9 P. M. Cohn's Completely Primal Elements

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1 INTRODUCTION

P.M. Cohn [6] introduced the notion of a completely primal element in an integral domain. He called a nonzero element x of an integral domain D *primal* if $x|ab$ implies $x = x_1x_2$ where $x_1|a$ and $x_2|b$, and called x *completely primal* if every factor of x is primal. He defined D to be a *Schreier domain* if D is integrally closed and every nonzero element of D is (completely) primal. (Later the second author [15] defined D to be *pre-Schreier* if every nonzero element of D is (completely) primal.) Cohn then observed that D is pre-Schreier if and only if the group of divisibility $G(D)$ of D is a Riesz group. This led the second author [17] to study completely primal elements in partially ordered groups. Here an element $x \in G^+$, G a partially ordered group, is *primal* if $x \leq a_1+a_2$ for $a_1, a_2 \in G^+$ implies $x = x_1+x_2$ where $0 \leq x_i \leq a_i$, and x is *completely primal* if each $y \in G$ with $0 \leq y \leq x$ is primal. The element $x \in D$ is easily seen to be (completely) primal if and only if $xU(D)$ is (completely) primal as an element of $G(D)$.

Fuchs [7] showed that a directed partially ordered group G is a Riesz group if and only if every element of G^+ is (completely) primal if and only if for every finite set of elements $a_1, \ldots, a_n \in G$, the set $U(a_1, \ldots, a_n) = \{g \in G \mid g \geq a_1, \ldots, a_n\}$ is lower directed. In Section 2 we "localize" this condition and show (Theorem 2.1) that an element $c \in G^+$ is completely primal if and only if $U(c, x)$ is lower directed for each $x \in G^+$. In Section 3 this result is applied to the group of divisibility of an integral domain D to show (Theorem 3.1) that an element $x \in D$ is completely primal if and only if $(x) \cap (y)$ is locally cyclic for all nonzero $y \in D$. (Recall that

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$(x) \cap (y)$ is *locally cyclic* if for $a_1, \ldots, a_n \in (x) \cap (y)$, there exists $a \in D$ with $(a_1, \ldots, a_n) \subseteq (a) \subseteq (x) \cap (y)$.) Thus an integral domain D is pre-Schreier if and only if for each pair of elements $x, y \in D$, $(x) \cap (y)$ is locally cyclic ([11], [15]). We show (Corollary 3.2) that for $x, y \in D^*$ with x completely primal, $(x) \cap (y)$ is principal if and only if $(x) \cap (y)$ is a finite type v-ideal. We also consider elements $x \in D^*$, called *extractors*, with the stronger property that $(x) \cap (y)$ is principal for all $y \in D^*$. Thus an extractor is completely primal.

Cohn [6] also proved the following analog of Nagata's UFD Theorem which we call Cohn's Theorem: Let D be an integral domain and let S be a subset of D which is multiplicatively generated by completely primal elements of D. If D_S is pre-Schreier, then so is D. This raises the question that if S is generated by extractors and if D_S is a GCD-domain, under what conditions is D a GCD-domain? This question has been considered in several papers including [1], [3], [10], and [12]. We consider this question in Section 4, where we take an alternative approach introduced by Nour El Abidine [13]. We first show (Theorem 4.1) that a finite type v-ideal containing an extractor is principal. This is used to show (1) (Theorem 4.2) that if S is generated by extractors and every v-ideal of D of the form $(a_1) \cap \cdots \cap (a_n)$ has finite type, then the natural map of t-class groups $Cl_t(D) \to Cl_t(D_S)$ is an isomorphism and (2) (Theorem 4.3) if S is generated by extractors and each ideal $(a) \cap (b)$ has finite type, then D_S is a GCD-domain implies that D is a GCD-domain.

We follow standard terminology and notation from the theory of partially ordered groups and from multiplicative ideal theory. For any undefined terms or results, the reader is referred to [9].

2 COMPLETELY PRIMAL ELEMENTS IN PARTIALLY ORDERED GROUPS

Throughout this section let G denote a not necessarily commutative, directed partially ordered group $(G, +, 0, \leq)$. An element x in G is *primal* if $x \geq 0$ and $x \leq a_1 + a_2$ for $a_1, a_2 \in G^+ = \{g \in G \mid g \geq 0\}$ implies that $x = x_1 + x_2$ where $0 \leq x_i \leq a_i$, and x is *completely primal* if every member of $[0, x] = \{g \in G \mid 0 \leq g \leq x\}$ is primal. Clearly if every element of G^+ is primal, then every element of G^+ is completely primal. Recall that G is a *Riesz group* if G satisfies the following interpolation property: For $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ for $1 \leq i, j \leq 2$, there exists $c \in G$ with $a_i \leq c \leq b_j$ for $1 \leq i, j \leq 2$. From Fuchs [7, Theorem 2.2(5)] it follows that G is a Riesz group if and only if every element of G^+ is (completely) primal.

Primal and completely primal elements in partially ordered groups were introduced by the second author in [17]. It was noted that the conjugate of a (completely) primal element is (completely) primal and that the sum of two (completely) primal elements is (completely) primal. It was further shown that if S is the set of all completely primal elements of G, then the subgroup $\langle S \rangle$ generated by S is an o-ideal of G (i.e., a normal, convex, directed subgroup of G) and a Riesz group.

An element $e \in G^+$ is an *extractor* if $e \wedge x$ exists for all $x \in G^+$ (or equivalently, if $e \vee x$ exists for all $x \in G^+$). It may easily be proved directly that an extractor is completely primal. However, this follows immediately from our characterization of

completely primal elements given in Theorem 2.1; see Corollary 2.2. Clearly G is lattice-ordered if and only if each $c \in G^+$ is an extractor. In [4], we showed that G is a Riesz group (respectively, a lattice-ordered group) if and only if for each convex, directed, normal subgroup $H \subsetneq G$, $G - H$ contains a completely primal element (respectively, an extractor).

Thus completely primal elements occupy a somewhat fundamental position in the theory of directed, partially ordered groups and hence can be studied for their own sake. The aim of this section is to provide, in the form of the following theorem, a characterization of completely primal elements which is remarkably similar to a characterization of Riesz groups [7, Theorem 2.2(2)]. In the next two sections we will indicate the applications to multiplicative ideal theory.

For $x, y \in G$, let $U(x, y) = \{g \in G \mid g \geq x, y\}$ and $L(x, y) = \{g \in G \mid g \leq x, y\}$. A nonempty subset $S \subseteq G$ is *lower directed* (respectively, *upper directed*) if for $s_1, s_2 \in S$, there exists $s \in S$ with $s \leq s_1, s_2$ (respectively, $s_1, s_2 \leq s$). Let $x, y \in G$. Certainly if $x \vee y$ exists (or equivalently, if $x \wedge y$ exists), then $U(x, y) = [x \vee y) = \{g \in G \mid g \geq x \vee y\}$ and $L(x, y) = (x \wedge y] = \{g \in G \mid g \leq x \wedge y\}$ and are lower and upper directed, respectively.

THEOREM 2.1. For an element $c \in G^+$, the following statements are equivalent.
(1) c is completely primal.
(2) For every $x \in G^+$, the set $U(c, x) = \{t \in G \mid t \geq c, x\}$ is lower directed.
(3) For every $x \in G^+$, the set $L(c, x) = \{t \in G \mid t \leq c, x\}$ is upper directed.

Proof. (1) $\Rightarrow$ (2). Suppose that c is completely primal. To show that $U(c, x)$ is lower directed assume that $\alpha_1, \alpha_2 \in U(c, x)$. Then $\alpha_1 = c + \delta_1 = x + \beta_1$ and $\alpha_2 = c + \delta_2 = x + \beta_2$ where $\beta_i, \delta_i \in G^+$. Since c is completely primal and $c \leq x + \beta_1$ we have $c = c_1 + c_2$ with $0 \leq c_1 \leq x$ and $0 \leq c_2 \leq \beta_1$. Let $x = c_1 + s_1$, $\beta_1 = c_2 + s_2$. Substituting for x and c in $\alpha_2 = c + \delta_2 = x + \beta_2$ we get $c_1 + c_2 + \delta_2 = c_1 + s_1 + \beta_2$ which gives $c_2 + \delta_2 = s_1 + \beta_2$.

Now as c_2 is primal and $c_2 \leq s_1 + \beta_2$ we have $c_2 = c_{21} + c_{22}$ where $0 \leq c_{21} \leq s_1$ and $0 \leq c_{22} \leq \beta_2$. Let $s_1 = c_{21} + t_1$ and $\beta_2 = c_{22} + t_2$. Thus we have $\alpha_1 = x + \beta_1 = c_1 + s_1 + c_2 + s_2 = c_1 + s_1 + c_{21} + c_{22} + s_2$ and $\alpha_2 = x + \beta_2 = c_1 + s_1 + \beta_2 = c_1 + s_1 + c_{22} + t_2$. If we write $h = c_1 + s_1 + c_{22}$ then $\alpha_1 = h + (-c_{22} + c_{21} + c_{22}) + s_2$ and $\alpha_2 = h + t_2$. Consequently we have $h \leq \alpha_1, \alpha_2$.

Further as $x = c_1 + s_1$ we have $h = x + c_{22} \geq x$ and as $s_1 = c_{21} + t_1$ we have $h = c_1 + c_{21} + t_1 + c_{22} = c_1 + c_{21} + c_{22} + (-c_{22} + t_1 + c_{22}) = c + (-c_{22} + t_1 + c_{22}) \geq c$. Thus $h \in U(c, x)$ and $h \leq \alpha_1, \alpha_2 \in U(c, x)$ and hence $U(c, x)$ is lower directed.

(2) $\Rightarrow$ (1). Suppose that for every $x \in G^+$, $U(c, x)$ is lower directed and let $c \leq a + b$ where $a, b \in G^+$. Then $a + b, a + c \in U(c, a)$. By the lower directed property there exists $h \in U(c, a)$ such that $h \leq a + b, a + c$ and clearly $-a + h \leq b, c$. Thus $c = c_1 + (-a + h)$. Now as $c \leq h$ we have $c_1 - a \leq 0$ giving $0 \leq c_1 \leq a$. Consequently $c = c_1 + (-a + h)$ where $0 \leq c_1 \leq a$ and $0 \leq -a + h \leq b$ and if we put $c_2 = -a + h$ we have $c = c_1 + c_2$ where $c_1 \leq a$ and $c_2 \leq b$. This shows that c is primal. For the completely primal part let $d \in [0, c]$; then $c = c' + d$ and $c' + U(d, x) = U(c' + d, c' + x) = U(c, c' + x)$. But $U(c, c' + x)$ is lower directed which leads to the conclusion that $c' + U(d, x)$ is lower directed. Now for $\alpha_1, \alpha_2 \in U(d, x)$ we have $c' + \alpha_1, c' + \alpha_2 \in c' + U(d, x)$ which leads to the existence of h in $c' + U(d, x)$ such that $h \leq c' + \alpha_1, c' + \alpha_2$. But then $-c' + h$ belongs to $U(d, x)$ and $-c' + h \leq \alpha_1, \alpha_2$ leading to the conclusions that $U(d, x)$ is lower directed and that d is primal.

(2) $\Rightarrow$ (3). Let $\alpha_1, \alpha_2 \leq c, x$. Now $-\alpha_1, -\alpha_2 \geq -c, -x$, so $c + x - \alpha_1, c + x - \alpha_2 \geq c + x - c, c + x - x = c$. So $c + x - \alpha_1, c + x - \alpha_2 \in U(c, c + x - c)$. So by (2) there exists $t \in G$ with $c, c + x - c \leq t \leq c + x - \alpha_1, c + x - \alpha_2$. So $-c, c - x - c \geq -t \geq \alpha_1 - x - c, \alpha_2 - x - c_1$. Thus $x = -c + c + x$, $c = c - x - c + c + x \geq -t + c + x \geq \alpha_1 - x - c + c + x = \alpha_1$, $\alpha_2 - x - c + c + x = \alpha_2$. So $-t + c + x \in L(c, x)$ and $-t + c + x \geq \alpha_1, \alpha_2$. So $L(c, x)$ is upper directed. (3) $\Rightarrow$ (2). Similar to the proof of (2) $\Rightarrow$ (3). □

COROLLARY 2.2. (1) If c is a completely primal element of G and if $d \in G^+$ is such that $c \wedge d \neq 0$, then there is h in $U(c, d)$ such that $h < c + d$.

(2) If x is a strictly positive completely primal element of G and $y \in G^+$ such that $x \wedge y \neq 0$, then there exists h such that $0 < h \leq x, y$.

(3) An element c is a completely primal element if and only if for every $x \in G^+$ and for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in U(c, x)$ there exists h in $U(c, x)$ such that $h \leq \alpha_1, \alpha_2, \ldots, \alpha_n$. A similar statement holds for $L(c, x)$.

(4) If $c \in G$ is an extractor, then c is completely primal.

Proof. (1) If $c \wedge d \neq 0$ then $U(c, d) \neq [c + d)$, for $U(c, d) = [c + d)$ if and only if $c \wedge d = 0$. Now suppose on the contrary that there is no such $h < c + d$. Then as not all $k \in U(c, d)$ satisfy $k \geq c + d$ there must be k in $U(c, d)$ such that $c + d$ and k are incomparable. By the lower directed property there is h in $U(c, d)$ such that $h \leq c + d, k$ leading to the conclusion that $h < c + d$.

(2) By (1) there is k in $U(x, y)$ such that $k < y + x$. Let $k = x + a = y + b$. Since x is completely primal and $x \leq y + b$ we have $x = x_1 + x_2$ such that $0 \leq x_1 \leq y$ and $0 \leq x_2 \leq b$. If $x_1 = 0$ then $x_2 = x$ and $x \leq b$. This gives $k = y + b \geq y + x$ contradicting the fact the $k < y + x$. Hence $x_1 \neq 0$. But then $0 < x_1 \leq x, y$.

(3) If c is completely primal then for every $x \in G^+$, $U(c, x)$ is lower directed. So for $\alpha_1, \alpha_2 \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq U(c, x)$ there is $h_1 \in U(c, x)$ such that $h_1 \leq \alpha_1, \alpha_2$. For $h_1, \alpha_3 \in U(c, x)$ there exists $h_2 \in U(c, x)$ such that $h_2 \leq h_1, \alpha_3$ which gives $h_2 \leq \alpha_1, \alpha_2, \alpha_3$. Similarly proceeding we arrive at $h \in U(c, x)$ such that $h \leq \alpha_1, \alpha_2, \ldots, \alpha_n$. The converse is obvious.

(4) While it can easily be proved directly that an extractor is completely primal, note that this is immediate from Theorem 2.1 since if c is an extractor, then $U(c, x) = [c \vee x) = \{g \in G \mid g \geq c \vee x\}$ and $L(c, x) = (c \wedge x] = \{g \in G \mid g \leq c \wedge x\}$ are clearly directed below and above, respectively. □

3 COMPLETELY PRIMAL ELEMENTS IN INTEGRAL DOMAINS

Let D be an integral domain with quotient field K and group of units $U(D)$. Cohn [6] called an element $x \in D^* = D - \{0\}$ *primal* if for $a, b \in D^*$ with $x | ab$, $x = x_1 x_2$ ($x_1, x_2 \in D$) where $x_1 | a$ and $x_2 | b$, and called x *completely primal* if every factor of x is primal. D is called a *pre-Schreier domain* [15] if every nonzero element of D is (completely) primal and D is a *Schreier domain* [6] if D is an integrally closed pre-Schreier domain.

An element $x \in D^*$ is called an *extractor* if $(x) \cap (y)$ is principal for each $y \in D^*$. Since $(x, y)^{-1} = (x^{-1}) \cap (y^{-1}) = \frac{1}{xy}((x) \cap (y))$, x is an extractor if and only if $(x, y)_v$ is principal for all $y \in D^*$. (Here for a nonzero fractional ideal I of D, $I_v = (I^{-1})^{-1} = \bigcap\{zD \mid z \in K \text{ with } zD \supseteq I\}$.) As in the partially ordered group case, it may easily be proved directly that an extractor is completely primal; or see Theorem 3.1. And certainly a completely primal element is primal. Later in this section we will give examples to show that neither implication can be reversed.

A fractional ideal I of D is said to be *locally cyclic* if for $a_1, \ldots, a_n \in I$, there exists $a \in I$ with $(a_1, \ldots, a_n) \subseteq (a) \subseteq I$. And the fractional ideal I of D is said to be *locally co-cyclic* if for $a_1, \ldots, a_n \in K^*$ with $(a_1) \cap \cdots \cap (a_n) \supseteq I$, there exists $a \in K^*$ with $(a_1) \cap \cdots \cap (a_n) \supseteq (a) \supseteq I$. Clearly in either of the above definitions, we can take $n = 2$.

The group $G(D) = K^*/U(D)$, partially ordered by $aU(D) \leq bU(D) \Leftrightarrow a | b$ in D, is called the *group of divisibility of* D. Note that $1U(D)$ is the identity element of $G(D)$ and that $G(D)^+ = \{zU(D) \mid z \in D^*\}$. We often write $G(D)$ additively, so $aU(D) + bU(D) = abU(D)$. Also note that $G(D)$ is order-isomorphic to the multiplicative group $\mathrm{Prin}(D)$ of nonzero principal fractional ideals $\{zD \mid z \in K^*\}$, ordered by reverse inclusion $xD \leq yD \Leftrightarrow yD \subseteq xD$, via the map $xU(D) \leftrightarrow xD$.

Let $x \in D^*$ be primal. If $xU(D) \leq a_1 U(D) + a_2 U(D)$ in $G(D)$ with $0 \leq a_1 U(D)$, $a_2 U(D)$, then $x | a_1 a_2$ in D, so $x = x_1 x_2$ where $x_1 | a_1$ and $x_2 | a_2$. Thus $xU(D) = x_1 U(D) + x_2 U(D)$ where $0 \leq x_i U(D) \leq a_i U(D)$, so $xU(D)$ is primal in $G(D)$. The converse is just as easily shown. Hence $x \in D^*$ is (completely) primal if and only if $xU(D)$ is (completely) primal as an element of $G(D)$. Now for $x, y \in K^*$,

$$\begin{aligned} U(xU(D), yU(D)) &= \{zU(D) \mid zU(D) \geq xU(D), yU(D)\} \\ &= \{zU(D) \mid zD \subseteq xD \cap yD\} \end{aligned}$$

and

$$\begin{aligned} L(xU(D), yU(D)) &= \{zU(D) \mid zU(D) \leq xU(D), yU(D)\} \\ &= \{zU(D) \mid zD \supseteq xD + yD\} \\ &= \{zU(D) \mid zD \supseteq (x, y)_v\}. \end{aligned}$$

So $U(xU(D), yU(D))$ is lower directed if and only if $xD \cap yD$ is locally cyclic while $L(xU(D), yU(D))$ is upper directed if and only if $xD + yD$ (or $(x, y)_v$) is locally co-cyclic. Also, $x \in D^*$ is easily seen to be an extractor if and only if $xU(D)$ is an extractor in $G(D)$.

From the remarks given in the previous paragraph, the characterizations of completely primal elements in partially ordered groups given in Theorem 2.1 easily translate to integral domains. We summarize these in the following theorem.

THEOREM 3.1. Let D be an integral domain and let $0 \neq x \in D$. Then the following statements are equivalent.

(1) x is completely primal (respectively, an extractor).

(2) $xU(D)$ is completely primal (respectively, an extractor) in $G(D)$.

(3) $xD \cap yD$ is locally cyclic (respectively, principal) for all $y \in D^*$.

(4) $(xD + yD)_v$ is locally co-cyclic (respectively, principal) for all $y \in D^*$. □

COROLLARY 3.2. Let x be a completely primal element of D and let $y \in D^*$. Then $(x) \cap (y)$ is principal if and only if there are $x_1, \ldots, x_n \in (x) \cap (y)$ such that $(x) \cap (y) = (x_1, \ldots, x_n)_v$. □

COROLLARY 3.3. ([11], [15]) An integral domain D is pre-Schreier if and only if for every pair of elements x, y in D^*, $xD \cap yD$ is locally cyclic if and only if for all $x, y \in K^*$, $(x, y)^{-1}$ is locally cyclic. □

Integral domains of current interest in multiplicative ideal theory mostly fall under the following two categories. (1) Atomic domains, that is, integral domains in which every nonzero nonunit is a product of atoms (i.e., irreducible elements). These domains include Noetherian domains, Krull domains, Mori domains, and rings of the type $K + XL[X]$ where $K \subseteq L$ are fields [2]. (2) Domains in which for each pair $x, y \in D^*$, $(x) \cap (y)$ is a finite type v-ideal. (Recall that a v-ideal I has *finite type* if there exist $a_1, \ldots, a_n \in I$ with $(a_1, \ldots, a_n)_v = I$.) These domains include coherent domains, Prüfer domains, GCD-domains, and Prüfer v-multiplication domains. (A Prüfer v-multiplication domain is a domain with the property that for each $x, y \in D^*$, there exists a finitely generated fractional ideal A with $((x,y)A)_v = D$. Hence $A_v = (x,y)^{-1} = \frac{1}{xy}((x) \cap (y))$, so $(x) \cap (y)$ has finite type.) We note that in both of these classes of integral domains, a completely primal element is actually an extractor. As Cohn [6] observed, a primal atom is prime, so a completely primal element that is a product of atoms is a product of principal primes and hence an extractor. This takes care of the atomic domain case. Corollary 3.2 shows that for the second class of domains, a completely primal element is an extractor.

We next give examples to show that none of the implications extractor $\Rightarrow$ completely primal $\Rightarrow$ primal can be reversed.

In [17] it was shown that if D is not a pre-Schreier domain and if K is its field of quotients, then in the ring $D + XK[X]$, the element X is primal but not completely primal (for each nonzero element of D is a factor of X). In fact, $D + XK[X]$ is pre-Schreier if and only if X is completely primal if and only if D is pre-Schreier. For the proof we may note that X is completely primal if and only if each nonzero element of D is primal. But this makes D pre-Schreier. But if D is pre-Schreier, $S = D - \{0\}$ consists of primal elements and $K[X] = (D + XK[X])_S$ is pre-Schreier, so by Cohn's Theorem $D + XK[X]$ is pre-Schreier. Next we give an example of a product of atoms that is primal but not a product of primes, and hence not completely primal. Thus, while a primal atom is prime, a primal product of atoms need not be a product of primes.

EXAMPLE 3.4. Let K_1 be a proper subfield of the field K_2 and let X be an indeterminate over K_2. Then in $D = K_1 + XK_2[X] = \{a_0 + \sum a_i X^i \mid a_0 \in K_1, a_i \in K_2\}$ the element X^2 is a primal product of atoms that is not completely primal.

According to [2], D is an atomic domain in which the only non-prime atoms are elements of the type αX where $\alpha \in K_2$. Moreover, every nonzero element of D can be written as $\alpha X^r(1 + Xf_1(X))$ where $r \geq 0$, $f_1(X) \in K_2[X]$, and $\alpha \in K_2$ unless $r = 0$, in which case $\alpha \in K_1$. Suppose that $X^2 | fg$ where $f, g \in D$. Then $f = \alpha X^r(1+Xf_1(X))$ and $g = \beta X^s(1+Xg_1(X))$. If $r \geq 3$, we can write $X^2 = X^2 \cdot 1$ where $X^2 | f$ and $1 | g$. Similarly we can deal with the case $s \geq 3$. Now suppose that $r = 2$. Two cases arise: (i) $s = 0$ and (ii) $s \geq 1$. If $s = 0$ then $\beta \in K_1$ and as $X^2 | fg$, $\alpha \in K_1$. Hence $X^2 = X^2 \cdot 1$ where $X^2 | f$ and $1 | g$. If $s \geq 1$, we can write $X^2 = \beta^{-1}X \cdot \beta X$. Clearly $\beta^{-1}X \mid \alpha X^2(1 + f_1(X))$ and $\beta X \mid \beta X^s(1 + Xg_1(X))$. The same reasoning applies when $s = 2$. This leaves the case $r = s = 1$. But as $X^2 | fg$ we must have $\alpha\beta \in K_1$. Thus $X^2 = \alpha X \cdot \alpha^{-1} X$ where $\alpha X | f$ and $\alpha^{-1} X | g$.

That X^2 is not completely primal follows from the fact that X is not a prime in $K_1 + XK_2[X]$.

If $K_1 \subset K_2$ is a finite extension, then $K_1 + XK_2[X]$ is Noetherian. Hence even in a Noetherian domain a primal element need not be completely primal. □

To get an example of a completely primal element that is not an extractor, it suffices to give an example of a (pre-)Schreier domain that is not a GCD-domain. Examples of such domains may be found in [11] and [15–16].

4 NAGATA-LIKE THEOREMS

Nagata showed that if D is a Noetherian domain and S is a multiplicatively closed subset of D generated by principal primes with D_S a UFD, then D itself is a UFD. Later it was observed that the same result holds for an atomic domain. This result is now called Nagata's Theorem. A related theorem of Nagata states that if D is a Krull domain and S is generated by principal primes, then the natural map of divisor class groups $\mathrm{Cl}(D) \to \mathrm{Cl}(D_S)$ is an isomorphism. Then Cohn [6] showed that if D is an integral domain and S is a multiplicatively closed subset of D generated by completely primal elements with D_S pre-Schreier, then so is D. We call this result Cohn's Theorem.

These results raise the question that if S is generated by extractors and if D_S is a GCD-domain, under what conditions is D a GCD-domain? Two previous papers [10] and [12] imposed conditions that would force each $d \in D^*$ to be a product $d = xy$ where $y \in \bar{S}$, the saturation of S, and x is coprime to each member of S. For a review of the scope of this approach, the reader may consult [3]. Other Nagata-like theorems involving the t-class group may be found in [1] and [8].

Nour El Abidine [13] took a different approach. A domain D *satisfies property* P^* if for all $x_1, \ldots, x_n \in D^*$ there exist $y_1, \ldots, y_m \in D^*$ such that $(x_1, \ldots, x_n)^{-1} = (y_1, \ldots, y_m)_v$. Thus D satisfies P^* if and only if for $a_1, \ldots, a_n \in D^*$, $(a_1) \cap \cdots \cap (a_n)$ is a finite type v-ideal. He showed that if S is generated by primes of D and if D has property P^*, then $\mathrm{Cl}_t(D_S) \approx \mathrm{Cl}_t(D)$ and that if D_S is a GCD-domain, then D is also a GCD-domain. Here $\mathrm{Cl}_t(D)$ is the t-class group of D which was studied in [5]. We briefly recall its definition. For a fractional ideal A of D, $A_t = \bigcup\{J_v \mid 0 \neq J \subseteq A$ is finitely generated$\}$. A nonzero fractional ideal A is t-invertible if $(AA^{-1})_t = D$. The set $T(D)$ of t-invertible t-ideals of D forms a group under the t-product: $A \times B = (AB)_t$. The set $\mathrm{Prin}(D)$ of nonzero principal fractional ideals of D is a subgroup of $T(D)$. The quotient group $T(D)/\mathrm{Prin}(D)$ is called the *t-class group of D*. (For D a Krull domain, $\mathrm{Cl}_t(D)$ is the usual divisor class group.) If S is any multiplicatively closed subset of D, then the map $\phi : \mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ given by $[A] \to [AD_S]$ is a homomorphism where $[A]$ represents the class of A in $\mathrm{Cl}_t(D)$. For more details concerning the t-class group, the reader is referred to [3] and [5].

In this section we establish that property P^* is so strong a condition that Nour El Abidine's condition that "S is generated by primes" can be replaced by "S is generated by a set of completely primal elements". (Of course, in the presence of property P^*, a completely primal element is an extractor.)

We first prove some results about extractors.

THEOREM 4.1. Let D be an integral domain.
(1) If $x \in D^*$ is an extractor and $d|x$, then d is an extractor.
(2) If $x, y \in D^*$ are extractors, then so is xy.
(3) $x \in D^*$ is an extractor if and only if for each finite type integral v-ideal A of D with $x \in A$, A is principal. Moreover, in this case $A = (a)$, where a is an extractor.

Proof. (1) Let $x = dt$. Now $t(d, y)_v = (td, ty)_v$, so $(d, y)_v$ is principal if and only if $(x, ty)_v$ is principal. Thus if x is an extractor, so is d.

(2) Suppose that $x, y \in D^*$ are extractors and let $z \in D^*$. Now $(x, z)_v = (d)$ for some $d \in D$, so $x = x_1 d$ and $z = z_1 d$ where $(x_1, z_1)_v = D$. Let $(y, z_1)_v = (k)$, so $y = y_2 k$ and $z_1 = z_2 k$ where $(y_2, z_2)_v = D$. Thus $(x_1, z_2)_v = D$. Then $(xy, z)_v = dk(x_1 y_2, z_2)_v = dkD$ since $D = (x_1, z_2)_v (y_2, z_2)_v = ((x_1, z_2)(y_2, z_2))_v \subseteq (x_1 y_2, z_2)_v \subseteq D$.

(3) ($\Leftarrow$) For this implication, we need only take $A = (x, y)_v$ where $y \in D^*$. ($\Rightarrow$) Let $A = (a_1, \ldots, a_n)_v$. Now since x is an extractor, $(a_1, x)_v = (a_1') \subseteq A$ for some a_1' and a_1' being a factor of x is an extractor. Now since a_1' is an extractor, $(a_1', a_2)_v = (a_2') \subseteq A$ for some a_2' and a_2' being a factor of a_1' is again an extractor. Continuing this process, we get $a_1', \ldots, a_n'$ where $(a_i', a_{i+1})_v = (a_{i+1}') \subseteq A$. Hence $(a_1, \ldots, a_n) \subseteq (a_n') \subseteq A$. Thus $(a_n') = A$. Moreover, a_n' is an extractor. □

THEOREM 4.2. Suppose that D is an integral domain and S is a multiplicatively closed subset of D generated by extractors. Then the natural map $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is injective. If D further satisfies property P^*, then $\mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ is surjective and hence an isomorphism.

Proof. The natural map $\phi : \mathrm{Cl}_t(D) \to \mathrm{Cl}_t(D_S)$ given by $\phi([A]) = [AD_S]$ is a homomorphism since AD_S is a t-invertible t-ideal whenever A is and $(AB)_t D_S = (AD_S BD_S)_t$ for t-invertible t-ideals A and B [5]. Suppose that $[A] \in \ker \phi$ where A is a t-invertible t-ideal of D. After multiplying by a suitable element of D, we may assume that A is integral. Then $AD_S = rD_S$ for some $r \in D^*$. Hence $r^{-1}AD_S = D_S$. Since $r^{-1}A$ is a finite type v-ideal there is an element $s \in S$ so that $sr^{-1}A$ is an integral t-invertible t-ideal of D with $sr^{-1}AD_S = D_S$. But then $sr^{-1}A \cap S \neq \emptyset$, so $sr^{-1}A$ contains an extractor. By Theorem 4.1(3) $sr^{-1}A$ is principal and hence $[A] = [sr^{-1}A] = [D]$.

For the surjectivity of ϕ it is sufficient, according to [8], to show that the map $T(D) \to T(D_S)$ given by $A \to AD_S$ is surjective. Let $\bar{A} \in T(D_S)$. Then there is a fractional ideal $K = (k_1, \ldots, k_n)$ of D such that $\bar{A} = ((k_1, \ldots, k_n)D_S)_v$. Since D satisfies property P^*, K^{-1} is of finite type and so $\bar{A} = (k_1, \ldots, k_n)_v D_S$ by [5, Lemma 2.5]. For the same reasons [14, Lemma 4], $D_S = ((KD_S)_v (KD_S)^{-1})_v = (KD_S K^{-1} D_S)_v = ((KK^{-1})D_S)_v = (KK^{-1})_v D_S$. Now $(KK^{-1})_v$ is integral and of finite type and $(KK^{-1})_v \cap S \neq \emptyset$, so again by Theorem 4.1(3) $(KK^{-1})_v$ is principal. Thus $K_v \in T(D)$ and $K_v D_S = \bar{A}$. So ϕ is surjective. □

Theorem 4.2 generalizes [1, Theorem 2.3]. To generalize the second of the previously mentioned results of Nour El Abidine, we do not need to assume the full force of property P^*.

THEOREM 4.3. Let D be an integral domain such that for all $a, b \in D^*$, $(a) \cap (b)$ is a finite type v-ideal and let S be generated by a set of extractors of D. If D_S is a GCD-domain, then so is D.

Proof. By Cohn's Theorem D_S is a GCD-domain (and hence pre-Schreier) implies that D is pre-Schreier. So every nonzero element of D is completely primal. But then by Corollary 3.2 every completely primal element of D is an extractor. Thus D is a GCD-domain. □

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