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PAIRS OF RINGS WITH THE SAME PRIME IDEALS

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1. Introduction. There are numerous instances in which the partners in an extension of commutative rings $R \subset T$ have the same prime ideals, i.e., in which $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. Although this equality is intended to be taken set-theoretically, the identification easily extends to the corresponding spaces endowed with their Zariski topologies (see Proposition 3.5(a)), but of course need not extend to an identification of $\operatorname{Spec}(R)$ and $\operatorname{Spec}(T)$ as affine schemes. Perhaps the most striking recent illustration of this phenomenon arises from the work of Hedstrom and Houston [**14**] in which R is a pseudo-valuation domain and T is a suitable valuation overring. Other examples may be found by means of the $D + M$ construction, either in its traditional form [**12**, p. 560] or in the generalized situation introduced by Brewer and Rutter [**5**]. Important as these examples are, our main purpose in this paper is to study the phenomenon $\operatorname{Spec}(R) = \operatorname{Spec}(T)$ without prior restrictions on the rings involved. Two byproducts for pseudo-valuation domains are a simple recovery of some results of [**14**] (see Corollaries 3.30 and A.5) and a unified approach to some recent work on coherence due to Dobbs [**9**] and Hedstrom and Houston [**15**] (see Proposition A.7 and Corollary A.11). The key tool for the latter work, which also serves to characterize the Noetherian case (in Corollary 3.29), is Lemma 3.27, which is an adaptation of methods introduced by Dobbs and Papick [**10**] and modified in [**5**]. Apart from the unifying and generalizing role of this article, it should be noted, however, that the condition $\operatorname{Spec}(R) = \operatorname{Spec}(T)$ permits new ideal-theoretic behavior not present in the examples mentioned above (see Example 3.14).

It must be admitted that the following types of (integral) domains cannot, except in the case of fields, be properly contained in domains with the same prime ideals: completely integrally closed (Corollary 3.16), Prüfer (Remarks 3.17(b)), GCD-also known as pseudo-Bézout (Corollary A.4), and domains containing a principal prime. In a more positive vein, Theorem 3.25 shows, given any domain R, how to determine all domains T which contain R and satisfy $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. The technique, which diverts attention from T's of the specified type and instead hunts for suitable fields, may be applied also to nondomains (see Remarks 3.33(b)). It yields information about intermediate rings (in

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Corollary 3.26—see also Proposition 3.31), and (in Examples 3.34) reveals the generally "non-pseudo-valuation-domain" flavor of proper extensions $R \subset T$ of Noetherian domains with $\operatorname{Spec}(R) = \operatorname{Spec}(T)$.

As to organization, we advise that Section 3 constitutes the main body of the paper. Theorem 3.10 presents a necessary and sufficient condition that a given pair of commutative rings $R \subset T$ have the same primes; namely that the set of maximal ideals of R be comparable to the set of maximal ideals of T. Although this condition may not be weakened in general (see Remark 3.9 and Example 3.12), simplifications are available in certain cases (see Corollaries 3.20 and 3.21). Section 2, which the reader may wish to consult only as needed, presents a summary of part of the material in [**14**] on pseudo-valuation domains (recast for our purposes in Proposition 2.5) and a brief description of the $D + M$ constructions. Note that the new pullback construction for pseudo-valuation domains in Proposition 2.6 suggests the pullback view of the construction in Theorem 3.25 and the above-mentioned examples issuing from it. Appendix A studies transfer of the coherence property between domains with the same primes; this and the above-mentioned application to GCD-domains are consequences of Corollary A.3 concerning finite-conductor domains. Appendix B treats similar transfer questions, for divided and going-down domains. Its main result, Proposition B.2, is independent of the rest of the paper and may be read for motivation at any time.

All rings are assumed to be commutative, with 1. Given rings $R \subset T$, we also assume that the 1 of the subring R is the same as the 1 of T. Unexplained material is standard, as in [**4**] or [**12**].

2. Some pullback constructions. This short section collects information about certain constructions which are used in examples later in the paper and, to motivate Theorem 3.25 and its applications, we also point out how to view these constructions as pullbacks.

The first construction is what we shall term the *classical* $D + M$ *construction*. Its data consist of a valuation domain of the form $V = K + M$, where K is a field and M is the nonzero maximal ideal of V, and a proper subring D of K. The ideal theory of the ring $D + M$ is well known (see [**12**, Theorem A, p. 560], summarized in [**2**, Theorem 2.1]), as is the catalogue of the overrings of $D + M$ (see [**2**, Theorem 3.1]). Henceforth, we assume familiarity with the references just cited. For later purposes, we pause to record an easy upshot of those references.

PROPOSITION 2.1. *In the context and notation of the "classical $D + M$ construction" described above,* $\operatorname{Spec}(D + M) = \operatorname{Spec}(K + M)$ *if and only if D is a field.*

A more general construction, herein termed the *generalized* $D + M$ *construction*, was introduced in [**5**]. Its data consist of a domain

$T = K + M$, where K is a field and M is a nonzero maximal ideal of T, and a proper subring D of K. The adjective "generalized" points out that T need not be a valuation domain; indeed, T need not be quasilocal. An analogue of Proposition 2.1 for the "generalized $D + M$ construction" will be given in Corollary 3.11 below.

We pause to remark how the "generalized $D + M$ construction" (and, as a special case, the "classical $D + M$ construction") may be viewed as a pullback in the category of (commutative) rings (with 1). In the context of the preceding paragraph, we claim that $D + M$ is canonically isomorphic to the pullback of the diagram

$$\begin{array}{ccc} & & K + M \\ & & \downarrow \\ D & \rightarrow & K \end{array}$$

in which the vertical map is projection onto the first direct summand and the horizontal map is the inclusion. We omit the routine verification of this assertion.

The last construction to be noted here concerns the notion of a pseudo-valuation domain (for short, a PVD), which was introduced by Hedstrom and Houston [**14**] and has been studied subsequently in [**8**], [**9**] and [**15**]. A domain R, with quotient field K, is said to be a PVD in case each prime ideal P of R is *strongly prime*, in the sense that whenever $x, y \in K$ satisfy $xy \in P$, then either $x \in P$ or $y \in P$. As the terminology suggests, any valuation domain is a PVD [**14**, Proposition 1.1]. Although the converse is false [**14**, Example 2.1], any PVD must, at least, be quasilocal [**14**, Corollary 1.3]. While many PVD's arise from the "classical $D + M$ construction" (see [**8**, Proposition 4.9]), it can be shown by minor modifications of the argument in [**9**, Remark 2.2] that the PVD introduced in [**14**, Example 3.6] cannot be the result of any "generalized $D + M$ construction."

For reference purposes, we record the following triviality.

PROPOSITION 2.2. *If domains* $R \subset T$ *satisfy* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$, *then* R *is a* PVD *if and only if* T *is a* PVD.

Proof. (modulo Lemma 3.1). Without loss of generality, R and T are not fields. Then, for easy general reasons explained in Lemma 3.1, R and T share the same quotient field. As they also share the same primes, the result follows from the above definitions of "strongly prime" and PVD.

Much of the motivation for this paper comes from the following result of Hedstrom and Houston. First, we recall some notation for conductors. If R is a domain with quotient field K and I and J are R-submodules of K, then $(I : J) = \{x \in K : xJ \subset I\}$.

PROPOSITION 2.3. ([**14**, Theorem 2.7, Proposition 2.8 and Corollary 2.9]) *The following conditions are equivalent for a quasilocal domain R with maximal ideal M:*

(1) *R is a* PVD*;*

(2) *R has a (unique) valuation overring V with maximal ideal M. Moreover, if R is a* PVD *but not a valuation domain, then M is not a principal ideal of R, the valuation domain V in* (2) *is given by $V = (R : M)$, and V satisfies* $\mathrm{Spec}(R) = \mathrm{Spec}(V)$.

The construction for V in Proposition 2.3 can be revisited so that the case in which R is a valuation domain is treated simultaneously. (See Proposition 2.5 below.) Instead of using $(R : M)$, we consider the ring $(M : M)$, which is in fact the largest overring of R in which M is an ideal. First, we give a useful lemma about conductors. As usual, a nonzero ideal I of a domain R is called *divisorial* if $I = (R : (R : I))$.

LEMMA 2.4. *Let R be a quasilocal domain with nonzero maximal ideal M. Then:*

(a) *$(R : M) = (M : M)$ if and only if M is not a principal ideal of R. In case $M = Rr$ for some $r \in R$, then $(R : M) = Rr^{-1}$ and $(M : M) = R$.*

(b) *If M is not a divisorial ideal of R, then $(R : M) = (M : M)$.*

Proof. As any nonzero principal ideal must be divisorial, (b) follows from (a). As for (a), note that $(M : M) \subset (R : M)$ in general. If $x \in (R : M)\backslash(M : M)$, the conditions $xM \subset R$ and $xM \not\subset M$ force $xM = R$, so that $M = Rx^{-1}$. The remaining assertions are evident.

PROPOSITION 2.5. *The following conditions are equivalent for a quasilocal domain R with maximal ideal M:*

(1) *R is a* PVD*;*

(2) *$(M : M)$ is a (the unique) valuation overring of R with maximal ideal M;*

(3) *R has a valuation overring V such that* $\mathrm{Spec}(R) = \mathrm{Spec}(V)$.

Proof. One need only combine Proposition 2.3 with Lemma 2.4(a), after noting that $(M : M) = R$ in case R is a valuation domain.

A pullback construction, based on [**13**], was developed in [**8**, Lemma 4.5 (iv), (v)] to describe the PVD's of Krull dimension exceeding 1. We close this section with a rather different pullback construction which serves to describe arbitrary PVD's.

PROPOSITION 2.6. *Pseudo-valuation domains are precisely the pullbacks in the category of (commutative) rings (with 1) of diagrams of the form*

$$\begin{tikzcd} & V \arrow[d] \\ k \arrow[r] & K \end{tikzcd}$$

in which V is a valuation domain having maximal ideal P and residue class field $K = V/P$, the vertical map is the canonical surjection, k is a subfield of K, and the horizontal map is the inclusion.

Sketch of proof. If R is a PVD with maximal ideal P, it may be shown that R is (canonically isomorphic to) the pullback of the diagram of the stipulated form in which $V = (P : P)$ and $k = R/P$. Conversely, if R is the pullback of a diagram of the above form, then R may be identified with $\{x \in V : x + P \in k\}$ as a subring of V. As this ring is easily shown to have unique maximal ideal P and $P \neq 0$ without loss of generality, it follows that V is an overring of R, so that Proposition 2.3 implies that R is a PVD, as required.

3. Rings with the same primes. Our main result, Theorem 3.10, will characterize the pairs of rings mentioned in the title. We begin by examining the role of conductors and overrings (shades of [**14**]!) and by showing that the rings of interest are quasilocal. First, recall that if R is a ring with total quotient ring K and if I is an ideal of R, then $(I : I)$ denotes the conductor $\{x \in K : xI \subset I\}$.

LEMMA 3.1. *Let $R \subset T$ be rings possessing a common nonzero ideal, I. If I contains a non-zerodivisor of R, then T is contained in the total quotient ring of R, and indeed $T \subset (I : I)$.*

Proof. Let y be a non-zerodivisor of R which is contained in I. For any $t \in T$, note that $x = ty \in I$, since I is an ideal of T. Hence $t = xy^{-1}$ belongs to the total quotient ring of R, and the assertions are evident.

LEMMA 3.2. *If R is a proper subring of a ring T, then R and T have at most one maximal ideal in common.*

Proof. We prove the contrapositive of the assertion. Let M and N be distinct maximal ideals common to R and T. As M and N are comaximal ideals of R, we have $M + N = R$. Similarly, since M and N are comaximal in T, $M + N = T$, and so $R = T$, as required.

Combining Lemmas 3.1 and 3.2 leads to

PROPOSITION 3.3. *Let $R \subset T$ be distinct rings such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Then R (and T) must be quasilocal. If, in addition, R is a domain but not a field, then $T \subset (M : M)$, where M denotes the maximal ideal of R.*

Remarks 3.4. (a) Instead of assuming that R is a domain but not a field for the second assertion of Proposition 3.3, we need only assume the presence of suitable non-zerodivisors, in the sense of Lemma 3.1; the same proof works. Of course, without the assumption about non-zerodivisors, the conclusion of Lemma 3.1 fails; for example, any proper field extension $F \subset L$ satisfies $\mathrm{Spec}(F) = \mathrm{Spec}(L)$, although L is not contained in (the total quotient ring of) F. To avoid technicalities (and with an eye toward the intended applications) we shall often treat the case of domains, leaving to the reader the reformulations (when valid) involving non-zerodivisors.

(b) The hypothesis of Lemma 3.2 cannot be weakened, as it is easy to find nonquasilocal domains $R \subset T$ with but one maximal ideal in common. For an example, use the "generalized $D + M$ construction" as follows. Corresponding to any proper field extension $L \subset K$, consider the polynomial ring $T = K[X] = K + M$, where $M = XT$; set $R = L + M$. Then T is an overring of R, the only maximal ideal common to R and T is M, and neither R nor T is quasilocal.

(c) Let R, T and M be as in the second assertion of Proposition 3.3. We claim that $T \subset (P : P)$ for each $P \in \mathrm{Spec}(R)$. Indeed, $(M : M) \subset (P : P)$; i.e., we claim that if $x \in (M : M)$ and $p \in P$, then $xp \in P$. For a proof, note that $x^2 p \in M$ since $x^2 \in (M : M)$ and $p \in M$, so that $(xp)^2 = (x^2 p)p \in MP \subset P$. Since P is prime, $xp \in P$, as required.

PROPOSITION 3.5. *Let $R \subset T$ be rings such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Then:*

(a) *The Zariski topologies on* $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(T)$ *coincide.*

(b) *If R is a domain, then $R_P = T_{T \setminus P}$ for each nonmaximal prime ideal P of R.*

Proof. (a) As $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, it follows that the set of nonunits of R coincides with the set of nonunits of T. Hence, the Zariski topologies on $\mathrm{Spec}(R)$ and $\mathrm{Spec}(T)$ have the same basic open sets (cf. [**4**, p. 99]).

(b) Trivially, $R_P \subset T_{T \setminus P}$. For the reverse inclusion, let M be the unique maximal ideal of R. (M exists by Proposition 3.3, as we may suppose $R \neq T$.) Select $m \in M \setminus P$. For the typical element of $T_{T \setminus P}$, expressed as ts^{-1} where $t \in T$ and $s \in T \setminus P$, observe that $ts^{-1} = (tm)(sm)^{-1} \in R_P$, since $tm \in M \subset R$ and $sm \in R \setminus P$.

The preceding result shows that if $\mathrm{Spec}(R)$ and $\mathrm{Spec}(T)$ are equal as sets (of prime ideals arising from the rings $R \subset T$), then $\mathrm{Spec}(R)$ and $\mathrm{Spec}(T)$ are equal as topological spaces. However, R and T need not be isomorphic in this case. (For an example, let R be any PVD which is not a valuation domain, let M be the maximal ideal of R, and let T be the valuation domain $(M : M)$.) Hence, $\mathrm{Spec}(R)$ and $\mathrm{Spec}(T)$ need not be equal as affine schemes (as Spec gives a duality between the category of (commutative) rings (with 1) and the category of affine schemes). If R is a domain (and $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ as sets), such an inequality qua schemes is manifested by an inequality of corresponding stalks of the structure sheaves at some prime; in this case, Proposition 3.5(b) shows that the maximal ideal is the only such prime.

In view of Proposition 2.3, Proposition 3.5(b) leads to another proof of the result [**14**, Proposition 2.6] that R_P is a valuation domain whenever P is a nonmaximal prime ideal of a pseudo-valuation domain R.

As usual, if A is a ring, we shall let $J(A)$ denote its Jacobson radical and $\operatorname{Max}(A)$ its set of maximal ideals.

LEMMA 3.6. *Let $R \subset T$ be rings. If $J(R)$ is an ideal of T, then $J(R) \subset J(T)$.*

Proof. It is enough to show that $J(R) \subset N$ for each maximal ideal N of T. If this condition fails, then $J(R) + N = T$, and so $r + n = 1$ for some $r \in J(R)$ and $n \in N$. As $n \in 1 + J(R)$, we have that n is a unit of R and, a fortiori, a unit of T, contradicting $n \in N$, to complete the proof.

COROLLARY 3.7. *Let $R \subset T$ be rings such that R is quasilocal and its maximal ideal M is also an ideal of T. Then $M \subset J(T)$.*

Proof. $J(R) = M$. Apply Lemma 3.6.

The inclusion asserted in Corollary 3.7 may be strict. For an example using the "classical $D + M$ construction" in the context of a valuation domain $K + M$, set $R = k + M$ and $T = D + M$, where $k \subset D \subset K$ such that k is a field and D is a quasilocal nonfield. Then $J(T) = J(D) + M$ properly contains M.

PROPOSITION 3.8. *Let $R \subset T$ be rings such that R is quasilocal with maximal ideal M. Then* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$ *if and only if* $M \in \operatorname{Max}(T)$.

Proof. The "only if" half is trivial. Conversely, suppose that $M \in \operatorname{Max}(T)$. By Corollary 3.7, $M \subset J(T)$, and so M is the only maximal ideal of T. Hence, $\operatorname{Spec}(T) \subset \operatorname{Spec}(R)$. For the reverse inclusion, let $P \in \operatorname{Spec}(R)$. For any $t \in T$ and $p \in P$, note that $(tp)^2 = pt^2p \in MTP = MP \subset P$, while $tp \in TM = M \subset R$. As $P \in \operatorname{Spec}(R)$, we have that $tp \in P$, and so P is an ideal of T. Finally, to see that P is prime in T, suppose $xy \in P$ for elements x and y of T; our task is to show that at least one of x and y is in P. This is evident in case both x and y are in M, since P is prime in R. For the remaining possibility, suppose without loss of generality that $x \in T\backslash M$. Then $x^{-1} \in T$ and $y = x^{-1}(xy) \in TP \subset P$, as required.

The above argument that $\operatorname{Spec}(R) \subset \operatorname{Spec}(T)$ if $M \in \operatorname{Max}(T)$ appears in [**14**, Theorem 2.7] for the case R a PVD and T a suitable valuation overring of R.

Remark 3.9. To show that Proposition 3.8 cannot be strengthened, we now produce a quasilocal domain R with maximal ideal M and an overring S of R such that $M \in \operatorname{Spec}(S)$ and $\operatorname{Spec}(R) \neq \operatorname{Spec}(S)$. Use the "classical $D + M$ construction" as follows: Let L be a field, set $K = L(X)$, and consider $T = K[[Y]] = K + M$, where $M = YT$. It is easy to check that $R = L + M$ and $S = L[X] + M$ have the asserted properties.

THEOREM 3.10. *For rings $R \subset T$, the following are equivalent:*
(1) $\operatorname{Spec}(R) = \operatorname{Spec}(T)$;
(2) $\operatorname{Max}(R) = \operatorname{Max}(T)$;
(3) $\operatorname{Max}(R) \subset \operatorname{Max}(T)$;
(4) $\operatorname{Max}(T) \subset \operatorname{Max}(R)$;
(5) *R and T have the same radical ideals.*
Moreover, if (any of) the above conditions hold and $R \neq T$, then R is quasilocal.

Proof. We shall prove (1) $\Rightarrow$ (5) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (4).

Since the radical ideals of a ring are just the intersections of families of the ring's prime ideals, (1) $\Rightarrow$ (5) is immediate. Moreover, (5) $\Rightarrow$ (2), as the maximal ideals of a ring are just the maximal elements (with respect to inclusion) of the ring's set of radical ideals. The implication (2) $\Rightarrow$ (3) is trivial. To show that (3) $\Rightarrow$ (1), we may assume $R \neq T$. By Lemma 3.2 and (3), R is quasilocal (thus disposing of the theorem's final assertion), and Proposition 3.8 may be applied to yield (1).

As (2) $\Rightarrow$ (4) trivially, it remains to show (4) $\Rightarrow$ (2). By Lemma 3.2, we may suppose, given (4), that T is quasilocal, say with maximal ideal M. It is enough to prove that R cannot have a maximal ideal $N \neq M$. Given such N, select $x \in M \backslash N$ and $y \in N \backslash M$. As $y \in T \backslash M$, it follows that y is a unit of T, and so $xy^{-1} \in MT = M$, whence $x = (xy^{-1})y \in MN \subset N$, the desired contradiction. This completes the proof.

We pause to answer a question which was left open in Section 2.

COROLLARY 3.11. *Let T be a ring of the form $K + M$, where K is a field and M is a maximal ideal of T. Let D be a proper subring of K, and set $R = D + M$. Then* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$ *if and only if T is quasilocal and D is a field.*

Proof. If N is any maximal ideal of D, then $N + M \in \operatorname{Max}(R)$ since $R/(N + M) \cong D/N$. Now, suppose that $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. Theorem 3.10 gives $\operatorname{Max}(R) = \operatorname{Max}(T)$, and so each such $N = 0$, whence D is a field. Moreover, since $R \neq T$, Proposition 3.3 shows that T is quasilocal. Conversely, if T is quasilocal and D is a field, then $M \in \operatorname{Max}(R)$ since $R/M \cong D$, and the conclusion $\operatorname{Spec}(R) = \operatorname{Spec}(T)$ now follows from the implication (4) $\Rightarrow$ (1) in Theorem 3.10.

Although the equality in condition (2) of Theorem 3.10 concerning maximal ideals may be weakened to an inclusion (see conditions (3) and (4)), no similar weakening is possible for condition (1) about prime ideals or condition (5) about radical ideals. For a family of examples using the "classical $D + M$ construction," abstract the construction in Remark

3.9 as follows: Consider a valuation ring $K + M$, arrange rings $k \subset D \subset K$ such that k is a field and D is not a field, and set $R = k + M$ and $T = D + M$. Then $\mathrm{Spec}(R) \subset \mathrm{Spec}(T)$ (and so each radical ideal of R is a radical ideal of T) but $\mathrm{Spec}(R) \neq \mathrm{Spec}(T)$ since $\mathrm{Max}(T) \not\subset \mathrm{Max}(R)$. Similarly, if A is a domain unequal to its quotient field B, then $\mathrm{Spec}(B) \subset \mathrm{Spec}(A)$ (and each radical ideal of B is a radical ideal of A) but $\mathrm{Spec}(A) \neq \mathrm{Spec}(B)$.

The situation for primary ideals is a bit more complicated. Certainly, if rings $R \subset T$ have the same (sets of) primary ideals, then passing to maximal elements yields $\mathrm{Max}(R) = \mathrm{Max}(T)$ and hence, by Theorem 3.10, $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. However, the converse is false. While one easily infers from $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ (arising from rings $R \subset T$) that each primary ideal of T is a primary ideal of R, not every primary ideal of R need be a primary ideal of T. The next example and proposition expand on this theme.

Example 3.12. Let $L \subset K$ be a proper field extension, and set $T = K[[X]] = K + M$, with $M = XT$. Let $R = L + M$ and $I = XR$. By the lore of the "classical $D + M$ construction," $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. Moreover, I is an M-primary ideal of R, since $M^2 \subset I$. However, I is not a primary ideal of T, as I is not an ideal of T.

PROPOSITION 3.13. *Let $R \subset T$ be distinct rings such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Let Q be a P-primary ideal of R. Then:*

(1) *If Q is an ideal of T, then Q is a P-primary ideal of T.*

(2) *If P is not the maximal ideal of R, then Q is a P-primary ideal of T.*

Proof. (1). It is enough to show that if $x, y \in T$ satisfy $xy \in Q$, then either $x \in Q$ or $y \in P$. Without loss of generality, x is not in R. Hence $x^{-1} \in T$, and so $y = x^{-1}(xy) \in TQ = Q \subset P$.

(2). By (1), it is enough to show that Q is an ideal of T, i.e., that if $t \in T$ and $q \in Q$, then $tq \in Q$. Select $x \in M \backslash P$, where M denotes the maximal ideal of R. Now, $(tq)x = (tx)q \in (TM)Q = MQ \subset Q$ and $tq \in TQ \subset TM = M \subset R$. As Q is a P-primary ideal of R and $x \in R \backslash P$, we have $tq \in Q$, as desired.

The proof of (2) in Proposition 3.13 was merely a reworking of the proof of [**12**, Theorem A(e), p. 564].

Having considered prime, maximal, radical and primary ideals, we now turn briefly to arbitrary (principal) ideals. Consider rings $A \subset B$. If $\mathrm{Spec}(A) = \mathrm{Spec}(B)$, then each proper ideal of B is an ideal of A; the rings $S \subset T$ in Remark 3.9 illustrate that the converse is false. On the other hand, if A contains a non-zerodivisor x of B such that Ax is an ideal of B, then (even without supposing $\mathrm{Spec}(A) = \mathrm{Spec}(B)$) we have $A = B$, for $Ax = BAx = Bx$. In particular, if A and B are domains such that $\mathrm{Spec}(A) = \mathrm{Spec}(B)$ and A has a nonzero principal prime ideal, then $A = B$. The case in which A is a principal ideal domain will be generalized in Corollary 3.16, Remark 3.17(b) and Corollary A.4 below.

It would be wrong to leave the impression that the ideal theory for a pair of rings with the same primes could be predicted on the basis of the known ideal theories for the classical $D + M$ construction [**12**] and for PVD's [**14**]. To illustrate this, recall from [**2**, p. 80] the following strengthening of [**12**, Theorem A (k), p. 562]. If $T = K + M$ is a valuation domain with maximal ideal M and $R = D + M$ where D is a proper subring of the field K such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, then for any ideal I of R, either I is an ideal of T or IT is a principal ideal of T. As shown in [**14**, Proposition 2.11], the same conclusion holds in case R is a PVD which is not a valuation domain and T is its unique valuation overring such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. However, the next example, of "generalized $D + M$" type, shows that the corresponding conclusion fails for arbitrary pairs of rings with the same primes.

EXAMPLE 3.14. *Given distinct fields $L \subset K$, consider $T = K[[X, Y]] = K + M$, where $M = XT + YT$. Set $R = L + M$ and $I = XR + YR$. Now, M is the unique maximal ideal of T, and so the implication* (4) $\Rightarrow$ (1) *in Theorem* 3.10 *yields* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *However, I is not an ideal of T, and $IT = M$ is (by an easy degree argument) not a principal ideal of T.*

Pursuing the theme suggested following Proposition 3.13, we next find additional instances of domains none of whose proper overrings has the same primes. One positive consequence, Proposition 3.19 and its corollaries, will be a counterbalance to Remark 3.9. First, for a domain R, let $C(R)$ and R' denote the complete integral closure of R and the integral closure of R, respectively. If R is not a field, $C(R)$ is the union of the conductors $(I : I)$ corresponding to the nonzero ideals I of R; the (typically smaller) union of conductors indexed by the nonzero finitely generated ideals is precisely R'.

PROPOSITION 3.15. *If $R \subset T$ are domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ *and R is not a field, then $C(R) = C(T)$.*

Proof. Note that $C(R) \subset C(T)$ in general, even without the assumption of equal prime spectra. For the reverse inclusion, let $x \in C(T)$; i.e., $x \in (I : I)$ for some nonzero ideal I of T. It is enough to show that $xJ \subset J$ for some nonzero ideal J of R, for then $x \in (J : J)$, as Lemma 3.1 shows that R and T have the same quotient field. In case $I \neq T$, then I is an ideal of R (since $\mathrm{Spec}(R) = \mathrm{Spec}(T)$), and choosing $J = I$ suffices. Finally, if $I = T$, then $x = x \cdot 1 \in T$, so that choosing J to be any nonzero proper ideal of T works, and completes the proof.

COROLLARY 3.16. *If R is a completely integrally closed domain which is* *not a field, then no domain T properly containing R may satisfy* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$.

Proof. As R is completely integrally closed, $R = C(R)$. Now, if $R \subset T$ and $\operatorname{Spec}(R) = \operatorname{Spec}(T)$, use Proposition 3.15 to get $T \subset C(T) = C(R) = R$, whence $R = T$, as required.

Remarks 3.17. (a) It is well known (cf. [**12**, Theorem 14.5(3)]) that a valuation domain V is completely integrally closed if and only if $\dim(V)$, the Krull dimension of V, is at most 1. Hence, combining Corollary 3.16 and Proposition 2.3, we see that if R is a completely integrally closed PVD, then R is a valuation domain and $\dim(R) \leqq 1$. If we weaken the "completely integrally closed" condition to "Archimedean," (where, as usual, a domain R is said to be *Archimedean* in case $\bigcap Rr^n = 0$ for each nonunit r of R), we see from the proof of [**3**, Proposition 3.5] that if R is an Archimedean PVD, then $\dim(R) \leqq 1$. However, an Archimedean PVD need not be a valuation domain, since any domain of Krull dimension 1 is Archimedean [**17**, Corollary 1.4], while 1-dimensional PVD's need not be valuation domains [**14**, Example 2.1].

(b) If R is a Prüfer domain which is not a field and T is a domain containing R such that $\operatorname{Spec}(R) = \operatorname{Spec}(T)$, then $R = T$. For a proof, we may assume $R \neq T$, and so Proposition 3.3 implies that R is quasilocal (i.e., a valuation domain) and T is an overring of R. If $\dim(R) \leqq 1$, the assertion follows from Corollary 3.16 and the first-cited result in (a). In general, $T = R_P$ for some $P \in \operatorname{Spec}(R)$. The maximal ideal of T is $PR_P = P$ and must coincide with the maximal ideal M of R, so that $T = R_M = R$, as asserted. (For another proof, note that any $x \in T \backslash R$ leads to the contradiction $1 = xx^{-1} \in TM = M$.)

(c) Another approach to the result in (b) will now be given. First, recall from [**19**, Theorem 4] that an integral domain R is a Prüfer domain if and only if each overring of R is a flat R-module. The second proof of (b) then follows from the assertion that whenever a proper overring T of a ring R satisfies $\operatorname{Spec}(R) = \operatorname{Spec}(T)$, T is not a flat R-module. Its proof is rather direct: for each $P \in \operatorname{Spec}(R)$, [**1**, Theorem 1] shows that the canonical map $R_P \rightarrow T_{T \backslash P}$ is an isomorphism. Hence, for each $t \in T$ and $P \in \operatorname{Spec}(R)$, the ideal $I = \{r \in R : rt \in R\}$ satisfies $I \not\subset P$, so that $I = R$, $t = 1 \cdot t \in R$ and $T = R$, as claimed. In case R and T are domains, the appeal to [**1**] in the proof may be replaced by a use of [**19**, Theorem 2].

The following companion of Corollary 3.16 is easily established. If R is an integrally closed quasilocal domain which is not a field, and if the maximal ideal M of R is finitely generated, then no domain T properly containing R may satisfy $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. (The proof is immediate from Proposition 3.3; as $R \neq T$, the equality of prime spectra would give $T \subset (M : M) \subset R' \subset R$, a contradiction.) As our next example shows, the hypothesis that M is finitely generated cannot be removed.

Example 3.18. Let $L \subset K$ be a nontrivial, purely transcendental field extension. Consider any valuation domain of the form $T = K + M$, with maximal ideal M; set $R = L + M$. Then R is integrally closed, T is a proper overring of R, and $\text{Spec}(R) = \text{Spec}(T)$. Of course, M is not a finitely generated ideal of R.

It will be convenient to let INC denote the incomparability property of a ring extension $R \subset T$, i.e., the property that whenever $Q_1 \subset Q_2$ are prime ideals of T satisfying $Q_1 \cap R = Q_2 \cap R$, then $Q_1 = Q_2$.

PROPOSITION 3.19. *Let* $R \subset T$ *be a ring extension satisfying* INC, *such that* R *is quasilocal with maximal ideal* M. *Then* $\text{Spec}(R) = \text{Spec}(T)$ *if and only if* $M \in \text{Spec}(T)$.

Proof. We need only prove the "if" half. Suppose $M \in \text{Spec}(T)$. If M were nonmaximal qua ideal of T, any maximal ideal N of T which contained M would satisfy $N \cap R = M$, contradicting INC. Thus, $M \in \text{Max}(T)$, and an application of Proposition 3.8 completes the proof.

COROLLARY 3.20. *Let* $R \subset T$ *be rings such that* R *is quasilocal with maximal ideal* M. *Assume that either*

(1) T *is an integral extension of* R, *or*

(2) T *is a domain and* M *is a finitely generated ideal of* R.

Then $\text{Spec}(R) = \text{Spec}(T)$ *if and only if* $M \in \text{Spec}(T)$.

Proof. (1) follows from Proposition 3.19 since any integral extension satisfies INC (cf. [**12**, Proposition 9.9]). By Proposition 3.3, (2) is just a special case of (1), since $(M : M) \subset R'$.

The next result may be viewed as a companion to Remark 3.17(b).

COROLLARY 3.21. *Let* R *be a quasilocal domain with maximal ideal* M, *such that* R' *is a Prüfer domain. Let* T *be an overring of* R. *Then* $\text{Spec}(R) = \text{Spec}(T)$ *if and only if* $M \in \text{Spec}(T)$.

Proof. According to [**18**, Proposition 2.26], the condition that R' be a Prüfer domain is equivalent to the requirement that $R \subset S$ satisfy INC for each overring S of R. Hence, Proposition 3.19 applies, ending the proof.

In Corollary 3.16 and the companion result stated following Remarks 3.17, the failure of the given domains R to sustain the existence of proper overrings with the same primes was due to the fact that $(M : M) = R$ in those cases. This equality is characterized in Proposition 3.23 below. First, we shall give an example showing that the inequality $(M : M) \neq R$ does not guarantee that R will have a proper overring with the same primes.

Example 3.22. Let K be a field, consider $T = K[[X]]$, and let $R = K[[X^2, X^3]]$, the subring consisting of those members of T whose coefficient of X is 0. Now, R is a quasilocal Noetherian domain of Krull dimension 1, whose only nonzero prime is $M = X^2R + X^3R$. Observe that $(M : M) = T(\neq R)$; that $\operatorname{Spec}(R) \neq \operatorname{Spec}(T)$, indeed that $\operatorname{Spec}(R) \cap \operatorname{Spec}(T) = \{0\}$; and that there is no ring contained properly between R and T. (To motivate the next result, also observe that $M = RX^{-1} \cap R$ is a divisorial ideal of R.)

PROPOSITION 3.23. *Let R be a quasilocal domain which is not a field. Let M be the maximal ideal of R. Then $(M : M) \neq R$ if and only if M is a nonprincipal divisorial ideal of R.*

Proof. Assume that $(M : M) \neq R$. Then M is nonprincipal, as $(Rx : Rx) = R$ for each nonzero $x \in R$. If M is nondivisorial, then $M \subset (R : (R : M)) \subset R$ forces $(R : (R : M)) = R$, whence

$$R = (R : M) \supset (M : M),$$

the desired contradiction.

Conversely, if M is nonprincipal and divisorial, then

$$M = (R : (R : M)) = (R : (M : M)),$$

the last equality coming from Lemma 2.4(a). As $M \neq R = (R : R)$, we have $(M : M) \neq R$, to complete the proof.

Combining Propositions 3.3 and 3.23 yields:

COROLLARY 3.24. *Let R be a quasilocal domain which is not a field. Assume that the maximal ideal of R is either principal or a nondivisorial ideal of R. Then no domain T properly containing R may satisfy* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$.

Our attention thus far has focused on whether a given pair of rings could have the same prime ideals. We now ask: given a ring R, how does one obtain all overrings T of R which satisfy $\operatorname{Spec}(R) = \operatorname{Spec}(T)$? For simplicity, we restrict attention to domains, for which case the next elementary result completely answers the question. In view of the constructions in Section 2, it should not be too surprising that the answer involves pullbacks.

THEOREM 3.25. *Let R be a quasilocal domain with quotient field K. Let M denote the maximal ideal of R, set $A = (M : M)$, and let $L = R/M$ and $B = A/M$. Let $\pi : A \to B$ be the canonical surjection. Then the set of domains T contained in K and comparable (under inclusion) with R such that* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$ *is in one-to-one correspondence with the set of fields F which are comparable with L and are contained in B. An explicit one-to-one correspondence, $T \leftrightarrow F$, is given by $T = \pi^{-1}(F)$. Such T contains (resp., is contained in) R if and if the corresponding F contains (resp., is contained in) L. Finally, another description of the correspondence $T \leftarrow F$*

arises since T is canonically isomorphic to the pullback in the category of (commutative) rings (with 1) of the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ F & \longrightarrow & B \end{array}$$

in which the vertical map is π and the horizontal map is the inclusion.

Proof. If T is a domain inside K such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, then M is an ideal of T, so that $T \subset A$. Now, $F = T/M$ is a field contained in B, and $\pi^{-1}(F) = T$. By the standard homomorphism theorem, R and T bear the same inclusion relation (if any) to each other as do L and F.

On the other hand, given a field F which is contained in B and is comparable with L, then $T = \pi^{-1}(F)$ is comparable with R, and $T/M \cong F$. Hence, $M \in \mathrm{Max}(T)$, and the implications (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) in Theorem 3.10 yield $\mathrm{Spec}(R) = \mathrm{Spec}(T)$.

A standard homomorphism theorem establishes the asserted one-to-one correspondence. Finally, for the alternate construction of T from F, note that the usual construction of the pullback for the given diagram is the ring $\{(a, f) \in A \times F : \pi(a) = f\}$ which, via first projection to A, is clearly isomorphic to $\pi^{-1}(F) = T$. This completes the proof.

The preceding theorem may be used to reduce questions about intermediate rings to questions about intermediate fields, as in the next corollary.

COROLLARY 3.26. *Let $R \subset T$ be distinct rings such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Let M be the unique maximal ideal of R. (M exists by Proposition 3.3.) Then* $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ *for each ring S contained between R and T if and only if the field extension $R/M \subset T/M$ is algebraic.*

Proof. Set $L = R/M$ and $B = T/M$. Now, $L \subset B$ is an algebraic extension if and only if each ring D contained between L and B is a field (cf. [**12**, Lemma 9.1]). As in the preceding proof, the set of such rings D is in one-to-one correspondence with the set of rings S contained between R and T; the correspondence $D \leftarrow S$ is given by $D = S/M$. Now, each D is a field if and only if $M \in \mathrm{Max}(S)$ for each S; by the implication (3) $\Rightarrow$ (1) in Theorem 3.10, the latter condition holds if and only if $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ for each S, which completes the proof.

It is easy to use the "classical $D + M$ construction" in order to produce an example for which the condition in Corollary 3.26 fails to hold. However, we shall soon see that the condition in Corollary 3.26 does hold in case R is Noetherian but not a field. The next preparatory result will also find use in Section 4. As usual, $[K : L]$ will denote the L-vector space dimension of K, given fields $L \subset K$.

LEMMA 3.27. *Let* $R \subset T$ *be distinct rings such that* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. *Let* M *be the maximal ideal of* R. *If some nonzero ideal* I *of* T *is finitely generated as an* R*-module, then* T *is finitely generated as an* R*-module and* $[T/M : R/M] < \infty$.

Proof. We sketch how to ape the proof given in [**5**, Lemma 1], itself an adaptation of the approach in [**10**, Lemma 1]. As Nakayama's lemma guarantees $I \neq MI$, it follows that I/MI is a nonzero finite-dimensional R/M-vector space. However, I/MI is also a direct sum of copies of T/M, whence $[T/M : R/M] < \infty$. If the cosets $t_1 + M, t_2 + M, \ldots, t_n + M$ form an R/M-basis of T/M, then $T = M + \sum Rt_i$, and so

$$\{t_1, t_2, \ldots, t_n, 1\}$$

generates T as an R-module, to complete the proof.

Combining Lemma 3.27 and either Corollary 3.26 or Corollary 3.20 leads to:

COROLLARY 3.28. *Let* $R \subset T$ *be distinct rings such that* R *is not a field and* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. *If the maximal ideal of* R *is finitely generated over* R, *then* $\operatorname{Spec}(R) = \operatorname{Spec}(S)$ *for each ring* S *contained between* R *and* T.

COROLLARY 3.29. *Let* $R \subset T$ *be distinct rings such that* R *is not a field and* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. *Let* M *be the maximal ideal of* R. *Then* R *is Noetherian if and only if both* T *is Noetherian and* $[T/M : R/M] < \infty$.

Proof. The proof of [**5**, Theorem 4] adapts easily. For the "if" half, argue as in the last step of the proof of Lemma 3.27 to get T finitely generated as an R-module, and then appeal to [**11**, Theorem 2]. The "only if" half follows readily from Lemma 3.27.

In view of Proposition 2.3 and [**8**, Proposition 4.2], the preceding result immediately gives an elementary proof of the next result of [**14**, Proposition 3.2 and Corollary 3.4]. Its proof in [**14**] depends on Krull's principal ideal theorem and Noether's conditions for a Dedekind domain.

COROLLARY 3.30. *If* R *is a Noetherian* PVD, *then* $\dim(R) \leqq 1$ *and* R' *is a discrete (rank* 1*) valuation domain.*

The next result about intermediate rings has analogues for the $D + M$ constructions ([**12**, Theorem A(c), p. 560], [**5**, Proposition 8]).

PROPOSITION 3.31. *Let* $R \subset T$ *be distinct rings such that* $\operatorname{Spec}(R) = \operatorname{Spec}(T)$. *Let* S *be a ring contained between* R *and* T. *If* M *is the maximal ideal of* R *and* I *is an ideal of* S, *then either* $I \subset M$ *or* $M \subset I$.

Proof. If the assertion fails, select $x \in M \backslash I$ and $y \in I \backslash M$. As $y \in T \backslash M$, we have $y^{-1} \in T$, and so $y^{-1}x \in TM = M$. Thus, $x = y(y^{-1}x) \in IM \subset IS = I$, a contradiction.

We proceed to a variation on the method in the proofs of Theorem 3.25 and Corollary 3.26.

LEMMA 3.32. *Let $R \subset A$ be rings. Then there exists a ring T maximal with respect to the properties* $\text{Spec}(R) = \text{Spec}(T)$ *and* $R \subset T \subset A$.

Sketch of proof. Zorn's lemma may be used. To verify the condition on chains, note in general that if B is the union of a chain of rings B_i and if I is an ideal of B, then $I = U(B_i \cap I)$.

Let R be a quasilocal domain which is not a field. Let A, L and B be as in the statement of Theorem 3.25. A domain T maximal with respect to the properties $\text{Spec}(R) = \text{Spec}(T)$ and $R \subset T$ is guaranteed to exist by combining Proposition 3.3 and Lemma 3.32. In the one-to-one correspondence established in Theorem 3.25, such maximal T corresponds to a field F maximal with respect to the property $L \subset F \subset B$. The nature of such maximal subfields seems worthy of further study. For the present, we note only that such F need not be unique, given R (cf. Remark 3.33(b) below), and so the maximal rings T need not be unique. Of course, in case R either is a PVD or arises from the "classical $D + M$ construction" with D a field, there will be a unique maximal overring T with the same primes.

We close this section with another remark about the "mod M" method of Theorem 3.25 and some Noetherian examples illustrating behavior unlike that of PVD's.

Remarks 3.33. (a) Let B be a ring containing the field L. In order to stimulate interest in the maximal subfields of B which contain L, we construct a quasilocal ring R whose maximal ideal M contains a non-zero-divisor, such that $R/M \cong L$ and $(M : M)/M \cong B$ (thus reversing the process considered prior to this remark). For the construction, set $A = B[[X]]$ and $M = XA$. (Caution: although $A = B + M$, the "generalized $D + M$ construction" context need not be present here, for $B \cong A/M$ is not a field in the interesting case.) Set $R = L + M$. Of course, $R/M \cong L$. We leave to the reader the straightforward verifications that R is a quasilocal ring with maximal ideal M containing the non-zerodivisor X and that $(M : M) = A$. (Note also that R is a domain if and only if B is a domain.)

(b) Next, as an application of (a), we shall construct a Noetherian ring R having (at least) two overrings maximal with respect to the property of having the same primes as R. For the construction, let L be a subfield of the real numbers, and view L as being contained naturally in

$$B = L[X, Y]/(X^2 + 1, Y^2 + 1) = L[x, y],$$

where x and y are the cosets represented by X and Y, respectively. Construct R as in (a). As the class of Noetherian rings is closed under homomorphic images and adjunction of individual (power series—or polynomial—) indeterminates, $A = B[[X]]$ is Noetherian. However, B is finite-dimensional over L (more about this below), and so A is a finitely generated R-module. Thus, by [**11**, Theorem 2], R is Noetherian. By the above "mod M" method, it therefore suffices to show that $L[x]$ and $L[y]$ are distinct maximal subfields of B (containing L).

First, we claim that $\{1, x, y, xy\}$ is linearly independent over L, i e., that

$$a + bX + cY + dXY = (X^2 + 1)f + (Y^2 + 1)g$$

for $a, b, c, d \in L$ and $f, g \in L[X, Y]$ implies $a = b = c = d = 0$. To prove the claim, let i be "the" square root of -1 in the complex numbers and note that $\{1, i\}$ is linearly independent over L. Accordingly, substituting $X = i$ and $Y = i$ into the above relation yields $a - d = 0 = b + c$. Similarly, substituting $X = i$ and $Y = -i$ leads to $a + d = 0 = b - c$. Thus, $a = b = c = d = 0$, as required. As $\{1, x, y, xy\}$ generates B as an L-vector space, it is then a basis, and so $\dim_L(B) = 4$.

Next, note that $L[x]$ and $L[y]$ are each fields, as each is the image of an appropriate (injective) ring-homomorphism from $L(i)$ to B. Moreover, B is not a field, as the polynomial $T^2 + 1 \in B[T]$ has (at least) four roots in B. If F is any field contained between $L[x]$ and B, then

$$4 = \dim_L(B) = \dim_F(B)\cdot[F : L[x]]\cdot[L[x] : L] \geqq 2\cdot[F : L[x]]\cdot 2,$$

so that $[F : L[x]] = 1$ and $L[x] = F$. Thus $L[x]$ and, similarly, $L[y]$ are maximal subfields of B. Finally, they are distinct, for $y \in L[x] = L + Lx$ would violate the L-linear independence of the basis treated above.

Examples 3.34. (a) We next give an example of a Noetherian domain R such that $\dim(R) = 1$, R is not a PVD, and there is exactly one proper overring T of R with the property $\mathrm{Spec}(R) = \mathrm{Spec}(T)$.

For the construction, start with a field extension $L \subset K$ such that $[K : L] = 2$, with L of characteristic other than 2. Set $T = K[[X^2, X^5]]$, the ring consisting of those formal power series over K whose coefficients of X and X^3 are 0. Then $T = K + M$, where $M = X^2T + X^5T$ is the unique maximal ideal of T; let $R = L + M$. Note that $(M : M) = K[[X^2, X^3]]$, a ring already encountered in Example 3.22. It will be shown below that $K(\cong T/M)$ is the only field which properly contains $L(\cong R/M)$ and is contained in $B = (M : M)/M$. Assuming this for the present, we have, by the comments following Lemma 3.32, that T is the only proper overring of R with the same primes as R. As T is module-finite over R and T is Noetherian, [**11**, Theorem 2] shows that R is Noetherian. As $R' = K[[X]]$ and integrality preserves Krull dimension, we have $\dim(R) = 1$. Finally, R is not a PVD, as M is not strongly prime, for $(X^3)^2 \in M$ although $X^3 \notin M$. (For the same reason, T is not a PVD. Note that $(M : M)$ is also not a PVD, for its maximal ideal contains X^2 but not X.)

It remains only to prove that K is the unique field such that $L \subsetneq K \subset B$. If x denotes the image of X in B, then $B = K + Kx^3$. Now, $x^3 \neq 0$ (as $X^3 \notin M$), although $(x^3)^2 = 0$, and so B is not a field. In particular, $\dim_K B = 2$ and $\{1, x^3\}$ is a K-basis of B. Reasoning as in Remarks 3.33 (b), we have $\dim_L B = 4$. Thus, if F is a field properly containing L and contained in B, it follows that $[F : L] = 2$. By the "quadratic formula," $F = L[u]$, with $u^2 \in L$. Write $u = a + bx^3$, with $a, b \in K$. As $(x^3)^2 = 0$ and $\{1, x^3\}$ is linearly independent over K, the requirement that $u^2 \in K$ leads to $2ab = 0$, so that either $a = 0$ or $b = 0$. If $a = 0$, then $u^2 = b^2x^6 = 0$, although u is nonzero in the field F, a contradiction. Thus $b = 0$, and $u = a \in K$, whence $F \subset K$. As $[F : L] = [K : L] = 2$, it follows that $F = K$, and the proof is complete.

(b) Finally, for each $n \geqq 1$, we give an example of a quasilocal Noetherian domain R with maximal ideal M such that $\dim(R) = n$ and $(M : M)$ is the unique maximal proper overring of R having the same primes as R. It will follow from [**14**, Example 2.1] that R is a PVD if $n = 1$, while by Corollary 3.30, R is not a PVD if $n > 1$.

We begin the construction with a proper finite-dimensional field extension $L \subset K$. Let $T = K[[X_1, \ldots, X_n]]$ be the ring of formal power series in n variables over K. Then $T = K + M$, where

$$M = X_1T + \ldots + X_nT$$

is the unique maximal ideal of T. It is standard that T is Noetherian, $\dim(T) = n$ and T is completely integrally closed (see [**12**, Theorem 12.9] for the last of these). Let $R = L + M$. Then R is Noetherian by [**11**, Theorem 2], since $[K : L] < \infty$ forces T to be a finitely generated R-module. By integrality and the above information about T, we have that $\operatorname{Spec}(R) = \operatorname{Spec}(T)$, so that R is quasilocal with maximal ideal M and $\dim(R) = n$. To complete the proof, Proposition 3.3 shows that it is enough to establish $(M : M) = T$. As M is an ideal of T, one inclusion is clear; for the other, recall that T is completely integrally closed, so that $T = C(T) \supset (M : M)$, as claimed.

Appendix A. This appendix treats some questions in the spirit of Corollaries 3.16 and 3.29 for the case of GCD-domains and finite-conductor domains (definitions recalled below). Similar inheritance properties for divided domains and going-down rings are the subject of Appendix B.

We begin by sharpening part of Proposition 3.23.

LEMMA A.1. *Let R be a quasilocal domain with maximal ideal M and quotient field K. If $x \in K \backslash R$ satisfies $xM = M$, then $M = Rx^{-1} \cap R$ (and so M is a divisorial ideal of R).*

Proof. As $xM = M$, it follows that $x^{-1}M = M$, and so $M \subset Rx^{-1} \cap R$. If the assertion fails, then $Rx^{-1} \cap R = R$, whence $x \in R$, the desired contradiction.

COROLLARY A.2. *Let $R \subset T$ be distinct domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Let M be the unique maximal ideal of R. (M exists by Lemma 3.2.) If $x \in T \backslash R$, then $M = Rx^{-1} \cap R$ (and so M is a divisorial ideal of R).*

Proof. As R and T have the same nonunits, $x^{-1} \in T$, whence $xM = M$ and Lemma A.1 applies, to complete the proof.

Recall that a domain R is termed *finite-conductor* in case $Rx \cap Ry$ is a finitely generated ideal for each $x, y \in R$. By [**4**, Exercise 12(g), p. 24], any coherent domain is finite-conductor. Moreover, any GCD-domain (*pseudo-Bézout domain* in [**4**, p. 551]) is also finite-conductor.

COROLLARY A.3. *Let $R \subset T$ be distinct domains such that R is finite-conductor and* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Let M be the maximal ideal of R. Then both M and T are finitely generated R-modules.*

Proof. By Corollary A.2, $M = Rx^{-1} \cap R$ for any $x \in T \backslash R$. As Proposition 3.3 gives $x = ab^{-1}$ for some $a, b \in R$ and as R is finite-conductor, M is isomorphic to the finitely generated ideal $Rb \cap Ra$, and so M itself is finitely generated. Hence, by Lemma 3.27, T is also a finitely generated R-module.

We next present a companion to Corollary 3.16, [**5**, Theorem 11] and [**2**, Theorem 3.13] which yields our third proof of the result in Remarks 3.17(b).

COROLLARY A.4. *If distinct domains $R \subset T$ satisfy* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, *then R may not be a* GCD-*domain.*

Proof. By Proposition 3.3, T is an overring of R. Consequently, if R were a GCD-domain (and hence, by an argument going back to Gauss, necessarily integrally closed), Corollary A.3 would force T to be integral over R, whence $R = T$, the desired contradiction.

COROLLARY A.5. [**14**, Proposition 2.2] *A domain R is a valuation domain if and only if R is both a* GCD-*domain and a* PVD.

Proof. The "only if" half is trivial. For the "if" half, combine Corollary A.4 and Proposition 2.3.

The proof (of the "if" half) of Corollary A.5 in [**14**] is quite different from that given above, inasmuch as the proof in [**14**] depends upon the fact that the primes in a PVD are linearly ordered. (More general cases of such "treed" domains are studied in Appendix B below.) As noted in [**14**], it follows from [**16**, Theorem 1] that a GCD-domain whose primes are linearly ordered must be a valuation domain. Alternate references for this fact include [**21**, Proposition A], [**20**, Corollary 3.8] and [**6**, Corollary 4.3].

Remark A.6. Another reason for considering finite-conductor domains is given next. The reader may have noted that the converse of Corollary 3.28 is false. In other words, if $R \subset T$ are distinct rings such that R is not a field and $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ for each ring S such that $R \subset S \subset T$, then the maximal ideal M of R need not be finitely generated over R. (For a simple example using the "classical $D + M$ construction," let $R = \mathbf{Q} + M$ arise from the valuation domain $T = F + M$, where F is the algebraic closure of the rational number field $\mathbf{Q}$. Lemma 3.27 shows that M is not finitely generated over R.) However, the converse does hold in case T is a domain and R is a finite-conductor domain. To prove this, observe that $M \in \mathrm{Spec}(R[u])$ for any $u \in T \backslash R$, whence $u \in (M : M)$. Since $M \subset I = \{r \in R : ru \in R\} \neq R$, we have $M = I$. As R is finite-conductor, I is a finitely generated ideal, completing the proof.

The remainder of Appendix A gives a unified approach to some recent results. We begin with the "coherent" analogue of Corollary 3.29.

PROPOSITION A.7. *Let $R \subset T$ be distinct domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Let M be the maximal ideal of R. Then R is coherent if and only if the following three conditions hold: T is coherent, M is finitely generated as an ideal of T, and $[T/M : R/M] < \infty$*

Proof. Modify the approach in [**10**, Theorem 3] or [**5**, Theorem 3]. Argue as in Remark A.6 to show that coherence of R forces M to be finitely generated over R, and then use Lemma 3.27 to establish the "only if" half. Further details are left to the reader.

COROLLARY A.8. [**10**, Corollary 5] *Let $V = K + M$ be a valuation domain with nonzero maximal ideal M, such that K is a field. Let $R = k + M$ where k is a proper subfield of K. Then R is coherent if and only if both $[K : k] < \infty$ and $M \neq M^2$.*

Proof. Proposition A.7 applies since $\mathrm{Spec}(R) = \mathrm{Spec}(V)$, V is coherent, and, as noted in the proof of [**10**, Corollary 5], the conditions $M \neq M^2$ and "M is a finitely generated ideal of V" are equivalent.

COROLLARY A.9. ([**9**, Proposition 3.5], [**15**, Theorem 1.6]) *Let R be a* PVD *which is not a valuation domain, with maximal ideal M. Let*

$V = (M : M)$. *Then R is coherent if and only if both* $[V/M : R/M] < \infty$ *and M is a principal ideal of V.*

Proof. By Proposition 2.5, V is a valuation domain and, hence, coherent. Since $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ by Proposition 2.5, Proposition A.7 applies, to complete the proof.

COROLLARY A.10. *Let $R \subset T$ be domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ *and T is coherent. Then R is coherent if and only if R is finite-conductor.*

Proof. The "only if" half is immediate. For the converse, $R \neq T$ without loss of generality, and so Proposition 3.3 implies that R has a unique maximal ideal, say M. By Corollary A.3, since R is finite-conductor, M and T are finitely generated over R. Thus, M is also finitely generated over T and $[T/M : R/M] < \infty$, so that Proposition A.7 applies, to complete the proof.

COROLLARY A.11. ([**9**, Proposition 3.5], [**15**, Theorem 1.6]) *Let R be a* PVD *with maximal ideal M. Then*:

(a) *R is coherent if and only if R is finite-conductor.*

(b) *If R is not a valuation domain, then R is coherent if and only if M is a finitely generated ideal of R.*

Proof. (a). The "only if" half is trivial. For the "if" half, Proposition 2.5 places R inside the (coherent) valuation domain $V = (M : M)$, with $\mathrm{Spec}(R) = \mathrm{Spec}(V)$. Apply Corollary A.10.

(b) Using the extension $R \subset V$ considered in the proof of (a), we immediately deduce the "only if" half from Corollary A.3. For the "if" half, again use $R \subset V$, note by Lemma 3.27 that $[V/M : R/M] < \infty$, and apply Proposition A.7.

Appendix B. This brief section is intended to provide additional motivation for studying pairs of rings with the same prime ideals. Its two results may be viewed as companions to Proposition 2.2 concerning PVD's. First, note directly from the definition of PVD's in terms of strongly prime ideals that, if R is a PVD then R is *divided* in the sense of [**7**], i.e., that $P = PR_P$ for each $P \in \mathrm{Spec}(R)$. Of course, a divided domain need not be a PVD, even in the integrally closed case [**8**, Remark 4.10(b)].

PROPOSITION B.1. *Let $R \subset T$ be domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Then R is divided if and only if T is divided.*

Proof. It is enough to show that $PR_P = PT_P$ for each $P \in \mathrm{Spec}(R)$. However, this condition is evident: in case P coincides with the maximal ideal M of R (also of T!), the assertion reduces to $M = M$, while the case $P \neq M$ is disposed of by using Proposition 3.9(b).

Divided domains play a key role, along with the concept of an "integral unibranched extension," in characterizing quasilocal going-down rings [**7**, Theorem 2.5]. (Recall that a domain R is called a *going-down* ring—and we write R is GD—in case the extension $R \subset S$ has the going-down property for each domain S containing R. Examples of going-down rings include all Prüfer domains, as well as the domains of Krull dimension 1.) Any divided domain is a quasilocal going-down ring [**7**, Proposition 2.1], while the converse holds in the root-closed case [**7**, Corollary 2.8]. As well as being an analogue of Proposition 2.2, our final result may be viewed as a "nonintegral" variant of [**7**, Lemma 2.3]: if $R \subset T$ is an integral unibranched extension of domains, then R is GD if and only if T is GD.

PROPOSITION B.2. *Let $R \subset T$ be domains such that* $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. *Then R is* GD *if and only if T is* GD.

Proof. The "only if" half is obvious, since the going-down property for extensions is transitive. For the converse, let T be GD, and let V be a domain containing R. Our task is to show that the extension $R \subset V$ has the going-down property; i.e., if $P_2 \subset P_1$ are primes of R and $Q_1 \in \mathrm{Spec}(V)$ satisfies $Q_1 \cap R = P_1$, then we must produce $Q_2 \in \mathrm{Spec}(V)$ such that $Q_2 \subset Q_1$ and $Q_2 \cap R = P_2$. First, by replacing V by its localization at Q_1, it is harmless to suppose that V is quasilocal; by "abus de langage," we now let Q_1 denote the maximal ideal of V. Next, consider the ring $S = TV$ (a procedure dubbed "the rectangle argument" in [**18**, p. 7]). Observe that $P_1S = (P_1T)V = P_1V \subset Q_1$, whence $1 \notin P_1S$. Consequently $P_1S \neq S$, and we may select $W_1 \in \mathrm{Spec}(S)$, minimal among primes of S which contain P_1S. As T is GD, the extension $T \subset S$ has the going-down property. Thus $W_1 \cap T = P_1$, and there exists $W_2 \in \mathrm{Spec}(S)$ such that $W_2 \subset W_1$ and $W_2 \cap T = P_2$. Set $Q_2 = W_2 \cap V$. Evidently, $Q_2 \in \mathrm{Spec}(V)$ and $Q_2 \cap R = P_2$. Moreover, $Q_2 \subset Q_1$, since we began by arranging that Q_1 be the unique maximal ideal of V, thus completing the proof.

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