S$. Since $x \nmid y$, x as we can reduce x/y so that x and y have no common factor from S, we conclude that $p \nmid y$. But then $x \mid ye_i$ for $i = 1, \ldots, n$ and consequently $p \mid e_i$ for all $i = 1, 2, \ldots, n$. This contradicts the assumption that s is a GCD of d_i in S. As this contradiction arises from the assumtion that $1 \notin (e_1, \ldots, e_n)_v$ we conclude that $(e_1, \ldots, e_n)_v = R$ and $A = a_1sR$.

In the more popular area we have the following result.

Corollary 1.12. With S as described in Lemma 1.11, if $\mathrm{Pic}(R_S) = 0$ then $\mathrm{Pic}(R) = 0$.

Illustration of Example 1.10. According to [21], $\dim(K_1 + XK_2[X]) = 1$ and every prime ideal other than $XK_2[X]$ is principal we conclude that for all $f(X) \in K_1 + XK_2[X]$ with $f(0) \neq 0$; $f(X)$ is a product of primes. So, $S = \{f(X) \in R \mid f(0) \neq 0\}$ meets the conditions of Lemma 1.11. But $R_S = (K_1 + XK_2[X])_{XK_2[X]}$ which is one dimensional again and so has class group zero. So by Lemma 1.11, $Cl(R) = 0$.

Remark 1.13. Lemma 1.11 seems to suggest the multiplicative mechanism behind the following well known result of Nagata.

Theorem. Let R be a Krull domain and let S be generated by primes of R. If R_S is a UFD then so is R.

Fr the proof we note that S meets the requirements of Lemma 1.11. So $Cl(R) = 0$; and for R Krull this means that R is a UFD.

2. The case when $G(R) = 0$

Theorem 2.1. Let R be an integral domain. Then the following are equivalent.

(i) $G(R) = 0$,
(ii) For all $I, J \in T(R)$; $IJ \in T(R)$,
(iii) For all $I, J \in T(R)$; $(IJ)_t = R$ implies that $IJ = R$,
(iv) For all $I, J \in T(R)$; $(IJ)^{-1} = I^{-1}J^{-1}$.

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious.
(iv) ⇒ (i). Assuming (iv) we show that every t-invertible t-ideal of R is in fact invertible. Let $I \in T(R)$. Then $I^{-1} \in T(R)$. By (iv) $(II^{-1})^{-1} = I^{-1}(I^{-1})^{-1} = I^{-1}I$. Now $II^{-1} \subseteq R$ and so $(II^{-1})^{-1} \supseteq R$, dut $II^{-1} = (II^{-1})^{-1}$ implies that $II^{-1} = R$.

Corollary 2.2. Let R be a $*$-domain; then $G(R) = 0$.

Proof. According to [21], R is a $*$-domain if and only if, for all finitely generated $A, B \in F(R)$; $(A_vB_v)^{-1} = (AB)^{-1} = A^{-1}B^{-1}$. Now because for all $I, J \in T(R)$, I and J are v-ideals of finite type, part (iv) of Theorem 2.1 applies.

According to [1] R is a **generalized GCD-domain (G-GCD domain)** if R satisfies one of the following equivalent conditions:

(i) Every finite intersection of non-zero principal fractional ideals is invertible,

(ii) Every v-ideal of finite type is invertible (i.e. $D_f(R) = \mathrm{Inv}(R)$).

Corollary 2.3. Let R be a PVMD. Then the following properties are equivalent for R.

(i) $G(R) = 0$,
(ii) R is a $*$-domain,
(iii) R is a G-GCD domain,
(iv) $D_f(R)$ is closed under the usual product of fractional ideals of R.

Proof. Because R is a PVMD, $D_f(R) = T(R)$ and so each $A \in D_f(R)$ is a finite intersection of principal fractional ideals. Now because $D_f(R) = T(R)$ (iv) ⇔ (i) by Theorem 2.1. Further, because a G-GCD domain is locally GCD [1] and because a locally GCD-domain is a $*$-domain [22, Theorem 2.1] we have (iii) ⇒ (ii). Moreover by Corollary 2.2, (ii) ⇒ (i). All that remains is to show that (iv) ⇒ (iii). But this is obvious because if $I \in D_f(R)$ then $I^{-1} \in D_f(R)$ and by (iv) $II^{-1} \in D_f(R)$. So $II^{-1} = (II^{-1})_v = R$. Consequently every v-ideal of finite type of R is invertible.

We note that $Cl(R) = 0$ implies $G(R) = 0$. Now recalling the exact sequence $0 \to Pic(R) \to Cl(R) \to G(R) \to 0$ we note that for a quasi local domain R, $Cl(R) \cong G(R)$; because in this case $Pic(R) = 0$.

So for a quasi local domain R, Cl(R) is trivial/torsion if and only if G(R) is.

Now an interesting sufficient conditions for G(R) to be trivial.

Proposition 2.4. Let R be an integral domain such that $G(R_M) = 0$ for each maximal ideal M. Then $G(R) = 0$.

The Proof is based on the following three lemmas.

Lemma 2.5. Let I be a v-ideal of finite type such that I^{-1} is also of finite type. Then $IR_S = (IR_S)_v$ (and so is a v-ideal of finite type) for every multiplicative set S.

Proof. If I and I^{-1} are v-ideals of finite type, there exist finitely generated $A, B \in F(R)$ such that $I = A_v$ and $I^{-1} = B_v$. Now by [18, Lemma 4], $(AR_S)^{-1} = A^{-1}R_S = I^{-1}R_S$. But as $(AR_S)^{-1} = ((AR_S)_v)^{-1} = ((A_vR_S)_v)^{-1}$ [18, Lemma 4] $= (A_vR_S)^{-1} = (IR_S)^{-1}$ we have $(IR_S)^{-1} = I^{-1}R_S$. Now $((IR_S)^{-1})^{-1} = (I^{-1}R_S)^{-1}$ and because I^{-1} is a v-ideal of finite type we have $(I^{-1}R_S)^{-1} = I_vR_S = IR_S$.

Lemma 2.6. Let I be a t-invertible t-ideal of R and let S be a multiplicative set of R. Then IR_S is a t-invertible t-ideal of R_S.

Proof. Since $(II^{-1})_v = ((II^{-1})_v)^{-1} = R$, these ideals are of finite type. Now using Lemma 2.5 we have $R_S = (II^{-1})_vR_S = ((II^{-1})_vR_S)_v$. Now as I and I^{-1} are both v-ideals of finite type we have finitely generated $A, B \in F(R)$ such that $I = A_v$ and $I^{-1} = B_v$. So $R_S = ((II^{-1})_vR_S)_v = ((A_vB_v)_vR_S)_v$ $= ((AB)_vR_S)_v = (ABP_v)_v$ (since AB is finitely generated) $= (AR_SBR_S)_v =$ $((AR_S)_v\,(BR_S)_v)_v = ((A_vR_S)_v\,(B_vR_S)_v)_v = (A_vR_S\,B_vR_S)_v = (IR_S \cdot I^{-1}R_S)_v$. So $R_S = (IR_SI^{-1}R_S)_v$ $(ABR_S)_v = (ABR_S)_t = (AR_SBR_S)_t = ((AR_S)_t\,(BR_S)_t)_t =$ $((AR_S)_v\,(BR_S)_v)_t = ((A_vR_S)_v\,(B_vR_S)_v)_t = ((IR_S)_v\,(I^{-1}R_S)_v)_t = (IR_S \cdot I^{-1}R_S)_t$.

Lemma 2.7. Let I be a v-ideal of finite type in R. Then I is invertible if and only if IR_M is principal for every maximal ideal M of R.

Proof. Let I be a v-ideal of finite type such that IR_M is principal for every maximal ideal M of R. Then for each M, $(IR_M)^{-1}$ is prisnipal and by Lemma 2.5,

$II^{-1}R_M = IR_MI^{-1}R_M = IR_M(IR_M)^{-1} = R_M$. But then $II^{-1} = \cap\,(II^{-1})R_M = R$; where M ranges over all maximal ideals of R.

Proof of Proposition 2.4. Let $I \in T(R)$. By Lemma 2.7 it is sufficient to show that IR_M is principal for every maximal ideal M. But IR_M is a t-invertible t-ideal by Lemma 2.6 and IR_M is invertible because $G(R_M) = 0$. Because in a quasi local domain invertible is principal, we conclude that IR_M is principal for each maximal ideal M of R and Lemma 2.7 applies.

Proposition 2.4 gives rise to a number of questions.

Question A. Do we have similar sufficient condition for $Cl(R)$ to be trivial?

The answer to this question is no, because, in a Dedekind domain R, for every maximal ideal M, $Cl(R_M) = 0$ but $Cl(R) \neq 0$ if R is not a PID. However if $Pic(R) = 0$ then, because $Cl(R) \cong G(R)$, we can state a positive result.

Corollary 2.8. Let R be such that $Pic(R) = 0$ then the following statements are equivalent.

(i) $Cl(R) = 0$

(ii) $G(R) = 0$

(iii) $Cl(R_M) = 0$ for each maximal ideal M,

(iv) $G(R_M) = 0$ for each maximal ideal M.

Corollary 2.9. Let R be a semi quasi local integral domain. If R satisfies any one of the following we have $Cl(R) = 0$.

(a) For every maximal ideal M, MR_M is a t-ideal (b) R is one dimensional (c) R is a locally GCD (d) R is a $*$-domain.

Question B. If $Cl(R) = 0$ what can be said about $Cl(R_S)$ for a multiplicative set S?

Question C. If $G(R) = 0$ what can be said about $G(R_S)$ for a given S?

It would be interesting to find some examples to indicate that the answers to Questions B and C are note straightforward. It would also be interesting to find the conditions, on R and S, under which $Cl(R) = 0$ $(G(R) = 0)$ should imply $Cl(R_S) = 0$ $(G(R_S) = 0)$.

Looking back, again, at the exact sequence
$0 \to Pic(R) \to Cl(R) \to G(R) \to 0$, we note that if $G(R) = 0$ then $Pic(R) = Cl(R)$. So the class of integral domains R with $G(R) = 0$ is interesting in that at least for these integral domains R, $Cl(R)$ is a decent and well-known group. For this reason we give a quick list of integral domains R with $G(R) = 0$.

Corollary 2.10. Let R be an integral domain. If one of the following is satisfied by R then $G(R) = 0$.

(a) R is Prufer.

(b) R is reflexive i.e. every ideal of R is a v-ideal [14].

(c) Every maximal ideal of R is a t-ideal e.g. R is one dimensional.
(d) R_M is a $*$-domain for every maximal ideal M of R, e.g. (i) R is locally pre-Schreier, GCD, factorial; because each of these is a $*$-domain [22].
(e) $D_f(R)$ is closed under the usual product of fractional ideals.

Proof. (a) If R is Prufer, then $T(R) = Inv(R)$.
(b) If R is reflexive, for all $I, J \in T(R)$ each of IJ is a v-ideal and hence a t-invertible t-ideal and we can apply Theorem 2.1.
(c) By Proposition 1.6, for each maximal ideal M of R, $G(R_M) = 0$. Now by Proposition 2.4, $G(R) = 0$.
(d) By Corollary 2.2, for each M, $G(R_M) = 0$ and by Proposition 2.4, $G(R) = 0$.
(e) $D_f(R)$ closed under the usual product implies that $T(R)$ is closed under the usual product and this, by Theorem 2.1, is equivalent to $G(R) = 0$.

Remark 2.11. None of the conditions (a), (b), (c) and (d) are necessary for R to have $G(R) = 0$. For if $R = K[X, Y]$, where K is a field, then because R is a UFD, $Cl(R) = 0$ and hence $G(R) = 0$? but R is neither reflexive nor Prufer and dim $R = 2$, whereas the maximal t-ideals of R are of rank one. Moreover $R = K[[X^2, X^3]]$, being one dimensional local, has the property that $Cl(R) = G(R) = 0$; but R is not a $*$-domain by [22, Example 2.8]. Finally, the condition (e) does not seem to be necessary but we cannot produce an example to support this view. It would be interesting to have an example of R with $G(R) = 0$ and $D_f(R)$ not closed under the usual product. In fact it would be interesting to study integral domains R with the property that for all $A, B \in F(R)$, $(AB)_v = A_vB_v$.

We close this section with the mention of a class of integral domains R with the property:

(CP) Whenever an ideal A is contained in the union of a family of primes $\{P_j\}$, $A \subseteq P_j$ for some j. These integral domains were discussed in a more general setting in [15] and [13]. We show that these integral domains R have $G(R) = 0$.

Recall that a prime ideal P minimal over an ideal of the form $0 \neq (a) : (b) = \{x \in R \mid xb \in (a)\}$ ($\neq R$) is called an associated prime of (a principal ideal of) R; see [4]. Obviously every non-zero prime ideal of R contains an associated prime of R. So if U(R) is the set of units of R we have $R - U(R) = \cup P$ where P ranges over associated primes of R. Thus if R has the (CP) property then every maximal ideal of R is an associatep. 1703] an associated prime is a t-ideal. So in a (CP)-domain every maximal ideal is a t-ideal. Now using (c) of Corollary 2.10 we have the following result.

Corollary 2.12. For a (CP)-domain R, $C(R) = 0$.

Remark 2.13. That an associated prime of R is a t-ideal, has been mentioned a number of times but it has not been adequately proved even once. In view of the basic nature of the result we include the proof indicated in [20].

Proof of the fact that an associated prime of R is a t-ideal.

Let P be an associated prime of R. Then by Lemma 6 of [18], PR_P is a t-ideal. Now to show that P is a t-ideal let A be a finitely generated ideal of R such that $A \subseteq P$. Then by Lemma of [18], $(AR_P)_v = (A_vR_P)_v$. If $A_v \nsubseteq P$ then $A_vR_P = R_P$ and so $(A_vR_P)_v = (AR_P)_v = R_P$. But PR_P is a t-ideal of R_P and AR_P a finitely generated ideal contained in PR_P. So $R_P = (AR_P)_v \subseteq PR_P$ a contradiction. So, for all finitely generated $A \subseteq P$, $A_v \subseteq P$.

3. The case when Cl(R) is torsion

Proposition 3.1. Let R be an integral domain. Then the following properties are equivalent:

(i) Cl(R) is torsion,

(ii) For any t-invertible t-ideal I, $(I^n)_v$ is principal for some n,

(iii) For any two finitely generated ideals I_1, I_2, if $(I_1I_2)_v = R$ then $(I_1^n)_v$ is principal for some $n \geqslant 1$.

Proof. (i) ⇒ (ii) ⇒ (iii) are obvious. For (iii) ⇒ (i) let $I \in T(R)$. Then $I^{-1} \in T(R)$ and $(II^{-1})_t = R$. So $(II^{-1})_v = R$ and by (iii) $(I^n)_v$ is principal for some $n \geqslant 1$.

The case of Cl(R) being torsion when R is a PVMD has been studied in [19]. The following result can be traced back to [19].

Theorem 3.1. For a PVMD, R, the following are equivalent:

(i) Cl(R) is torsion,

(ii) For each pair $x, y \in R$ there exists $n(x, y) \in N$ such that $(x^n) \cap (y^n)$ is principal,

(iii) For every finitely generated ideal I, $(I^n)_v$ is principal for some $n \geqslant 1$,

(iv) $Cl(R[X])$ is torsion.

In the study of torsion groups it is of interest to know the cases where the property of being torsion causes the group to collapse.

Proposition 3.3. Let (R, M) be aquasi local domain with the property that for all t-invertible t-ideals I, $(I^n)_v = (I_v)^n$ then $Cl(R)$ torsion implies that $Cl(R)$ is trivial.

Proof. If for $I \in T(R)$, $(I^n)_v$ is principal then say $(I^n)_v = dR$. But then $(I_v)^n = dR$; which implies that I_v is invertible and hence principal.

4. The case when G(R) is torsion

Proposition 4.1. The following statements are equivalent for an integral domain R.

(i) $G(R)$ is torsion

(ii) For any two finitely generated ideals I_1, I_2 of R, if $(I_1 I_2)_v = R$ then $(I_1^n)_v (I_2^n)_v = R$ for some $n \geqslant 1$,

(iii) For every pair of finitely generated $I_1, I_2 \in F(R)$, if $(I_1, I_2)_v = R$ then $(I_1^n)_v (I_2^n)_v = R$ for some $n \geqslant 1$.

The proof is obvious.

Theorem 4.2. The following statements are equivalent for a PVMD, R:

(i) $G(R)$ is torsion,

(ii) For every finitely generated $I \in F(R)$, $(I^n)_v$ is invertible for some $n \geqslant 1$,

(iii) For $a_1, \ldots, a_m \in K - \{0\}$ and for some $n \geqslant 1$, $\cap (a_i^n)$ is invertible.

Proof. (i) $\Rightarrow$ (ii). Because in a PVMD every v-ideal of finite type is t-invertible, we have the implication.

(ii) $\Rightarrow$ (iii). Let $a_1, \ldots, a_m \in K - \{0\}$. Then $\cap (a_i) = I$ for some v-ideal of finite type. By (ii) $(I^n)_v$ is invertible for some $n \geqslant 1$. So by (iii) of Corollary 3.2 of [20], $\cap (a_i^n) = \big(\cap (a_i^n)\big)_v = (I^n)_v$ is invertible.

(iii) $\Rightarrow$ (i). If I is a t-invertible t-ideal then $I = \cap (a_i)$ where $a_i \in K - \{0\}$ and using $\big((\cap (a_i))^n\big)_v = \cap (a_i^n)$ we conclude the proof.

Remark. This paper is a revised version of "On the class group" by these authors (1984/85). Since then the work on class groups has gone on. What was introduced as a mere facility for PVMD's is being checked for its uses in general integral domains. Of these we mention a result which is relevant to our work in this paper.

Given that R is an itegral domain and x an indeterminate over R there is an injective homomorphism $\varnothing: Cl(R) \to Cl(R[x])$ defined by $A \to A[x]$. Recently Gabelli [6] has proved that Q is an isomorphism if and only if R is integrally closed. Recall that R is a **finite conductor** domain if for all a,

$b \in R - \{0\}$, $aR \cap bR$ is finitely generated. **Noetherian** and **Coherent** domains are finite conductor domains.

Proposition. Given that R is a finite conductor domain, $Cl(R[x]) = 0$ implies that R is a GCD-domain.

Proof. Because $A \to A[x]$ is an injective homomorphism from $Cl(R) \to Cl(R[x])$ we conclude that $Cl(R) = 0$. But then $Cl(R) \cong Cl(R[X])$. This, by [G], means that R is integrally closed. Now an integrally closed finite conductor domain is a PVMD [18] and a PVMD, R with $Cl(R) = 0$ is a GCD-domain [19].

REFERENCES

[1] Anderson, D.D. and D.F. Anderson, Generalized GCD-domains, Comment. Math. Univ. St. Pauli, 28(2) (1979) 215-221.

[2] Bouvier, A., The local class group of a Krull domain, Canad. Math. Bull. 26(1) (1983) 13-19.

[3] ———, Le groupe des classes d'un anneaux integre, 107 eme Congres des Societes Savantes, Brest fasc. IV (1982) 85-92.

[4] Brewer, J. and W. Heinzer, Associated primes of principal ideals, Duke Math. J. 41 (1974) 1-7.

[5] Cohn, P., Bezout rings and their subrings, Proc. Camb. Phil. Soc. 64 (1968) 251-264.

[6] Gabelli, S., On divisorial ideals in polynomial rings over Mori domains, Comm. Algebra 15(11) (1987), 2349-2370.

[7] Fossum, R., The Divisor Class Group of a Krull Domain, Springer-Verlag, band 74 (1973).

[8] Gilmer, R., Multiplicative Ideal Theory, Dekker (1972).

[9] Griffin, M., Some results on v-multiplication rings, Canad. J. Math., 19 (1967) 710-722.

[10] Jaffard, P., Les Systeme d'Ideaux, Dunod, Paris (1960).

[11] Nagata, M., On Krull's conjecture concerning valuation rings, Nagoya Math. J., 4 (1952) 29-33.

[12] ———, Corrections to my paper, "On Krull's conjecture concerning valuation rings", Nagoya Math. J., 9 (1955) 209-212.

[13] Pakala, J., and T. Shores, On compactly packed rings, Pacific J. Math., 97(1) (1981) 197-201.

[14] Querre, J., Sur les anneaux reflexifs, Canad. J. Math., 27 (6) (1975) 1222-1228.

[15] Reis, C., and T. Viswanathan, A compactness property of prime ideals in noetherian rings, Proc. Amer. Math. Soc. 25 (1970) 353-356.

[16] Ryckaert, A., Sur le groupe des classes et le groupe local des classes d'um anneau integre, Thesis, Claude Bernard University Lyon 1 (1986).

[17] Storch, U., Fasfaktorielle ringe, Schriftenreihe Math. Inst. Univ. Munster, Heft 36 (1967).

[18] Zafrullah, M., On finite conductor domains, Manuscripta Math., 24 (1978) 191-204.

[19] ______, A general theory of almost factoriality, Manuscripta Math., 51 (1985) 29-62.

[20] ______, The v-operation and intersections of quotient rings of integral domains, Comm. Algebra 13 (8) (1985) 1699-1712.

[21] ______, On generalized Dedekind domains Mathematika, 33 (1986) 285-295.

[22] ______, On a property of pre-Schreier domains Comm. Algebra, 15 (9) (1987) 1895-1920.

[23] ______, Rings between D [X] and K [X] (preprint).

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